Proofs for Lecture 13

Ariel Procaccia

Theorem 1. Let there be a CSP with $|D| = d$ and arity $r$ (each constraint having at most $r$ variables). If it is strong $(d(r - 1) + 1)$-consistent, then it is globally consistent.

Proof. For simplicity we provide the proof for the special case of $r = 2$.

We will prove the theorem by showing that strong $(d+1)$-consistent binary CSPs are $(d+i+1)$-consistent for any $i \geq 1$.

According to the definitions, we need to show that if $\bar{x} = (x_1, \ldots, x_{d+i})$ is any locally consistent subtuple of the subset of variables $\{X_1, \ldots, X_{d+i}\}$, and if $X_{d+i+1}$ is any additional variable, then there is an assignment $x_{d+i+1}$ to $X_{d+i+1}$ that is consistent with $\bar{x}$.

We call an assignment to a single variable a unary assignment and we view $\bar{x}$ as a set of such unary assignments. With each value $j \in D$ we associate a subset $A_j$ that contains all unary assignments in $\bar{x}$ that are consistent with the assignment $X_{d+i+1} = j$. Since variable $X_{d+i+1}$ may take on $d$ possible values $1, 2, \ldots, d$ this results in $d$ such subsets, $A_1, \ldots, A_d$.

We claim that there must be at least one set, say $A_1$, that contains the set $\bar{x}$. If this were not the case, each subset $A_j$ would be missing some member, say $x'_j$, which means that the tuple generated by taking a missing unary assignment from each of the $A_j$'s, i.e. $\bar{x}' = (x'_1, x'_2, \ldots, x'_d)$ whose length is $d$ or less (there might be repetitions), could not possibly be consistent with any of $X_{d+i+1}$’s values.

This leads to a contradiction because as a subset of $\bar{x}$, $\bar{x}'$ is locally consistent, and from the assumption of strong $(d+1)$-consistency, this tuple should be extensible by any additional variable including $X_{d+i+1}$.

Note that we need not assume that the $x'_i$’s are distinct unary assignments because strong $(d+1)$-consistency renders the argument applicable to subtuples $\bar{x}'$ of length less than $d$.

We found a subset, without loss of generality $A_1$, that contains the set $\bar{x}$. From the definition of $A_1$, it is consistent with $X_{d+i+1} = 1$. Hence, we found a value consistent with $\bar{x}$.

Theorem 2. Let there be a CSP with arity $r$. Let $t$ be an upper bound on the number of constraints each variable appears in. Let $q$ be a lower bound on the probability of choosing a satisfying assignment for a constraint. If $q \geq 1 - \frac{1}{e(l(r-1)+1)}$ then there is a solution to the CSP.

Lemma 3 (Lovász Local Lemma). We denote by $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ the series of events such that each event occurs with probability at most $p$ and such that each event is independent of all the other events except for at most $m$ of them. If $ep(m+1) \leq 1$ (where $e = 2.718\ldots$), then there is a nonzero probability that none of the events occur, $\Pr[\bigcap_{i=1}^n \bar{\mathcal{E}}_i] > 0$.

*Based on lecture notes by Zvi Vlodavsky and Bracha Hod.
Proof of Theorem 2. Let there be a random assignment of variables. \( \mathcal{E}_i \) is the event of \( C_i \) not being satisfied. Since a constraint has at least a \( q \) probability of being satisfied, \( \Pr[\mathcal{E}_i] \leq 1 - q \). Since a constraint has at most \( r \) variables, each appearing in at most \( (t - 1) \) other constraints, \( \mathcal{E}_i \) is independent of all other events except for at most \( r(t - 1) \) events.

According to The Lovász Local Lemma, by assigning \( p = 1 - q \), \( m = r(t - 1) \), if \( e(1-q)(r(t-1)+1) \leq 1 \) then \( \Pr[\bigcap_{i=1}^{n} \mathcal{E}_i] > 0 \). Hence, there is a solution to the CSP. \( \square \)