CMU 15-251
Probability I

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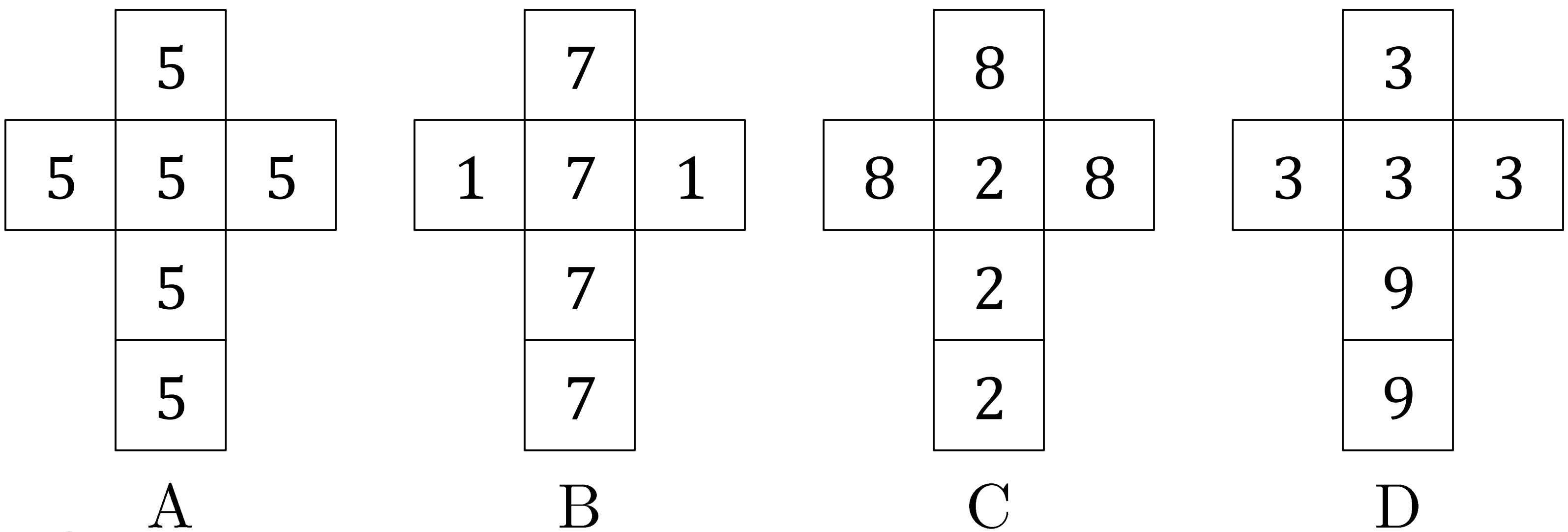
Course overview

	Old	New
Not CS	Today	Number theory
CS	Running time	Most of the course

Gambling 101

- You choose a die first, I choose second
- We both throw; higher number wins
- Which die would you choose?



Gambling 101

- Antoine Gombaud (1607-1684) made history for being a loser

> I will roll a die four times; I win if I get a 1

- After a while no one would take the bet

- $1 - \left(\frac{5}{6}\right)^4 = 0.518$

Gambling 101

- Gombaud invented a new scam:

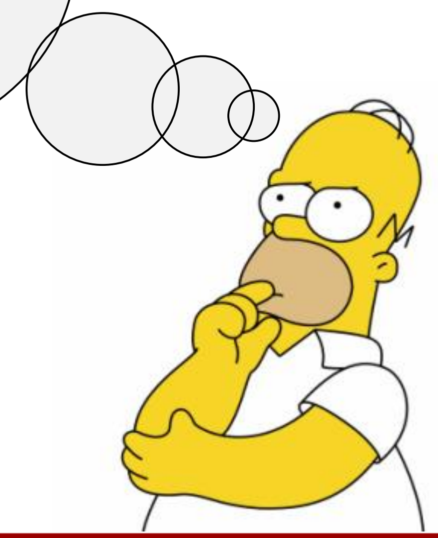
> I will roll two dice 24 times;
> I win if I get a double 1

- Why was he losing money?
- $1 - \left(\frac{35}{36}\right)^{24} = 0.491$
- Gombaud wrote to Pascal and Fermat, who subsequently created probability theory

Gambling 101

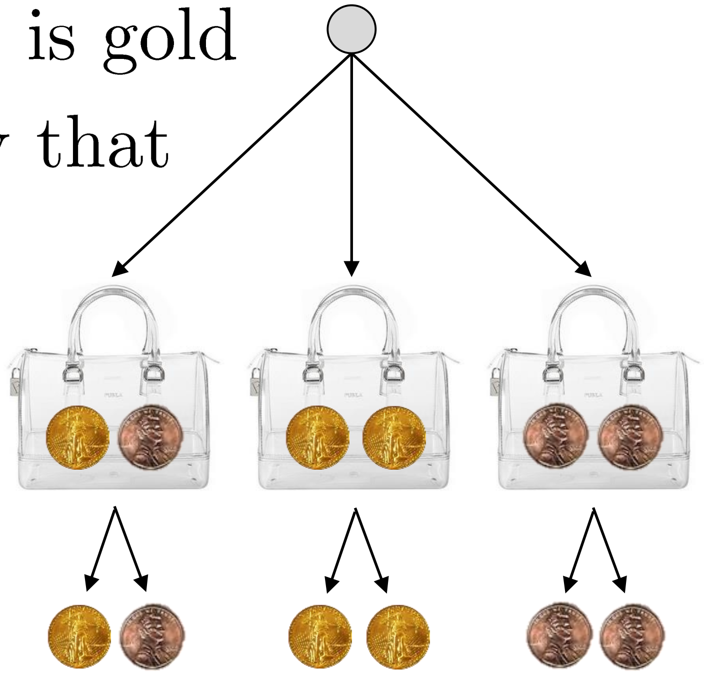
Morale of the story:

Analyzing gambling is not a side-benefit of probability theory; probability theory was created to analyze gambling!



Pennies and gold

- Three bags contain two gold coins, two pennies, and one of each
- Bag is chosen at random, and one coin from it is selected at random; the coin is gold
- Poll 1: What is the probability that the other coin is gold?
  1. 1/6
  2. 1/3
  3. 2/3
  4. 1



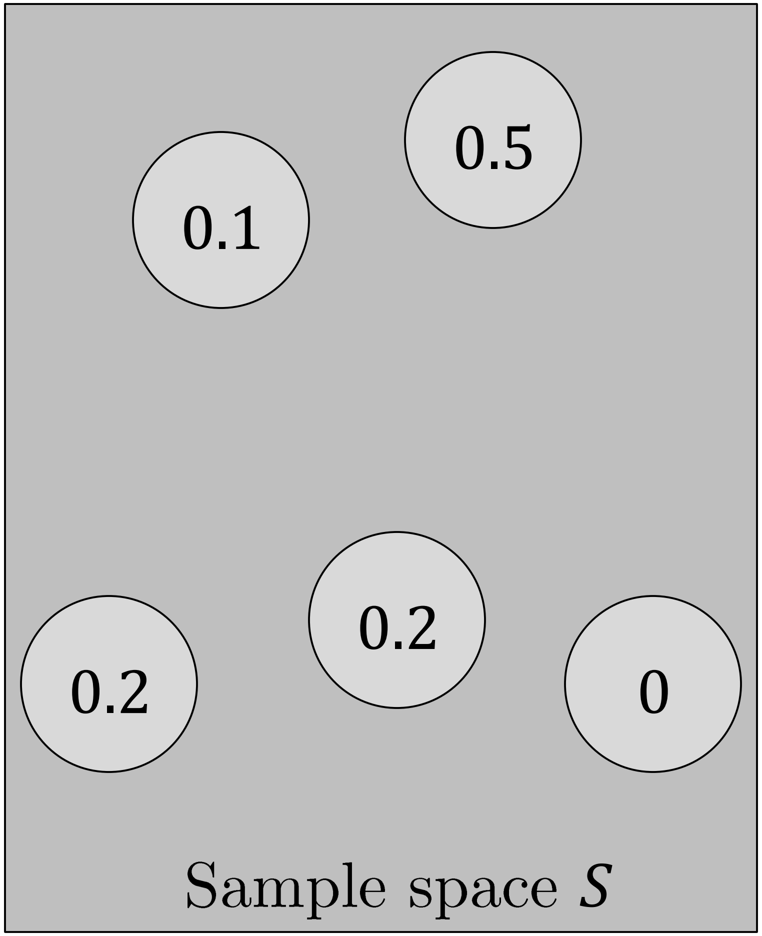
Language of probability

Probability can be counterintuitive; we need a formal language!

Language of probability

- The sample space is a finite set of elements S
- A probability distribution p assigns a non-negative real probability to each element, such that

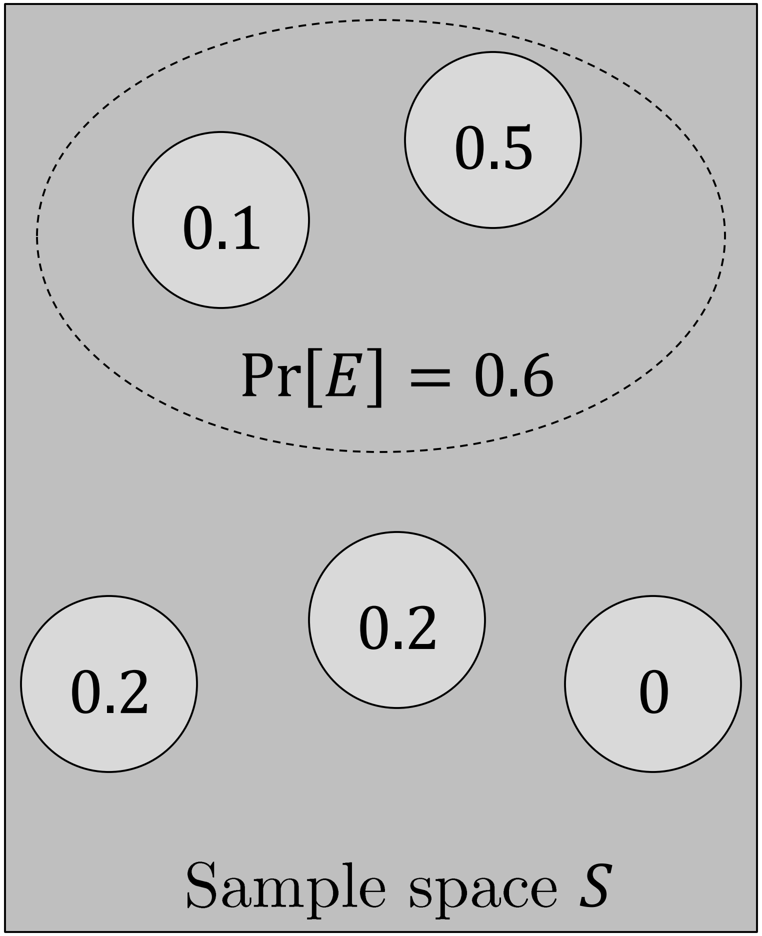
$$\sum_{x \in S} p(x) = 1$$



Language of probability

- An event is a subset $E \subseteq S$
- $\Pr[E] = \sum_{x \in E} p(x)$
- If each element $x \in S$ has equal probability, the distribution is uniform:

$$\Pr[E] = \sum_{x \in E} p(x) = \frac{|E|}{|S|}$$

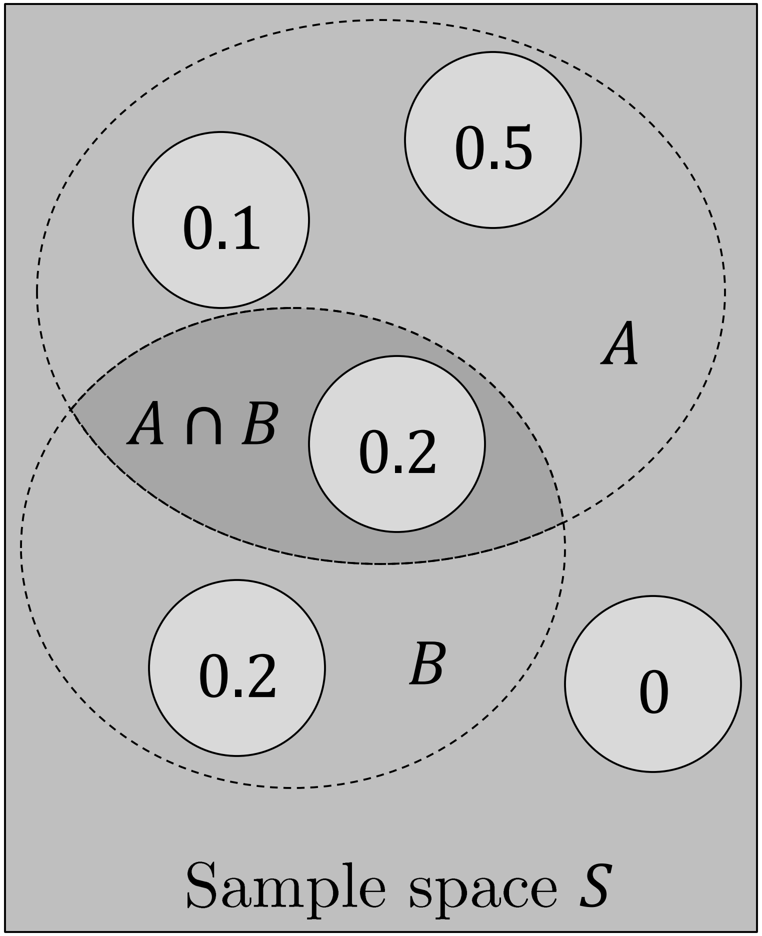


Language of probability

- We roll a white die and black die
- $S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$
  $(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
  $(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
  $(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
  $(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$
  $(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$
- Poll 2: Probability that the sum is 7 or 11?
  1. 1/9
  2. 2/9
  3. 3/9
  4. 4/9

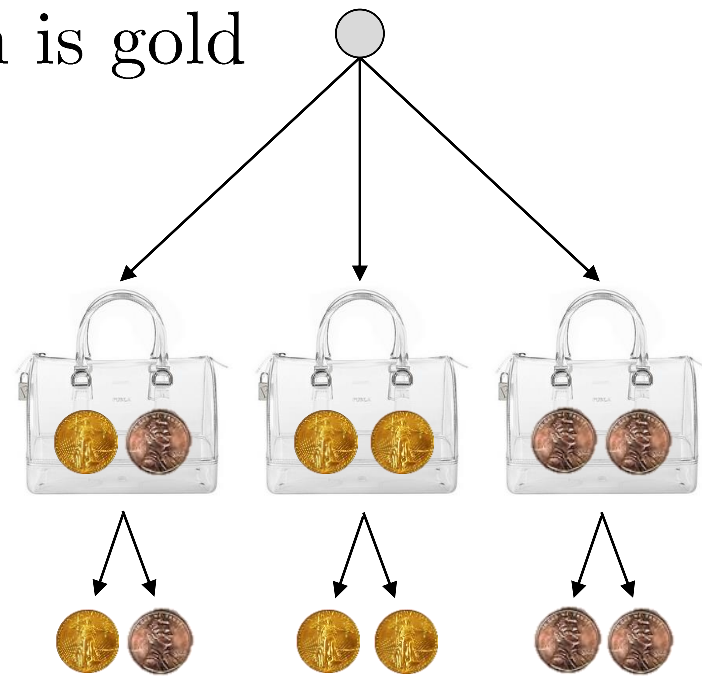
Conditional probability

- The probability of event A given event B is defined as
  $$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$
- Think of it as the proportion of $A \cap B$ to B



Pennies and gold, revisited

- Three bags contain two gold coins, two pennies, and one of each
- Bag is chosen at random, and one coin from it is selected at random; the coin is gold
- G_i: coin $i \in \{1,2\}$ is gold
- $\Pr[G_1] = \frac{1}{2}, \Pr[G_1 \cap G_2] = \frac{1}{3}$
- $\Pr[G_2|G_1] = \frac{1/3}{1/2} = \frac{2}{3}$



Conditional probability

- $\Pr[A \cap B] = \Pr[B] \times \Pr[A|B]$
- Interpretation: For A and B to occur, B must occur, and A must occur given that B occurred
- Applying iteratively:
  $\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \times \Pr[A_2|A_1] \times \cdots \Pr[A_n|A_1, \cdots, A_{n-1}]$

This is called the
chain rule

BAYES' RULE

- $\Pr[\mathrm{B}] \times \Pr[\mathrm{A}|\mathrm{B}] = \Pr[A \cap B] = \Pr[A] \times \Pr[B|A]$

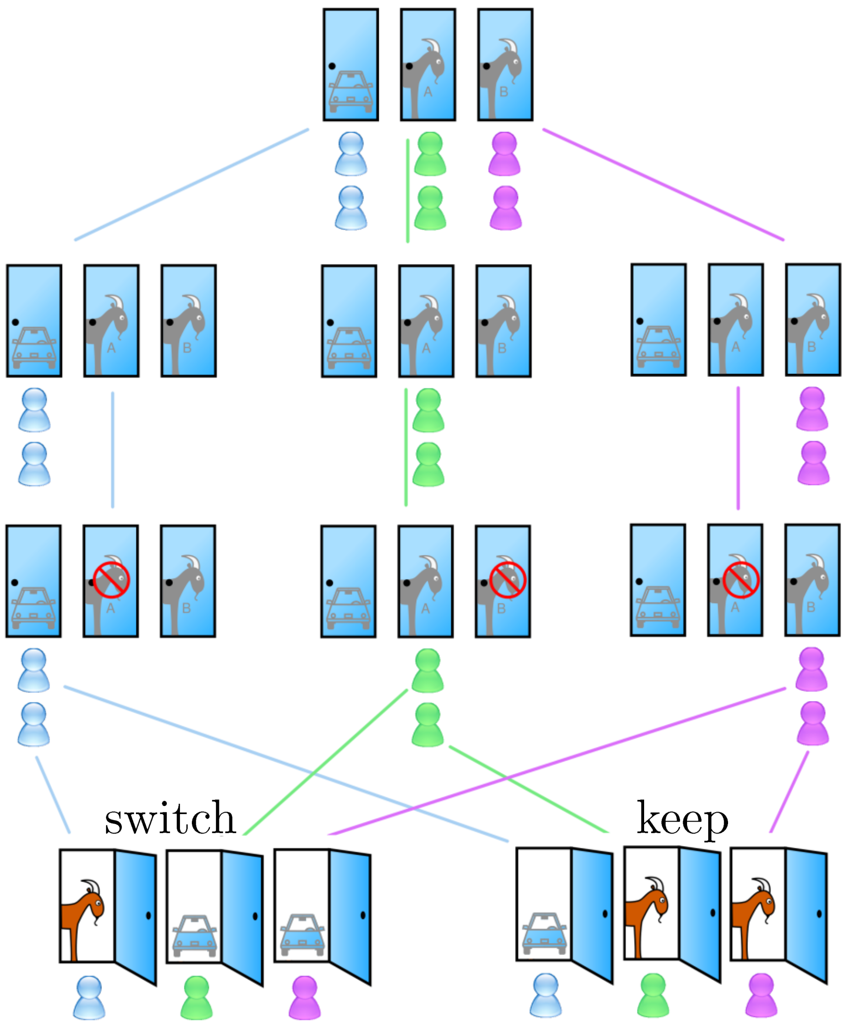
Bayes' rule:

$$\Pr[A|B] = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$$

Monty Hall problem

- Announcer hides prize behind one of three doors at random
- You choose a door
- Announcer opens a door with no prize
- Should you stay with your choice or switch?

Monty Hall problem



Monty Hall problem

- Choose door 1, door 2 opens
- $\Pr[P_3|O_2] = \frac{\Pr[P_3]\Pr[O_2|P_3]}{\Pr[O_2]}$
- $\Pr[P_3] = \frac{1}{3}, \Pr[O_2|P_3] = 1,$ $\Pr[O_2] = 1/2$
- Therefore, $\Pr[P_3|O_2] = 2/3$
- Poll 3: Assuming there are five doors, what is the probability of winning when switching?
  1. $3/15$
  2. $4/15$
  3. $5/15$
  4. $6/15$

$$\Pr[A|B] = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$$

Independence

- Events A and B are independent if and only if $\Pr[A|B] = \Pr[A]$
- Poll 4: Which of the following events are independent when rolling black die and white die?
  1. Black die is 1, white die is 1
  2. Sum is 2, sum is 3
  3. Black die is 1, product is 2
  4. Black die is 1, sum is 2

The birthday paradox

- m people in a room; suppose all birthdays are equally likely (excluding Feb 29); what is the probability that two people have the same birthday?
- $S = \{1, \dots, 365\}^m$, sample $\vec{x} = (x_1, \dots x_m)$
- $E = \{\vec{x} \in S \mid \exists i, j, \text{s.t. } x_i = x_j\}$

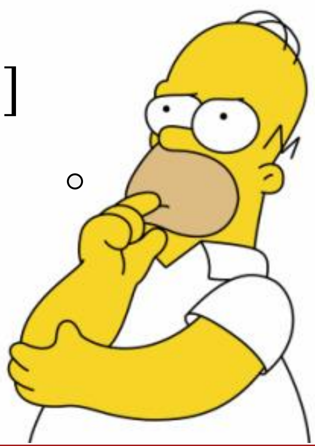
Apply the chain rule!

The birthday paradox

- E is the event that two people share a birthday
- We will compute $\bar{E}$
- Let A_i be the event that person i's birthday differs from the birthdays of $1, \dots, i-1$
- $\bar{E} = A_1 \cap \cdots \cap A_n$
- Using the chain rule:

$$\Pr[\bar{E}] = \Pr[A_1] \times \Pr[A_2 | A_1] \times \cdots \Pr[A_n | A_1, \cdots, A_{n-1}]$$

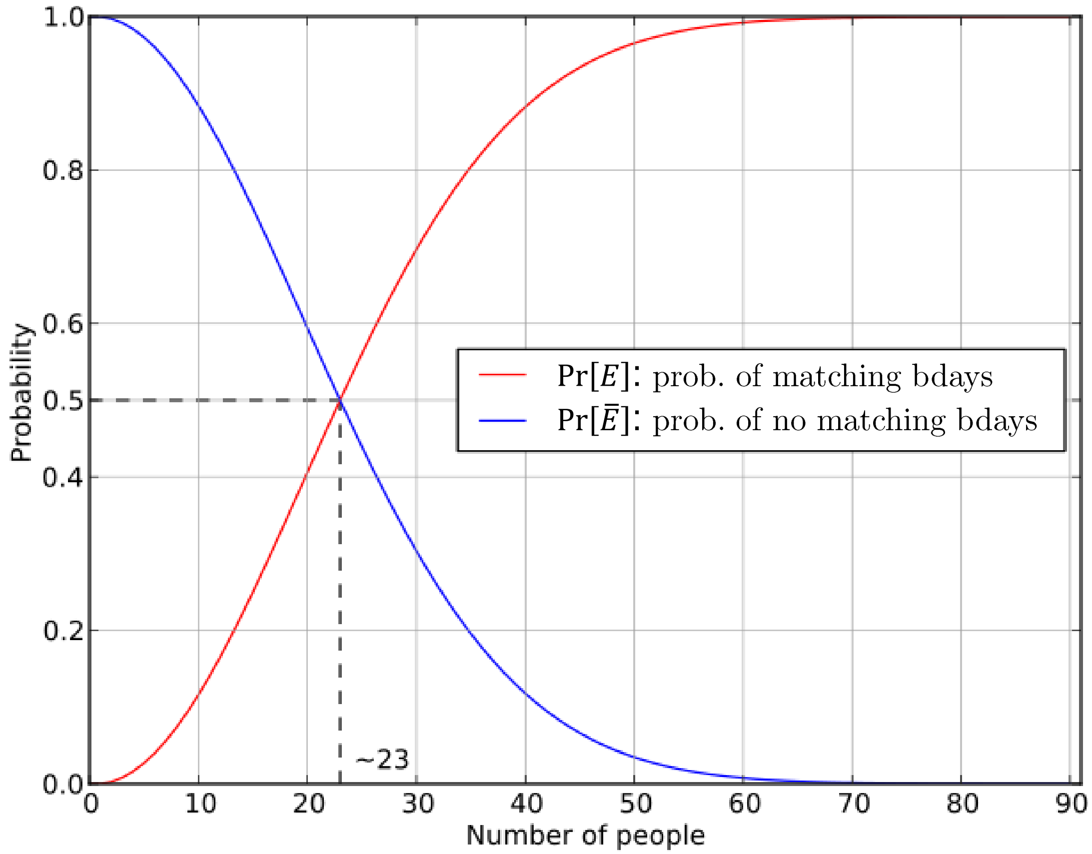
So what is $\Pr[A_i | A_1, \dots, A_{i-1}]$?



The birthday paradox

- $A_1 \cap \cdots \cap A_{i-1}$ means first $i-1$ students had different birthday
- $i-1$ out of 365 occupied when ith birthday is chosen
- $\Pr[A_i | A_1, \ldots, A_{i-1}] = \frac{365-(i-1)}{365} = 1 - \frac{i-1}{365}$
- $\Pr[\bar{E}] = 1 \times \left(1 - \frac{1}{365}\right) \times \cdots \times \left(1 - \frac{m-1}{365}\right)$
- $\Pr[E] = 1 - \Pr[\bar{E}]$

The birthday paradox



The birthday paradox

- Poll 5: What is the probability that two people have the same birthday if there are 730 people?

1. 1/2
2. 0.75
3. 0.9999999999997
4. 1

Birthday attack *

- A cryptographic hash function 'scrambles' a string S into a k-bit hash $f(S)$
- It should be hard to find a collision: two strings S_1, S_2 such that $f(S_1) = f(S_2)$
- Application: digital signatures
  - Alice wants Bob to sign a message m
  - They compute $f(m)$ and it is signed using Bob's secret key
  - Bad collision: Alice can find a fair contract m and a fraudulent contract m' such that $f(m) = f(m')$

* Just for fun

Birthday attack *

- The SHA-1 cryptographic hash function uses 160 bits
- To find a collision for SHA-1, take a huge number of strings, hash them all, hope that two hash to the same one
- If SHA-1 is really safe, each hash f(S) should be uniform in $\{1, \ldots, 2^{160}\}$
- This is like the Birthday Problem with 2^{160} days of the year!
- To find a collision you would need roughly $\sqrt{2^{160}} = 2^{80}$ strings

* Just for fun

Birthday attack *

- A crypto hash function is considered broken if you can beat the birthday attack
- SHA-1 collisions can be found using "only" 2^{63} strings
- Still used in SSL, PGP,...

* Just for fun

Summary

- Definitions / facts
  - Language of probability
  - Conditional probability
  - Independence
- Principles / problem solving
  - Chain rule
  - Bayes' rule