Graduate AI
Lecture 13:
Constraint Satisfaction 1

Teachers:
Martial Hebert
Ariel Procaccia (this time)
What are CSPs?

• A constraint satisfaction problem (CSP) consists of:
  o Variables \( \{X_1, \ldots, X_n\} \)
  o Domains \( \{D_1, \ldots, D_n\} \)
  o A set of constraints: defined on subsets of variables, give allowable tuples of values

• Consider (possibly partial) assignments of values to variables

• Solution = complete + consistent assignment
Example: Map coloring

Variables = \{WA, NT, SA, Q, NSW, V, T\}

\[ D_i = \{\text{red, yellow, blue}\} \]

Constraints: adjacent regions have different colors

Constraint graph

Carnegie Mellon University
Example: Cryptarithmetic

\[
\begin{array}{c}
\text{TWO} \\
+ \text{TWO} \\
\hline
\text{FOUR}
\end{array}
\]
Other examples

• Sudoku
• Assignment problems, e.g., who teaches which class
• Scheduling problems, e.g., meetings, transportation, manufacturing
• SAT (on Monday)
Binary CSPs

• Binary CSP = constraints involve exactly two variables
• Map coloring is binary, cryptarithmetic is not
• Any CSP can be transformed into a binary CSP; how?
• Hint: add new variable for each non-unary constraint
Complexity of CSPs

- CSPs are NP-complete
- Clearly in NP: given an assignment, check that it is legal
- Graph coloring is a special case
CSPs vs. Search

• Informed search:
  o State is a black box
  o Heuristics are problem-specific

• In contrast, CSPs:
  o States are assignments; have structure
  o General-purpose algorithms that do not require domain-specific knowledge
Search formulation

- Initial state: empty assignment
- Successor: consistent assignment to unassigned variable
- Goal test: assignment is complete
- Variable assignments are commutative
  \[ \Rightarrow \text{At each node, only consider legal assignments to one of the variables} \]
Example: backtracking
Improving efficiency

• Which variable to assign next?
• Most constrained variable: least # legal values
• Which value to assign next?
• Least constraining: largest # legal values for remaining variables
• Weird?
**K-CONSISTENCY**

- A CSP is **k-consistent** if for every $Y_1, \ldots, Y_k$, any legal assignment for $Y_1, \ldots, Y_{k-1}$ can be extended to a legal assignment for $Y_1, \ldots, Y_k$.
- Strong k-consistency $= k'$ consistency for every $k' \leq k$.
- Global consistency $= \text{strong n-consistency}$.
- Global consistency $\Rightarrow$ backtrack-free search.
- Practitioners usually enforce 2-consistency.
Solvability via global properties

- We prove the following statements on the board / in the addendum
- Assume $D_i=D$ for all $i$, denote $|D|=d$, and assume the CSP has arity $r$ (each constraint has at most $r$ variables). Then strong $(d(r-1)+1)$-consistency $\Rightarrow$ global consistency
- Let there be a CSP with arity $r$. Let $t$ be an upper bound on the number of constraints each variable appears in. Let $q$ be a lower bound on the probability of choosing a satisfying assignment for a constraint. If $q \geq 1 - 1/e(r(t-1)+1)$ then there is a satisfying assignment