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Lecture 16

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1 Overview

In this lecture we consider the problem of maximizing social welfare in combinatorial auctions. We will consider the setting with n items and m buyers. Buyer i has valuation function v_i and we assume each buyer is only interested in what he/she gets (v_i only depends on buyer i's allocation). We already know that VCG mechanisms maximize the social welfare, but this can be computationally hard. Instead we prefer a simpler scheme in which we set prices to each item.

2 Winner Determination and Valuation Functions

First, let us forget about prices and incentives, and only worry about the winner determination problem: given a set of v_i's, we want to allocate the items so that the social welfare is maximized. There are several classes of valuation functions that we can consider:

1. Additive valuation functions: $\forall i,\ v_i(S) = \sum_{x \in S} v_i(\{x\})$. We can maximize social welfare simply by giving each item to the buyer who values it the most.
2. Unit demand valuation functions: $\forall i,\ v_i(S) = \max_{x \in S} v_i(\{x\})$. In this case we can reduce the winner determination problem to a weighted bipartite matching problem (add an edge of weight $v_i(\{x\})$ between buyer i and item x).
3. Single minded valuation functions: $\forall i,\ \exists S_i,\ \forall S \not\supseteq S_i,\ v_i(S) = 0$. In other words each buyer has a single set that he/she wants. In this case maximizing social welfare becomes the set packing problem.
4. Subadditive valuation functions: $\forall i, S, T\ v_i(S \cup T) \leq v_i(S) + v_i(T)$.
5. Submodular valuation functions: $\forall S, x, y, i,\ v_i(S \cup \{x, y\}) - v_i(S \cup \{x\}) \leq v_i(S \cup \{y\}) - v_i(S)$ (another equivalent definition is $\forall S, T, i,\ v_i(S) + v_i(T) \geq v_i(S \cup T) + v_i(S \cap T)$).

We note that maximizing social welfare is NP-hard in the last three cases.

3 Setting Prices

We study the following simple mechanism. We put the *same* price on each item, buyers come in an arbitrary order, and buy whatever they want at these prices from what still remains on the shelf. If we want to view this as a direct-revelation mechanism, we would ask buyers to submit their valuations and then just go through them in some order, acting on their behalf (so it would be trivially incentive-compatible).

Definition 1 *A set S_i is supported at price p for buyer i if for all $W \subseteq S_i$, $v_i(W) \geq p|W|$.*

Claim 2 *Suppose buyer i is shown a set T_i and buys S_i (i.e. $S_i = \arg\max_{S \subseteq T_i} v_i(S) - p|S|$). If v_i is subadditive, then S_i must be supported at price p.*

Proof: Suppose for a contradiction that the claim does not hold. Then there exists $W \subseteq S_i$ such that $v_i(W) < p|W|$. Then since v_i is subadditive,

$$\begin{aligned} v_i(S_i) &\leq v_i(S_i \setminus W) + v_i(W) \\ v_i(S_i) &< v_i(S_i \setminus W) + p|W| \\ v_i(S_i) - p|S_i| &< v_i(S_i \setminus W) - p|S_i \setminus W|. \end{aligned}$$

Thus $S_i \setminus W$ is preferred to S_i and this is a contradiction. ∎

Let $T_1, \ldots, T_m$ denote the social welfare maximizing allocation, and assume $1 \leq \max_S v_i(S) \leq H$ for some maximum value H. Suppose we pick a random price p among $\{H, H/2, H/4, \ldots, 1/4n\}$ and suppose we somehow could require that buyer i only select from T_i (we can't do this, and don't even know what the T_i are, but just suppose). Let $L_{i,p}$ be the set buyer i would choose. In the last lecture we showed that

$$E\left[p|L_{i,p}|\right] = \Omega\left(\frac{v_i(T_i)}{\log(nH)}\right).$$

This leads us to the following mechanism. Pick p at random from $\{H, H/2, H/4, \ldots, 1/2n\}$ and set the price of each item to $p/2$. Let S_i be the set that buyer i purchases, and let $W_i = L_{i,p} \setminus (\cup_{j=1}^{i-1} S_j)$. Since $L_{i,p}$ is supported at price p, we have

$$\begin{aligned} p|W_i| &\leq v_i(W_i) \\ \frac{p}{2}|W_i| &\leq v_i(W_i) - \frac{p}{2}|W_i| \\ \frac{p}{2}|W_i| &\leq v_i(S_i) - \frac{p}{2}|S_i| \\ v_i(S_i) &\geq \frac{p}{2}|W_i| + \frac{p}{2}|S_i| \end{aligned}$$

Summing over all buyer, we get

$$\sum_i v_i(S_i) \geq \frac{p}{2}\sum_i (|S_i| + |W_i|)$$
$$\geq \frac{p}{2}\sum_i |L_{i,p}|.$$

The last inequality follows from the fact that every item in $L_{i,p}$ is either still there when buyer i comes in, in which case it is counted in $|W_i|$, or is bought by some buyer $j < i$, in which case it is counted in $|S_j|$. Finally, we recall from last lecture that

$$E\left[\frac{p}{2}\sum_i |L_{i,p}|\right] \geq \frac{\text{max social welfare}}{2\log(nH)}.$$

So, we have a simple pricing mechanism that gets within a logarithmic factor of the maximum social welfare.

4 Walrasian Equilibrium (Market Equilibrium)

We now allow different items to have different prices.

Definition 3 *Consider some pricing $p_1, \ldots, p_n$ on items. We define the demand set D_i to be* $\arg\max_S v_i(S) - p(S)$ *where $p(S) = \sum_{i \in S} p_i$.*

Definition 4 *A Walrasian equilibrium is a set of prices $p_1, \ldots, p_n$ and allocations $S_1, \ldots, S_m$ such that S_i is a demand set for buyer i. Furthermore, any unallocated item has price zero.*

Note that if there is no ties in $\max_S v_i(S) - p(S)$, all the buyers can come in at once and buy what they want, and there will be no contention (even if there are ties we can still assign sets to buyers so that they are as happy as they can be). However, a Walrasian equilibrium does not always exist. For example, suppose there are two buyers and one thousand items. The first buyer is single minded and wants everything and values that set at 1000. The second buyer has unit demand and has value 2 on any one item. Suppose the total price of the items is less than 1000, then there will be contention (we cannot give each buyer their demand set), as the first buyer will want everything and the second buyer will want at least one item. If the total price is greater than 1000, then the first player does not want anything, and the second player wants at most one thing costing at most 2. Thus there are unallocated item with nonzero prices.

Theorem 5 *If a Walrasian Equilibrium exists, then the allocation maximizes social welfare.*

Proof: Here is a LP-relaxation of the social welfare maximization problem. Let x_{iS} indicate whether we allocate set S to buyer i.

$$\begin{aligned}
\max \sum_i \sum_S & x_{iS} v_i(S), \\
\text{s.t.} \sum_{S \ni j} \sum_i & x_{iS} \leq 1 \qquad \forall \text{ item } j \\
\sum_S & x_{iS} = 1 \qquad \forall \text{ player } i \\
& x_{iS} \geq 0 \qquad \forall i, S
\end{aligned}$$

Let $S_1^*, \ldots, S_n^*$ be the allocation at Walrasian equilibrium and let $\{x_{iS}^*\}$ be the optimal LP solution. Then for all set S and buyer i

$$\begin{aligned}
v_i(S_i^*) - p(S_i^*) &\geq v_i(S) - p(S) \\
v_i(S_i^*) - p(S_i^*) &\geq \sum_S x_{iS}^* \left(v_i(S) - p(S) \right).
\end{aligned}$$

The second inequality follows from the fact that $\sum_S x_{iS}^* = 1$. Summing over all buyers,

$$\sum_i \left(v_i(S_i^*) - p(S_i^*) \right) \geq \sum_i \sum_S x_{iS}^* \left(v_i(S) - p(S) \right)).$$

$$(\text{social welfare at equilibrium}) - \sum_j p_j \geq (\text{optimal social welfare}) - \sum_j p_j.$$

where for the LHS above we use the fact that all non-allocated items have price zero in a Walrasian equilibrium (so $\sum_i p(S_i^*) = \sum_j p_j$), and for the RHS we use the "$\forall$ item j" constraints in the LP. ■