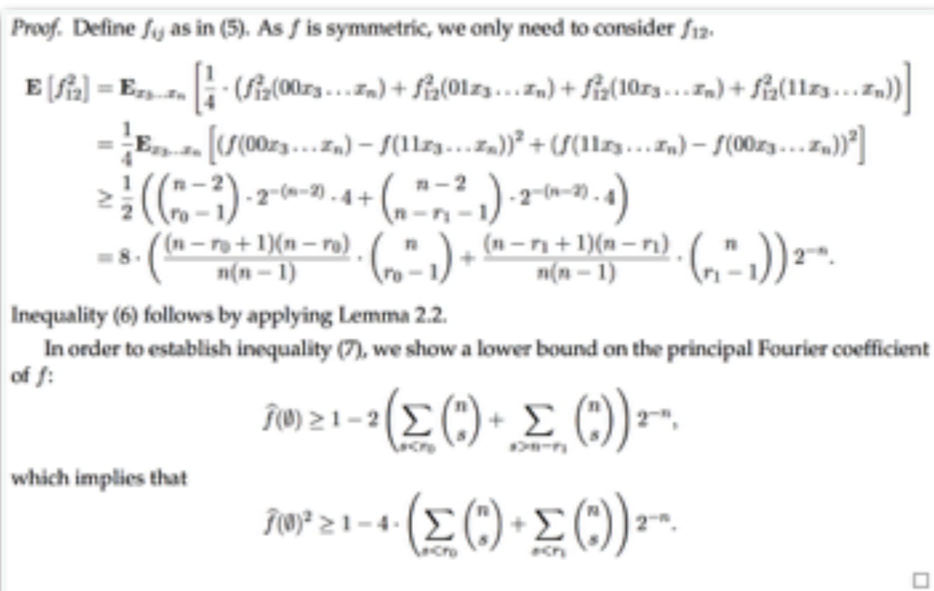
15-251
Great Theoretical Ideas in Computer Science

Lecture 2:
On Proofs and Pancakes



Proof. Define f_{ij} as in (5). As f is symmetric, we only need to consider f_{12}.

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\mathbf{E}\left[f_{12}^2\right] &= \mathbf{E}_{x_3\ldots x_n}\left[\frac{1}{4}\cdot\left(f_{12}^2(00x_3\ldots x_n)+f_{12}^2(01x_3\ldots x_n)+f_{12}^2(10x_3\ldots x_n)+f_{12}^2(11x_3\ldots x_n)\right)\right]\\
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\end{aligned}$$

Inequality (6) follows by applying Lemma 2.2.

In order to establish inequality (7), we show a lower bound on the principal Fourier coefficient of f:

$$\hat{f}(\emptyset) \ge 1-2\left(\sum_{s<r_0}\binom{n}{s}+\sum_{s>n-r_1}\binom{n}{s}\right)2^{-n},$$

which implies that

$$\hat{f}(\emptyset)^2 \ge 1-4\cdot\left(\sum_{s<r_0}\binom{n}{s}+\sum_{s<r_1}\binom{n}{s}\right)2^{-n}.$$

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September 3rd, 2015

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1. What is a proof ?
2. How do you find a proof ?
3. How do you write a proof ?

Is this a legit proof?

Proposition:

Start with any number.
If the number is even, divide it by 2.
If it is odd, multiply it by 3 and add 1.
If you repeat this process, it will lead you to 4, 2, 1.

Proof:

Many people have tried this, and no one came up with a counter-example.

□

Is this a legit proof?

Proposition:

$313(x^3 + y^3) = z^3$ has no solution for $x, y, z \in \mathbb{Z}^+$.

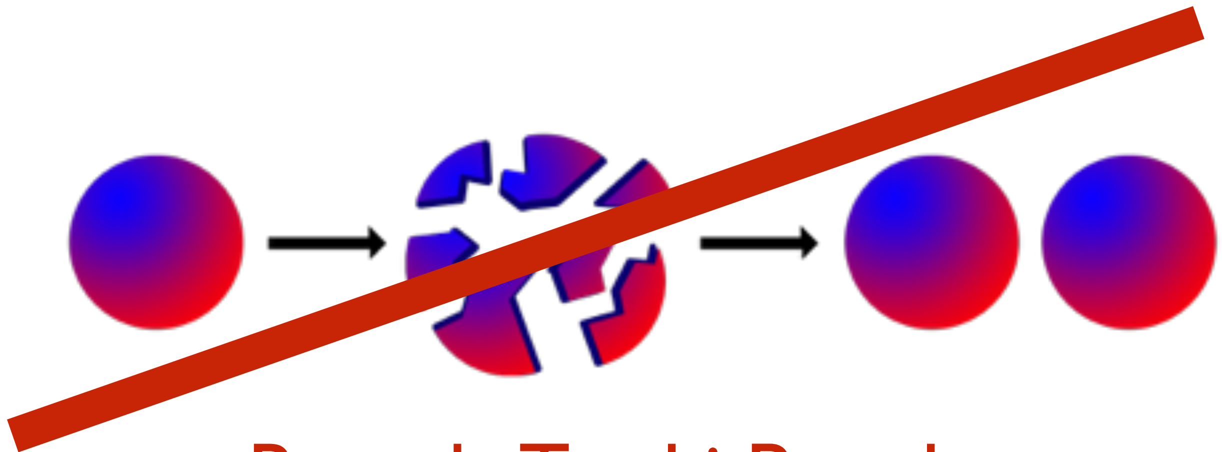
Proof:

Using a computer, we were able to verify that there is no solution for numbers with < 500 digits. $\square$

Is this a legit proof?

Proposition:

Given a solid ball in 3 dimensional space, there is no way to decompose it into a finite number of disjoint subsets, which can be put together to form two identical copies of the original ball.



Banach-Tarski Paradox

Proof:

Obvious.

□

Is this a legit proof?

Proposition:

$1 + 1 = 2$

Proof:

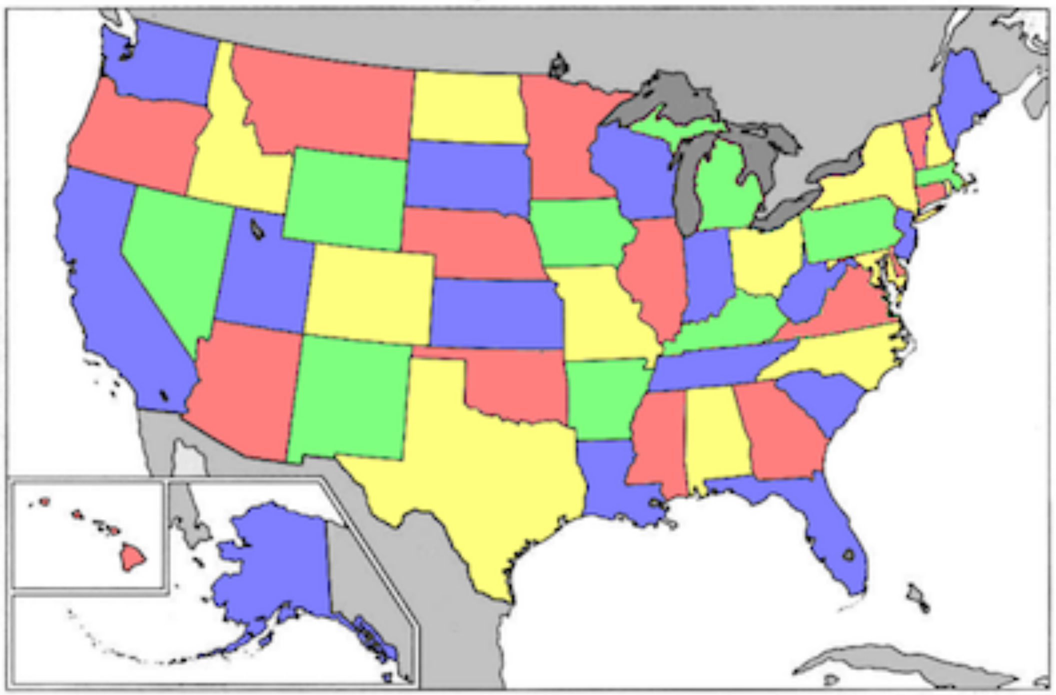
Obvious.

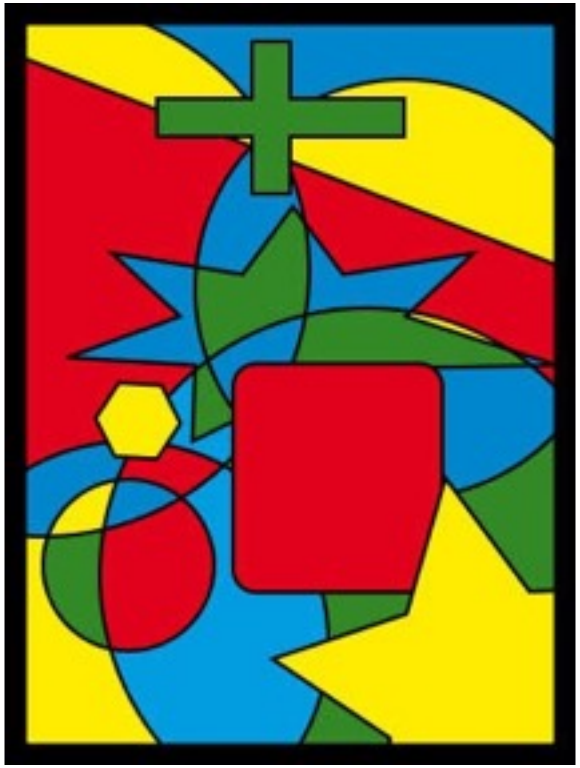
$\square$

The story of 4 color theorem

1852 Conjecture:

Any 2-d map of regions can be colored with 4 colors so that no adjacent regions get the same color.





The story of 4 color theorem

1879: Proved by Kempe in *American Journal of Mathematics* (was widely acclaimed)

1880: Alternate proof by Tait in *Trans. Roy. Soc. Edinburgh*

1890: Heawood finds a bug in Kempe's proof

1891: Petersen finds a bug in Tait's proof

1969: Heesch showed the theorem could in principle be reduced to checking a large number of cases.

1976: Appel and Haken wrote a massive amount of code to compute and then check 1936 cases.
(1200 hours of computer time)

The story of 4 color theorem

Much controversy at the time. Is this a proof?

What do you think?

Arguments against:

- no human could ever hand-check the cases
- maybe there is a bug in the code
- maybe there is a bug in the compiler
- maybe there is a bug in the hardware
- no "insight" is derived

1997: Simpler computer proof by
Robertson, Sanders, Seymour, Thomas

What is a mathematical proof?

inference rules like $$\frac{P, \quad P \implies Q}{Q}$$

A mathematical proof of a proposition is
a chain of logical deductions starting from a set of axioms
and leading to the proposition.

propositions accepted to be true

a statement that is true or false

Euclidian geometry

5 AXIOMS

1. Any two points can be joined by exactly one line segment.

2. Any line segment can be extended into one line.

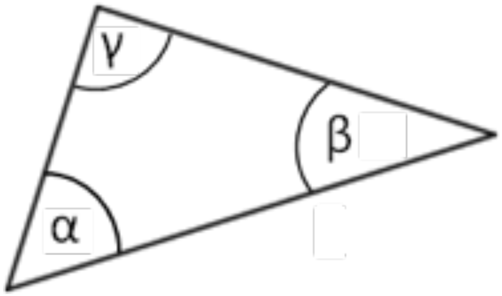
3. Given any point P and length r, there is a circle of radius r and center P.

4. Any two right angles are congruent.

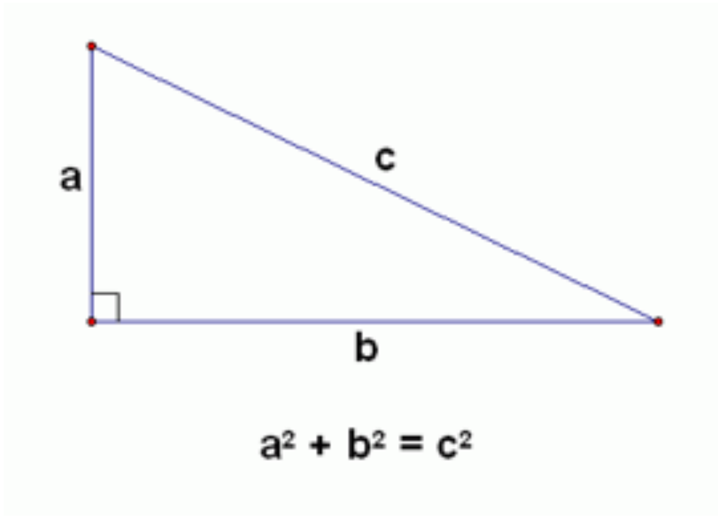
5. If a line L intersects two lines M and N, and if the interior angles on one side of L add up to less than two right angles, then M and N intersect on that side of L.

Euclidian geometry

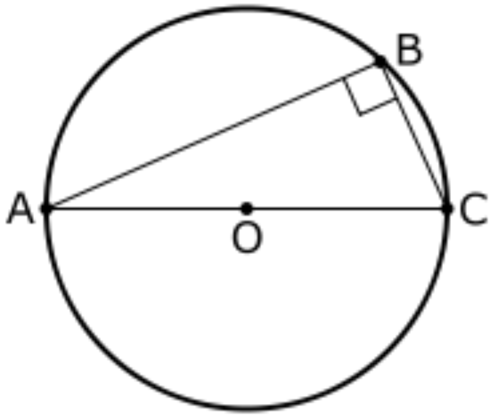
Triangle Angle Sum Theorem



Pythagorean Theorem

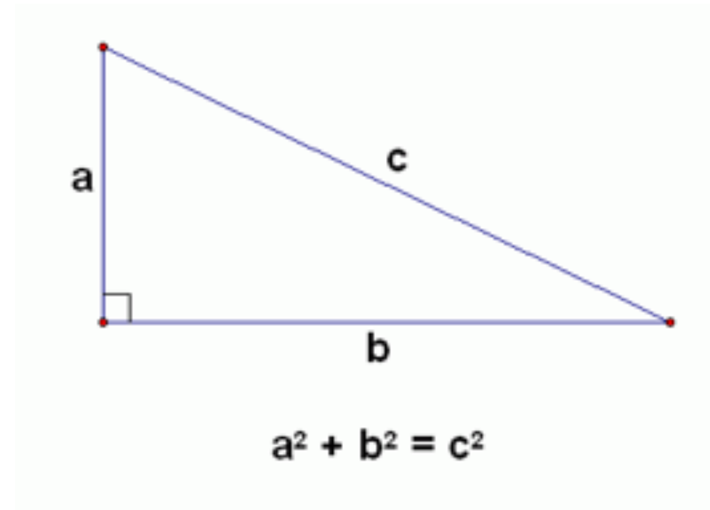


Thales' Theorem

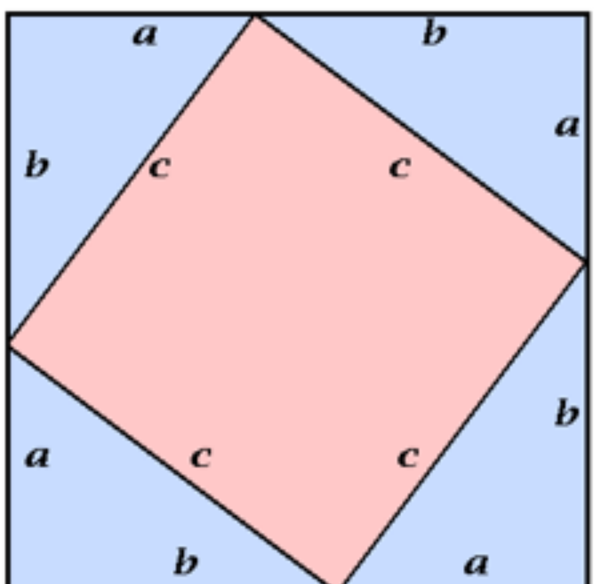


Euclidian geometry

Pythagorean Theorem



Proof:



$$
\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
&= 2ab + c^2
\end{aligned}
$$

Proof that square-root(2) is irrational

1. Suppose $\sqrt{2}$ is rational.
   Then we can find $a, b \in \mathbb{N}$ such that $\sqrt{2} = a/b$.
2. If $\sqrt{2} = a/b$ then $\sqrt{2} = r/s$,
   where r and s are not both even.
3. If $\sqrt{2} = r/s$ then $2 = r^2/s^2$.
4. If $2 = r^2/s^2$ then $2s^2 = r^2$.
5. If $2s^2 = r^2$ then r^2 is even, which means r is even.
6. If r is even, $r = 2t$ for some $t \in \mathbb{N}$.
7. If $2s^2 = r^2$ and $r = 2t$ then $2s^2 = 4t^2$ and so $s^2 = 2t^2$.
8. If $s^2 = 2t^2$ then s^2 is even, and so s is even.

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Proof that square-root(2) is irrational

5a. r^2 is even. Suppose r is odd.

5b. So there is a number t such that $r = 2t + 1$.

5c. So $r^2 = (2t+1)^2 = 4t^2 + 4t + 1$.

5d. $4t^2 + 4t + 1 = 2(2t^2 + 2t) + 1$, which is odd.

5e. So r^2 is odd.

5f. Contradiction is reached.

Odd number means not a multiple of 2.

Is every number a multiple of 2 or
one more than a multiple of 2?

Proof that square-root(2) is irrational

Odd number means not a multiple of 2.

Is every number a multiple of 2 or
one more than a multiple of 2?

5b1. Call a number r good if $r = 2t$ or $r = 2t + 1$ for some t.

If $r = 2t$, $r + 1 = 2t + 1$.

If $r = 2t + 1$, $r + 1 = 2t + 2 = 2(t + 1)$.

Either way, $r + 1$ is also good.

5b2. 1 is good since $1 = 0 + 1 = (0 \cdot 2) + 1$.

5b3. Applying 5b1 repeatedly, $2, 3, 4, \ldots$ are all good.

Proof that square-root(2) is irrational

Axiom of induction:

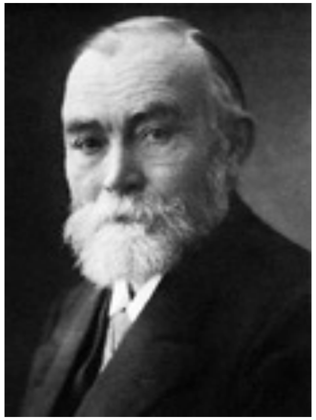
Suppose for every positive integer n, there is a statement $S(n)$.

If $S(1)$ is true, and $S(n) \implies S(n+1)$ for any n,

then $S(n)$ is true for every n.

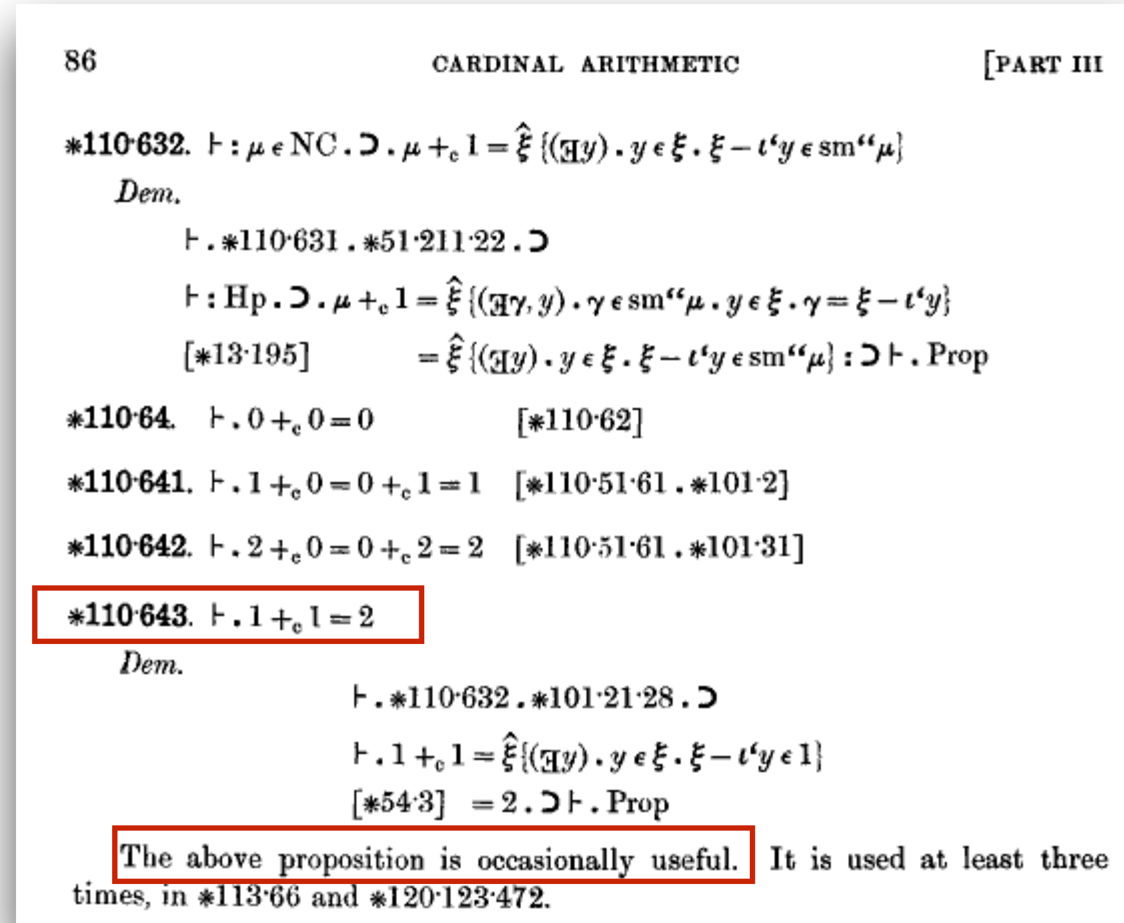
Can every mathematical theorem be derived from a set of agreed upon axioms?

Formalizing math proofs



Frege

Principia Mathematica
Volume 2



*110·632. $\vdash : \mu \in \mathrm{NC} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota{}^{\backprime}y \in \mathrm{sm}{}^{\backprime\backprime}\mu\}$

Dem.

$\vdash . *110{\cdot}631 . *51{\cdot}211{\cdot}22 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists \gamma, y) . \gamma \in \mathrm{sm}{}^{\backprime\backprime}\mu . y \in \xi . \gamma = \xi - \iota{}^{\backprime}y\}$

[*13·195] $= \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota{}^{\backprime}y \in \mathrm{sm}{}^{\backprime\backprime}\mu\} : \supset \vdash . \mathrm{Prop}$

*110·64. $\vdash . 0 +_c 0 = 0$ [*110·62]

*110·641. $\vdash . 1 +_c 0 = 0 +_c 1 = 1$ [*110·51·61 . *101·2]

*110·642. $\vdash . 2 +_c 0 = 0 +_c 2 = 2$ [*110·51·61 . *101·31]

*110·643. $\vdash . 1 +_c 1 = 2$

Dem.

$\vdash . *110{\cdot}632 . *101{\cdot}21{\cdot}28 . \supset$

$\vdash . 1 +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota{}^{\backprime}y \in 1\}$

[*54·3] $= 2 . \supset \vdash . \mathrm{Prop}$

The above proposition is occasionally useful. It is used at least three times, in *113·66 and *120·123·472.

Russell

Whitehead

Writing a proof like this
is like writing a computer program in machine language.

Interesting consequences:

Proofs can be found mechanically.

And can be verified mechanically.

What does this all mean for 15-251?

A proof is an argument that can withstand all criticisms from a highly caffeinated adversary (your TA).

Lord Wacker von Wackenfels

(1550 - 1619)

Kepler Conjecture

1611: Kepler as a New Year's present (!) for his patron, Lord Wacker von Wackenfels, wrote a paper with the following conjecture.

The densest way to pack oranges is like this:

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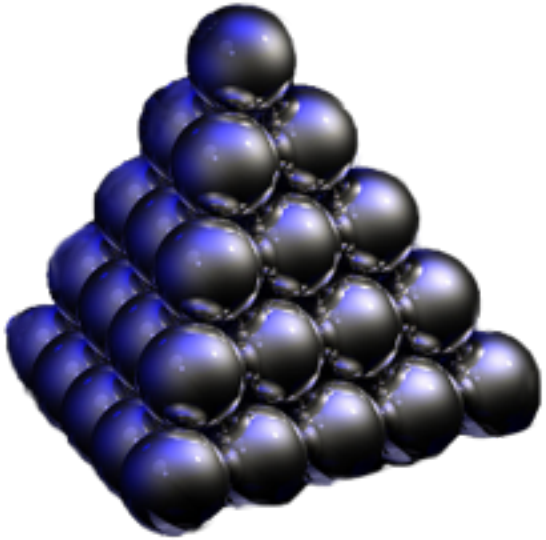

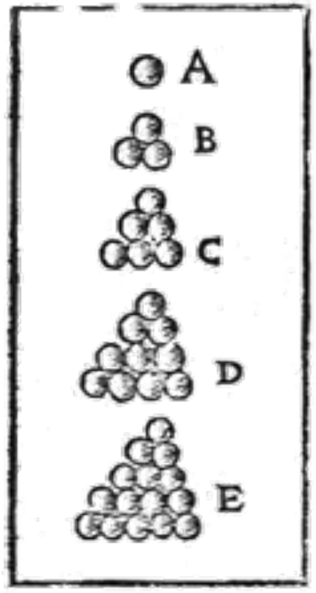

Kepler Conjecture

2005: Pittsburgher Tom Hales submits a 120 page proof in *Annals of Mathematics*.

Plus code to solve 100,000 distinct optimization problems, taking 2000 hours computer time.

Annals recruited a team of 20 refs.
They worked for 4 years.
Some quit. Some retired. One died.
In the end, they gave up.

They said they were "99% sure" it was a proof.

Kepler Conjecture

Hales: "I will code up a completely formal axiomatic deductive proof, checkable by a computer."

2004 - 2014: Open source "Project Flyspeck":

2015: Hales and 21 collaborators publish "A formal proof of the Kepler conjecture".

Formally proved theorems

Fundamental Theorem of Calculus *(Harrison)*

Fundamental Theorem of Algebra *(Milewski)*

Prime Number Theorem *(Avigad @ CMU, et al.)*

Gödel's Incompleteness Theorem *(Shankar)*

Jordan Curve Theorem *(Hales)*

Brouwer Fixed Point Theorem *(Harrison)*

Four Color Theorem *(Gonthier)*

Feit-Thompson Theorem *(Gonthier)*

Kepler Conjecture *(Hales++)*

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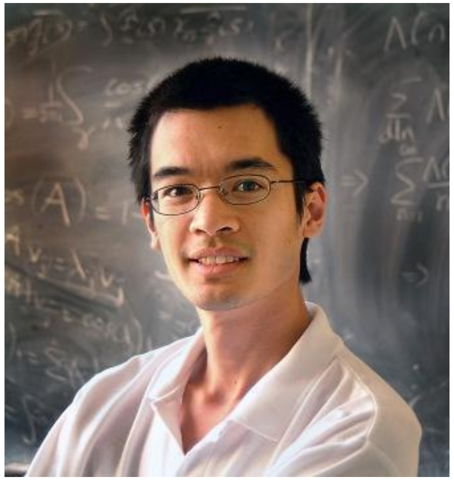
How do you find a proof?

No Eureka effect



Terence Tao

(Fields Medalist, "MacArthur Genius", ...)

I don't have any magical ability. ... When I was a kid, I had a romanticized notion of mathematics, that hard problems were solved in 'Eureka' moments of inspiration. [But] with me, it's always, 'Let's try this. That gets me part of the way, or that doesn't work. Now let's try this. Oh, there's a little shortcut here.' You work on it long enough and you happen to make progress towards a hard problem by a back door at some point. At the end, it's usually, 'Oh, I've solved the problem.'

How do you find a proof?

Some suggestions:

Make 1% progress for 100 days.
(Make 17% progress for 6 days.)

Give breaks, let the unconscious brain do some work.

Figure out some meaningful special cases (e.g. $n=1$, $n=2$).

Put yourself in the mind of the adversary.
(What are the worst-case examples/scenarios?)

Develop good notation.

Use paper, draw pictures.

Collaborate.

How do you find a proof?

Some suggestions:

Try different proof techniques.

- contrapositive $P \implies Q \iff \neg Q \implies \neg P$
- contradiction
- induction
- case analysis

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□

1. What is a proof ?
2. How do you find a proof ?
3. How do you write a proof ?

How do you write a proof?

- A proof is an essay, not a calculation!
- State your proof strategy.
- For long/complicated proofs, explain the proof idea first.
- Keep a linear flow.
- Introduce notation when useful. Draw diagrams/pictures.
- Structure long proofs.
- Be careful using the words "obviously" and "clearly". (obvious: a proof of it springs to mind immediately.)
- Be careful using the words "it", "that", "this", etc.
- Finish: tie everything together and explain why the result follows.

Question

If there are n pancakes in total (all in different size), what is the max number of flips that we would ever have to use to sort them?

$P_n =$ the number described above

What is P_n ?

Understanding the question

$$P_n = \max_S \min_A \# \text{ flips when sorting } S \text{ by } A$$

$\min_A$: over all strategies/algorithms for sorting

$\max_S$: over all pancake stacks of size n

Number of flips necessary to sort the worst stack.

Is it always possible to sort the pancakes?

Yes!

A sorting strategy (algorithm):

- Move the largest pancake to the bottom.
- Recurse on the other n-1 pancakes.

Playing around with an example

Introducing notation:

- represent a pancake with a number from 1 to n.
- represent a stack as a permutation of {1,2,…,n}
  e.g. (5 2 3 4 1)

  ↓ top ↓ bottom

Let X = number of flips needed for (5 2 3 4 1)

What is X ?

Playing around with (5 2 3 4 1)

$$? \leq X \leq 4$$

Need an argument for a lower bound.

A strategy/algorithm for sorting gives us an upper bound.

$$0 \leq X \leq 4$$

$$1 \leq X \leq 4$$

$$2 \leq X \leq 4$$

$$3 \leq X \leq 4$$

$$4 \leq X \leq 4$$

Playing around with (5 2 3 4 1)

Proposition: $X = 4$

Proof: We already showed $X \leq 4$.

We now show $X \geq 4$. The proof is by contradiction.

So suppose we can sort the pancakes using 3 or less flips.

Observation: Right before a pancake is placed at the bottom of the stack, it must be at the top.

Claim: The first flip must put 5 on the bottom of the stack.
Proof: If the first flip does not put 5 on the bottom of the stack, then it puts it somewhere in the middle of the stack.
After 3 flips, 5 must be placed at the bottom.
Using the observation above, 2nd flip must send 5 to the top.
Then after 2 flips, we end up with the original stack.
But there is no way to sort the original stack in 1 flip.
The claim follows. $\square$

Playing around with (5 2 3 4 1)

Proposition: $X = 4$

Proof continued:

So we know the first flip must be: $(5\ 2\ 3\ 4\ 1) \longrightarrow (1\ 4\ 3\ 2\ 5)$.

In the remaining 2 flips, we must put 4 next to 5.

Obviously 5 cannot be touched.

So we can ignore 5 and just consider the stack $(1\ 4\ 3\ 2)$.

We need to put 4 at the bottom of this stack in 2 flips.

Again, using the observation stated above,
the next two moves must be:

$$(1\ 4\ 3\ 2) \longrightarrow (4\ 1\ 3\ 2) \longrightarrow (2\ 3\ 1\ 4)$$

This does not lead to a sorted stack,
which is a contradiction since we assumed we could sort the stack in 3 flips. $\square$

Playing around with (5 2 3 4 1)

$$X = 4$$

What does this say about P_n ?

Pick one that you think is true:

- $P_n = 4$
- $P_n \leq 4$
- $P_n \geq 4$
- $P_5 = 4$
- $P_5 \leq 4$
- $P_5 \geq 4$
- None of the above.
- Beats me.

Playing around with (5 2 3 4 1)

$$X = 4$$

What does this say about P_n ?

$$P_5 = \max_S \min_A \text{ \# flips when sorting } S \text{ by } A$$

↳ all stacks of size 5

all stacks:	$(5\ 2\ 3\ 4\ 1)$	$(5\ 4\ 3\ 2\ 1)$	$(1\ 2\ 3\ 4\ 5)$	$(5\ 4\ 1\ 2\ 3)$	$\cdots$
min # flips:	4	1	0	2	

$P_5 =$ max among these numbers

$P_5 =$ min # flips to sort the "hardest" stack

So: $X = 4 \implies P_5 \geq 4$

Playing around with (5 2 3 4 1)

In fact: (will not prove)

$$5 \leq P_5 \leq 5$$

Find a specific "hard" stack. Show any method must use 5 flips.

Find a generic method that sorts any 5-stack with 5 flips.

Good progress so far:

- we understand the problem better
- we made some interesting observations

Ok what about P_n for general n ?

P_n for small n

$$P_0 = 0$$

$$P_1 = 0$$

$$P_2 = 1$$

$$P_3 = 3$$

lower bound:

(1 3 2) requires 3 flips.

upper bound:

- bring largest to the bottom in 2 flips
- sort the other 2 in 1 flip (if needed)

A general upper bound: “Bring-to-top” alg.

if n = 1: do nothing

else:

- bring the largest pancake to bottom in 2 flips
- recurse on the remaining n-1 pancakes

A general upper bound: "Bring-to-top" alg.

if n = 1: do nothing

else if n = 2: sort using at most 1 flip

else:

- bring the largest pancake to bottom in 2 flips
- recurse on the remaining n-1 pancakes

$T(n)$ = max # flips for this algorithm

$$T(1) = 0$$

$$T(2) \leq 1$$

$$T(n) \leq \boxed{2} + \boxed{T(n-1)} \qquad \text{for } n \geq 3$$

$$\implies T(n) \leq 2n - 3 \qquad \text{for } n \geq 2$$

A general upper bound: "Bring-to-top" alg.

Theorem: $P_n \leq 2n - 3 \quad \text{for } n \geq 2\,.$

Corollary: $P_3 \leq 3\,.$

Corollary: $P_5 \leq 7\,.$

(So this is a loose upper bound, i.e. not tight.)

A general lower bound

How about a lower bound?

You must argue against **all** possible strategies.

What is the worst initial stack?

A general lower bound

Observation:

Given an initial stack, suppose pancakes i and j are adjacent.

They will remain adjacent if we never insert the spatula in between them.

$$(\boxed{5\ 2}\ 3\ 4\ 1)$$

So:

If i and j are adjacent and $|i-j|>1$,

then we **<u>must</u>** insert the spatula in between them.

Definition:

We call i and j a bad pair if

- they are adjacent
- $|i-j|>1$

A general lower bound

Lemma (Breaking-apart argument):

A stack with b bad pairs needs at least b flips to be sorted.

e.g. $(\boxed{5\ 2}\,3\,\boxed{4\ 1})$ requires at least 2 flips.

In fact, we can conclude it requires 3 flips. Why?

Bottom pancake and plate can also form a bad pair.

A general lower bound

Theorem: $P_n \geq n$ for $n \geq 4$.

Proof:

Take cases on the parity of n.

If n is even, the following stack has n bad pairs:

$(2\ 4\ 6\ \cdots\ n-2\ n\ 1\ 3\ 5\ \cdots\ n-1)$ (assuming $n \geq 4$)

If n is odd, the following stack has n bad pairs:

$(1\ 3\ 5\ \cdots\ n-2\ n\ 2\ 4\ 6\ \cdots\ n-1)$ (assuming $n \geq 4$)

By the previous lemma, both need n flips to be sorted.

So $P_n \geq n$ for $n \geq 4$. $\square$

Where did we use the assumption $n \geq 4$?

So what were we able to prove about P_n ?

Theorem: $n \leq P_n \leq 2n - 3$ for $n \geq 4.$

Best known bounds for P_n

Jacob Goodman 1975: what we saw

William Gates and Christos Papadimitriou 1979:

$$\frac{17}{16}n \leq P_n \leq \frac{5}{3}(n+1)$$

Currently best known: $$\frac{15}{14}n \leq P_n \leq \frac{18}{11}n$$

Best known bounds for P_n

n	P_n
4	4
5	5
6	7
7	8
8	9
9	10
10	11
11	13
12	14
13	15
14	16
15	17
16	18
17	19
18	20
19	22

$$P_{20} = ?$$

23 or 24

Why study pancake numbers?

Perhaps surprisingly, it has interesting applications.

- In designing efficient networks that are resilient to failures of links.

  Google: pancake network

- In biology.

  Can think of chromosomes as permutations.

  Interested in mutations in which some portion of the chromosome gets flipped.

Lessons

Simple problems may be hard to solve.

Simple problems may have far-reaching applications.

By studying pancakes, you can be a billionaire.

Analogy with computation

input: initial stack

output: sorted stack

computational problem: (input, output) pairs
pancake sorting problem

computational model: specified by the allowed operations on the input.

algorithm: a precise description of how to obtain the output from the input.

computability: is it always possible to sort the stack?

complexity: how many flips are needed?