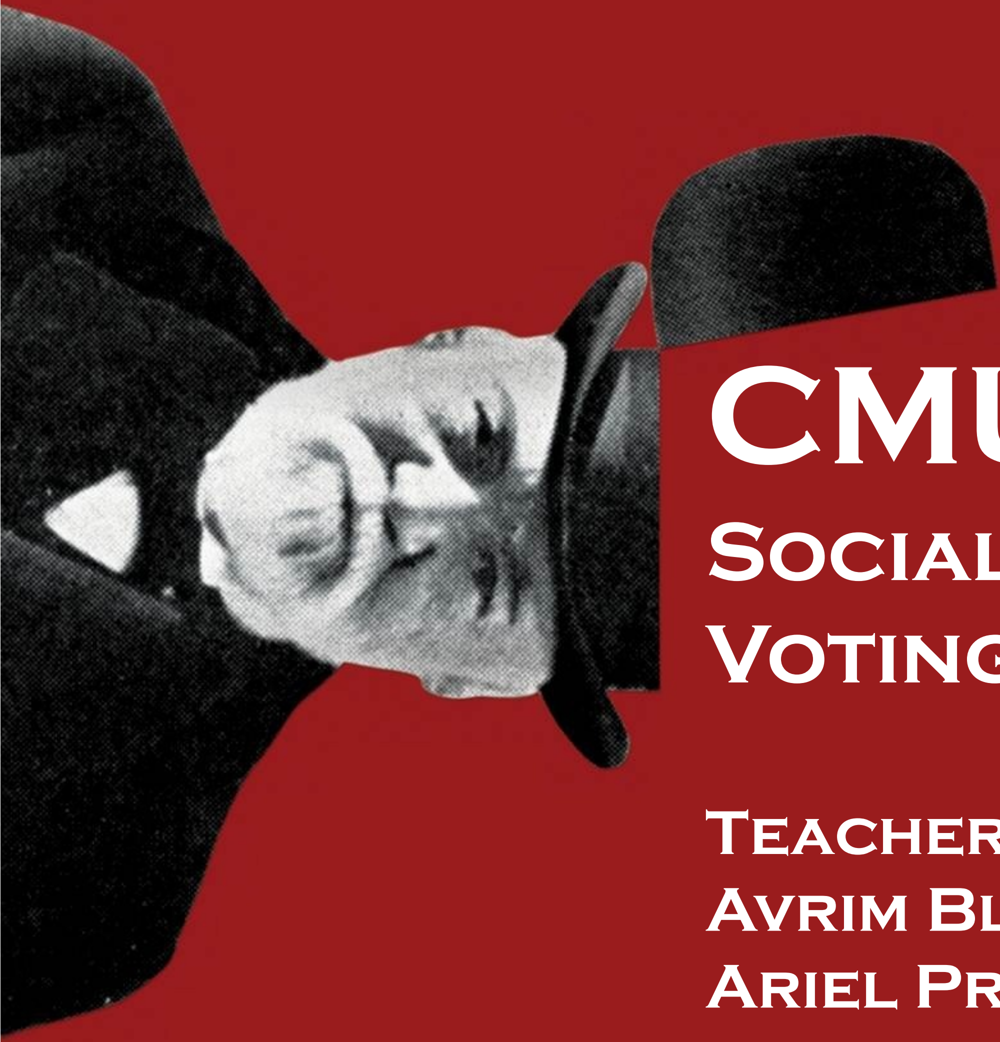

CMU 15-896

Social choice: Voting rules as MLEs

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History

- For Condorcet [1785], the object of voting is not merely to balance subjective opinions; it is a collective quest for the truth
- Enlightened voters try to judge which alternative best serves society
- For $m = 2$ the majority opinion will very likely be correct
- Realistic in trials by jury or the pooling of expert opinions — or in human computation!

Motivation: EteRNA

- Developed at CMU (Adrien Treuille) and Stanford
- Choose 8 RNA designs to synthesize
- Some designs are truly more stable than others
- The goal of voting is to compare the alternatives by true quality

Condorcet's noise model

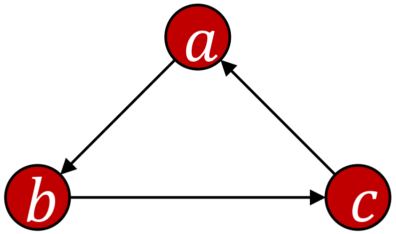
- True ranking of the alternatives
- Voting pairwise on alternatives, each comparison is correct with prob. $p > 1/2$
- Results are tallied in a voting matrix

	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-

Condorcet's 'solution'

- Condorcet's goal: find "the most probable" ranking
- Condorcet suggested: take the majority opinion for each comparison; if a cycle forms, "successively delete the comparisons that have the least plurality"
- In example, we delete $c \succ a$ to get $a \succ b \succ c$

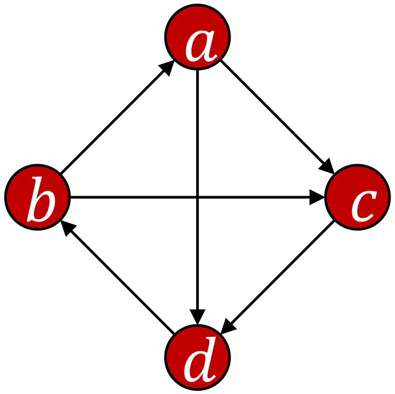
	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-



Condorcet's 'solution'

- With four alternatives we get ambiguities
- In example, order of strength is $c \succ d$, $a \succ d$, $b \succ c$, $a \succ c$, $d \succ b$, $b \succ a$
- Delete $b \succ a \Rightarrow$ still cycle
- Delete $d \succ b \Rightarrow$ either a or b could be top-ranked

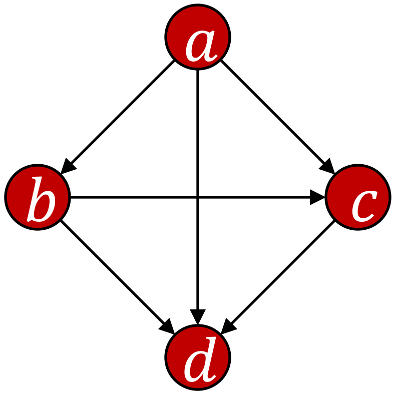
	a	b	c	d
a	-	12	15	17
b	13	-	16	11
c	10	9	-	18
d	8	14	7	-



Condorcet's 'solution'

- Did Condorcet mean we should reverse the weakest comparisons?
- Reverse $b \succ a$ and $d \succ b \Rightarrow$ we get $a \succ b \succ c \succ d$, with 89 votes
- $b \succ a \succ c \succ d$ has 90 votes (only reverse $d \succ b$)

	a	b	c	d
a	-	12	15	17
b	13	-	16	11
c	10	9	-	18
d	8	14	7	-



Exasperation?

- "The general rules for the case of any number of candidates as given by Condorcet are stated so briefly as to be hardly intelligible . . . and as no examples are given it is quite hopeless to find out what Condorcet meant" [Black, 1958]
- "The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends ... no amount of examples can convey an adequate impression of the evils" [Todhunter 1949]

Young's solution

- Suppose true ranking is $a \succ b \succ c$; prob of observations:
$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{11} p^{11}(1-p)^2$$
- For $a \succ c \succ b$ prob. is:
$$\binom{13}{8} p^8(1-p)^5 \cdot \binom{13}{6} p^6(1-p)^7 \cdot \binom{13}{2} p^2(1-p)^{11}$$
- Coefficients are identical
- Exponent of p is #agreements, exponent of $1-p$ is #disagreements

	a	b	c
a	-	8	6
b	5	-	11
c	7	2	-

Young's solution

- M = matrix of votes
- $\Pr[\succ|M] = \frac{\Pr[M|\succ] \cdot \Pr[\succ]}{\Pr[M]}$
- Assume uniform prior over $\succ$, $\Pr[\succ] = \frac{1}{m!}$
- Must maximize $\Pr[M|\succ]$, do this by minimizing #disagreements with observed votes on pairs of alternatives
- This is the Kemeny rule (NP-hard!)

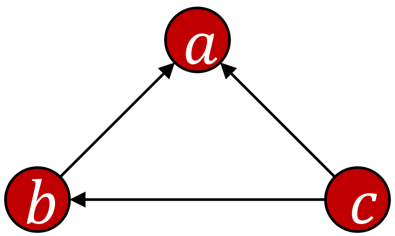
Condorcet vs. Borda

- Borda was a contemporary of Condorcet
- Noted for work in hydraulics, mechanics, optics, and the design of navigational instruments
- His voting rule was used by the French Academy of Sciences
- Condorcet held Borda's work in low esteem, but...

Which alternative is best?

- The top-ranked alternative of the MLE ranking may not be the most likely best alternative
- $c \succ b \succ a$ is MLE ranking
- c is best if $c \succ a$ and $c \succ b$
- Let M_{xy} be the votes for x, y
- $\Pr[c \succ a | M_{ca}] = \frac{p^{31}(1-p)^{29}}{p^{31}(1-p)^{29}+p^{29}(1-p)^{31}}$
- $\Pr[c \succ b | M_{cb}] = \frac{p^{31}(1-p)^{29}}{p^{31}(1-p)^{29}+p^{29}(1-p)^{31}}$

	a	b	c
a	-	23	29
b	37	-	29
c	31	31	-



Which alternative is best?

- $\Pr[c|M_{ca} \wedge M_{cb}] = \frac{[p^{31}(1-p)^{29}]^2}{[p^{31}(1-p)^{29}+p^{29}(1-p)^{31}]^2}$
- $\Pr[b|M_{ba} \wedge M_{bc}] =$
$$\frac{p^{37}(1-p)^{23}p^{29}(1-p)^{31}}{[p^{37}(1-p)^{23}+p^{23}(1-p)^{37}]\cdot[p^{29}(1-p)^{31}+p^{31}(1-p)^{29}]}$$
- $\Pr[a|M_{ab} \wedge M_{ac}] =$
$$\frac{p^{23}(1-p)^{37}p^{29}(1-p)^{31}}{[p^{23}(1-p)^{37}+p^{37}(1-p)^{23}]\cdot[p^{29}(1-p)^{31}+p^{31}(1-p)^{29}]}$$
- Vote: who is best when $p \sim 1$?

Which alternative is best?

- What about $p \sim 1/2$?
- **Theorem [Young 1995]:** When p is sufficiently close to $\frac{1}{2}$, Borda is MLE for the best alternative (assuming individual rankings)
- Vote: who is the Borda winner?

	a	b	c
a	-	23	29
b	37	-	29
c	31	31	-

Ten years later...

- Noise model = distribution over preference profiles for each true winner/ranking
- Which voting rules have a noise model for which they are MLEs of the true ranking (MLER) or true winner (MLEW)? [Conitzer and Sandholm, 2005]
- Vote: Neutral rule is MLER/MLEW for some noise model?
- Assume: votes are i.i.d.
- We focus on MLEWs

Scoring rules as MLEWs

- **Theorem [Conitzer and Sandholm 2005]:** any scoring rule is an MLEW.
- **Proof:**
  - w = true winner
  - The probability that a voter i ranks w in position $r_i(w)$ is proportional to $2^{s_{r_i(w)}}$, and the other alternatives are ranked randomly
  - $\Pr[M|w] \propto \prod_{i=1}^{n} 2^{s_{r_i(w)}} = 2^{\sum s_{r_i(w)}}$ ■

Maximin is not an MLEW

- **Lemma:** If there exist preference profiles $\succ^1$ and $\succ^2$ such that $f(\succ^1) = f(\succ^2) \neq f(\succ^3)$, where $\succ^3$ is their union, then f is not an MLEW
- **Proof:** $\Pr[\succ^3 |x] = \Pr[\succ^1 |x] \cdot \Pr[\succ^2 |x]$ ■
- **Lemma:** Any pairwise comparison graph whose weights are even-valued can be realized via votes
- **Proof:** To increase the weight on the edge (a, b), add the votes $a \succ b \succ x_1 \succ \cdots \succ x_{m-2}$ and $x_{m-2} \succ \cdots \succ x_1 \succ a \succ b$ ■

Maximin is not an MLEW

- **Theorem [Conitzer and Sandholm 2005]:** Maximin is not an MLEW
- **Proof:**