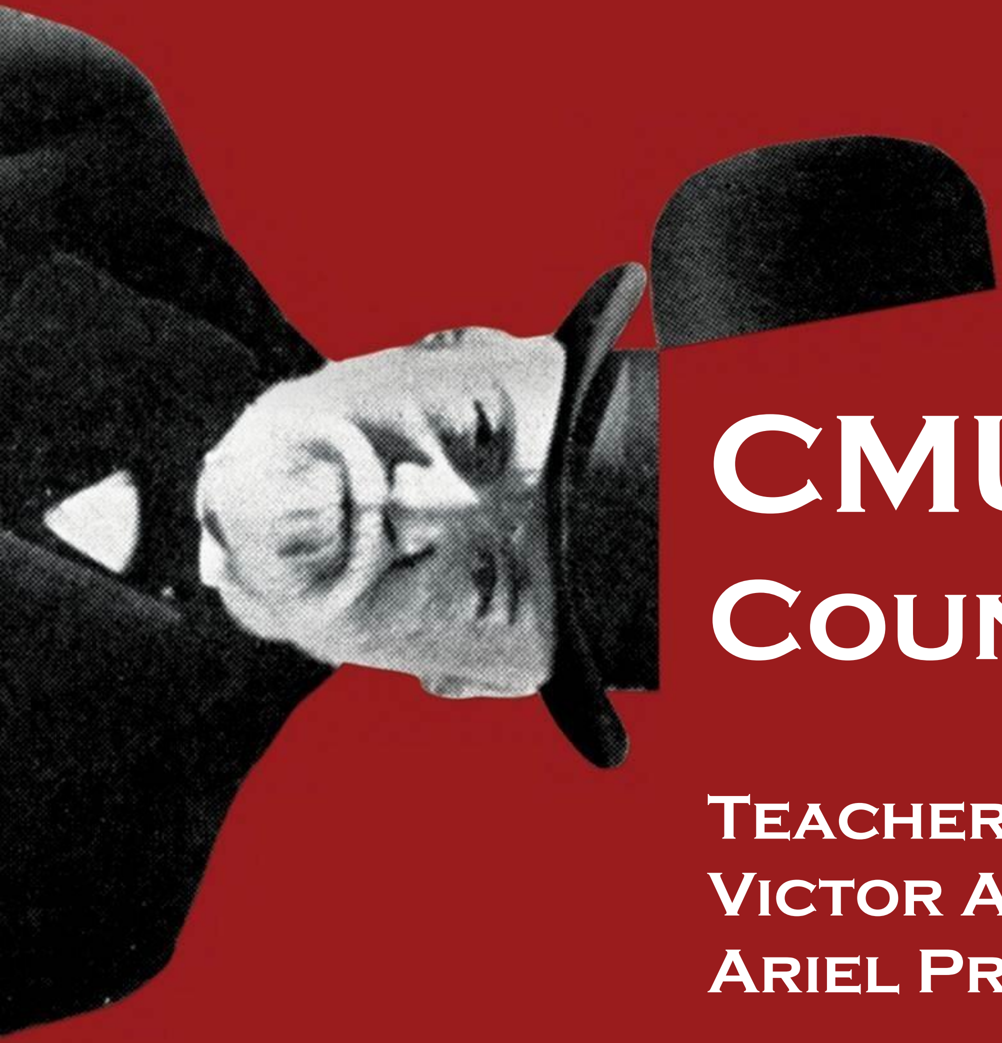

CMU 15-251

COUNTING I

TEACHERS:
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NOT A TRICK QUESTION

- I have 14 teeth on the top and 12 teeth on the bottom. How many teeth do I have?

ADDITION RULE

- If A and B are disjoint (finite) sets,
$$|A \cup B| = |A| + |B|$$
- If $A_1, \ldots, A_n$ are disjoint (finite) sets,
$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i|$$
- If A and B are (any, finite) sets,
$$|A \cup B| = |A| + |B| - |A \cap B|$$

PARTITION METHOD

- To count the elements of the set, partition it into disjoint subsets
- $A =$ outcomes of throwing white die and black die
- $A_i =$ outcomes when white die is i
- $|A| = \sum_{i=1}^{6} |A_i| = 6 \cdot 6$

PARTITION METHOD

- $A =$ all outcomes where white $\neq$ black
- $A_i =$ all outcomes where white is i and black is $j \neq i$
- $|A| = \sum_{i=1}^{6} |A_i| = 6 \cdot 5$

Partition method

- $A =$ all outcomes where white $\neq$ black
- $B =$ all outcomes where white $=$ black
- $|A \cup B| = |A| + |B| = 36$ and $|B| = 6$

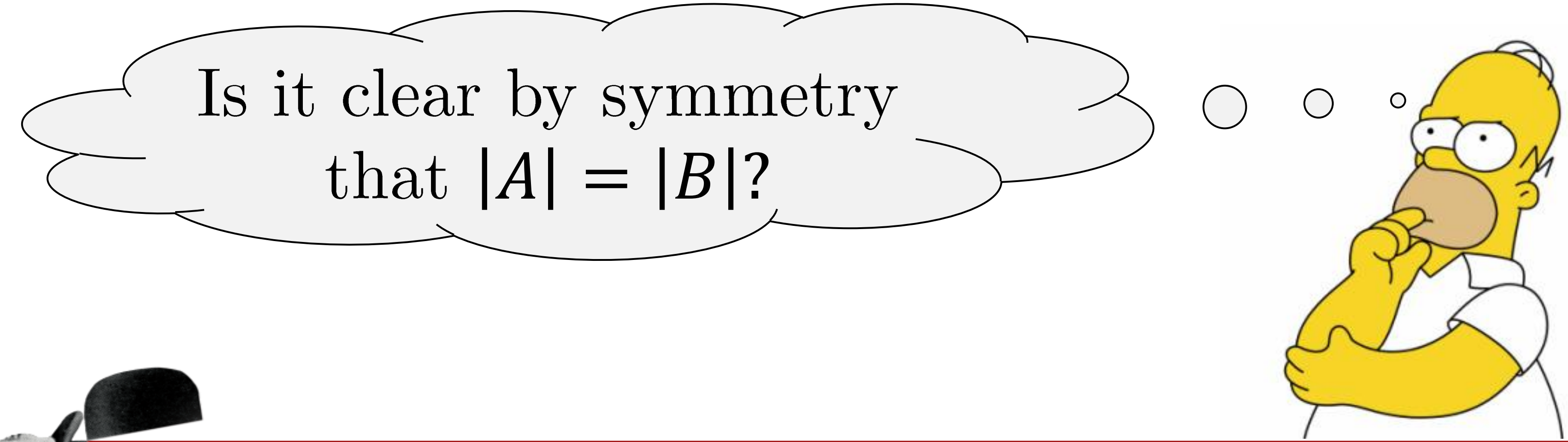
$$\Rightarrow |A| = 36 - 6 = 30$$

PARTITION METHOD

- $A =$ all outcomes where black $<$ white
- Vote: $|A| = ?$
  1. 9
  2. 12
  3. [✓ 15]
  4. 18

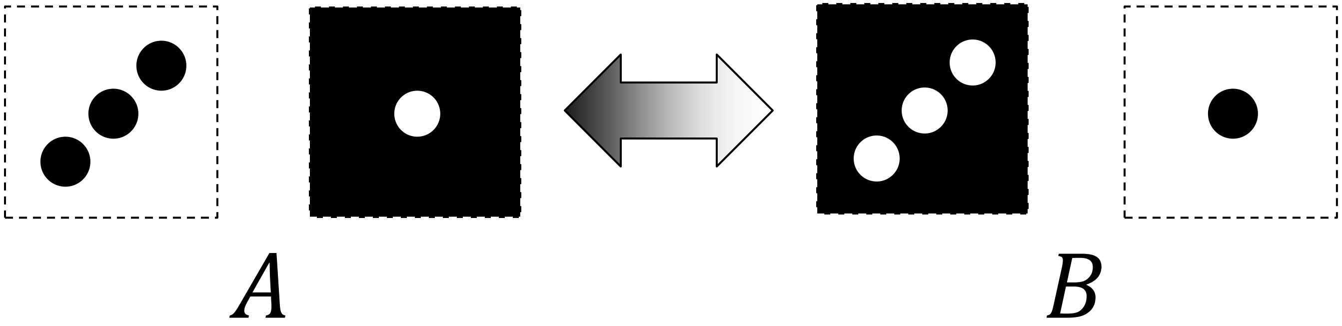
SYMMETRY

- $A =$ all outcomes where black $<$ white
- $B =$ all outcomes where black $>$ white
- $|A| + |B| = 30$
- By symmetry: $|A| = |B| \Rightarrow |A| = 15$


Is it clear by symmetry that $|A| = |B|$?

SYMMETRY VIA CORRESPONDENCE

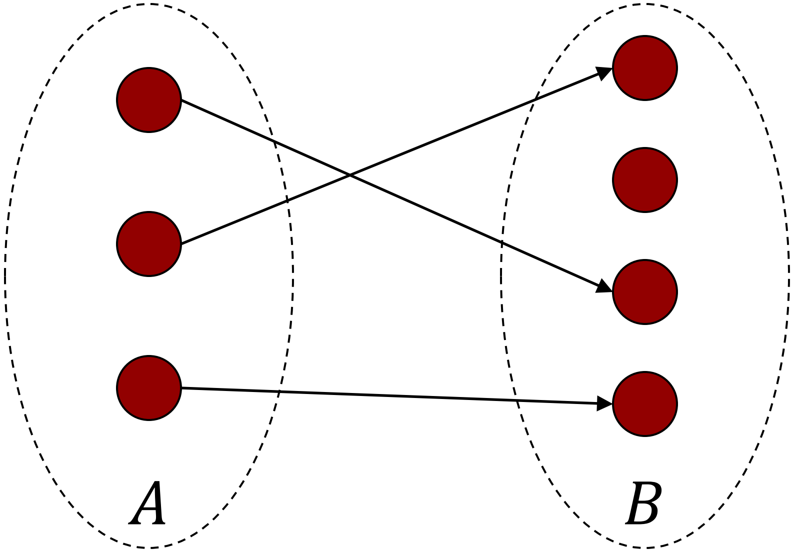
- Each outcome in A corresponds to outcome in B by swapping colors



- Each outcome in A matched with a different outcome in B, with none left over
- Thus, $|A| = |B|$

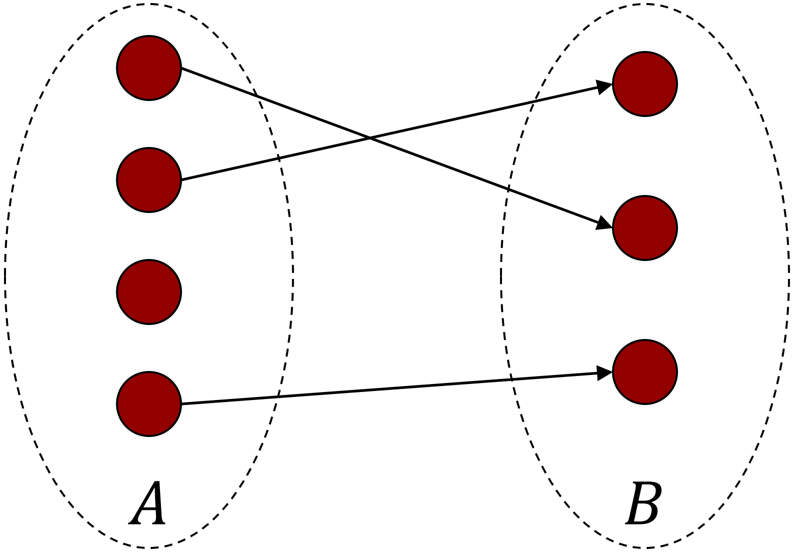
INJECTIVE FUNCTIONS

- $f: A \rightarrow B$ is injective if and only if
  $$\forall x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$$
- $\exists$ injective $f: A \rightarrow B \Rightarrow |A| \leq |B|$



SURJECTIVE FUNCTIONS

- $f: A \to B$ is surjective if and only if
  $$\forall y \in B\ \exists x \in A \text{ s.t. } f(x) = y$$
- $\exists$ surjective $f: A \to B \Rightarrow |A| \geq |B|$



CORRESPONDENCE PRINCIPLE

- $f: A \to B$ is bijective if and only if f is injective and surjective
- $\exists$ surjective $f: A \to B \Rightarrow |A| = |B|$

This is called the correspondence principle; it's one of the most important mathematical ideas of all time!

COUNTING SUBSETS

- How many n-bit sequences are there?
- $A = \{a_1, a_2, a_3, a_4, a_5\}$ has many subsets: $\{a_1, a_5\}, \{a_1, a_2, a_3, a_4, a_5\}, \{a_3, a_4, a_5\}, \emptyset, \ldots$
- These subsets correspond to 5-bit strings

a	b	c	d	e
0	1	1	0	1

$\{ \quad b \quad c \quad e \quad \}$

- 1 means "take it", 0 means "leave it"

Counting subsets

- Define a bijective f from n-bit sequences $\boldsymbol{b} = (b_1, \dots, b_n)$ to subsets of $A = \{a_1, \dots, a_n\}$:

$$f(\boldsymbol{b}) = \{a_i \mid b_i = 1\}$$

- f is injective:
  - If $\boldsymbol{b} \neq \boldsymbol{b}'$ then $\exists \boldsymbol{i}$ s.t. $b_i \neq b_i'$
  - a_i is in exactly one of $f(\boldsymbol{b}), f(\boldsymbol{b}') \Rightarrow f(\boldsymbol{b}) \neq f(\boldsymbol{b}')$
- f is surjective:
  - For any $S \subseteq A$ let $b_i = 1$ if and only if $a_i \in S$
  - $f(\boldsymbol{b}) = S$

COUNTING SUBSETS

An n-element set has 2^n subsets!

COUNTING MEALS

- A restaurant has 2 appetizers, 4 entrees, and 3 deserts
- How many items on the menu?

$$2 + 4 + 3 = 9$$

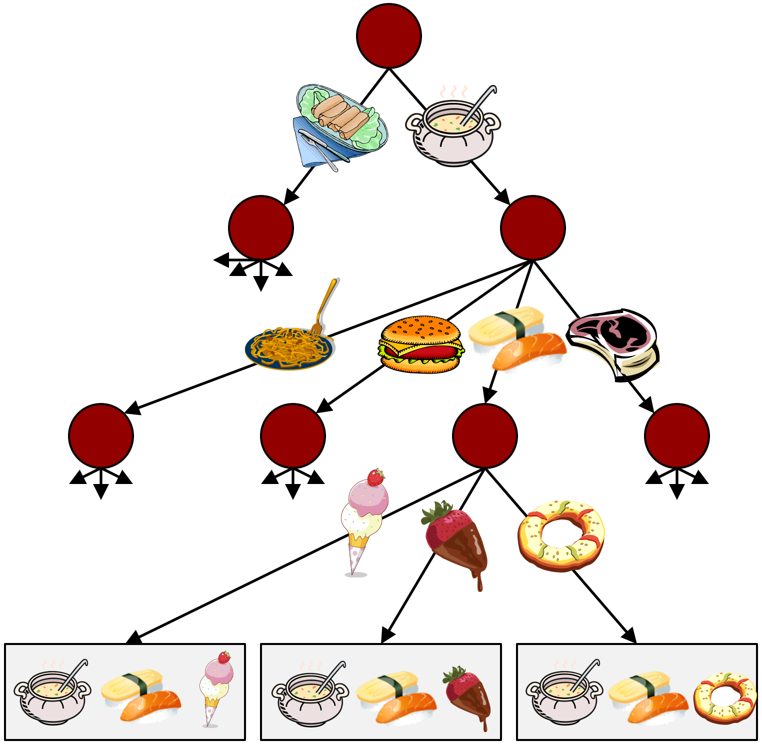
- How many ways to choose a complete meal?

$$2 \cdot 4 \cdot 3 = 24$$

- Vote: How many ways to choose a meal if I am allowed to skip some (or all) of the courses?
  1. [✓ 60]
  2. 145
  3. $2^9 = 512$
  4. $2^{24} = 16{,}777{,}216$

Choice trees

- Choice tree representation of set S
- Leaves correspond to elements of S
- If T has height h and each node at depth i has P_i children then $\#\text{leaves} = \mathrm{P}_1 \times \cdots \times P_h$



Product rule

- Suppose that every element of S can be constructed by a sequence of n choices with P_1 options for the first choice, P_2 options for the second, etc.
- If:
  - Each sequence of choices creates an element of S
  - No two sequences create the same element
- Then: $|S| = P_1 \times \cdots \times P_n$

PERMUTATIONS

- How many orderings of 52 cards?
  - 52 choices for first card
  - 51 choices for second card
  - …
  - 1 choice for the last card
  - By product rule: $52 \times 51 \times \cdots \times 1 = 52!$
- Permutation = ordering
- The number of permutations of n objects is $n!$

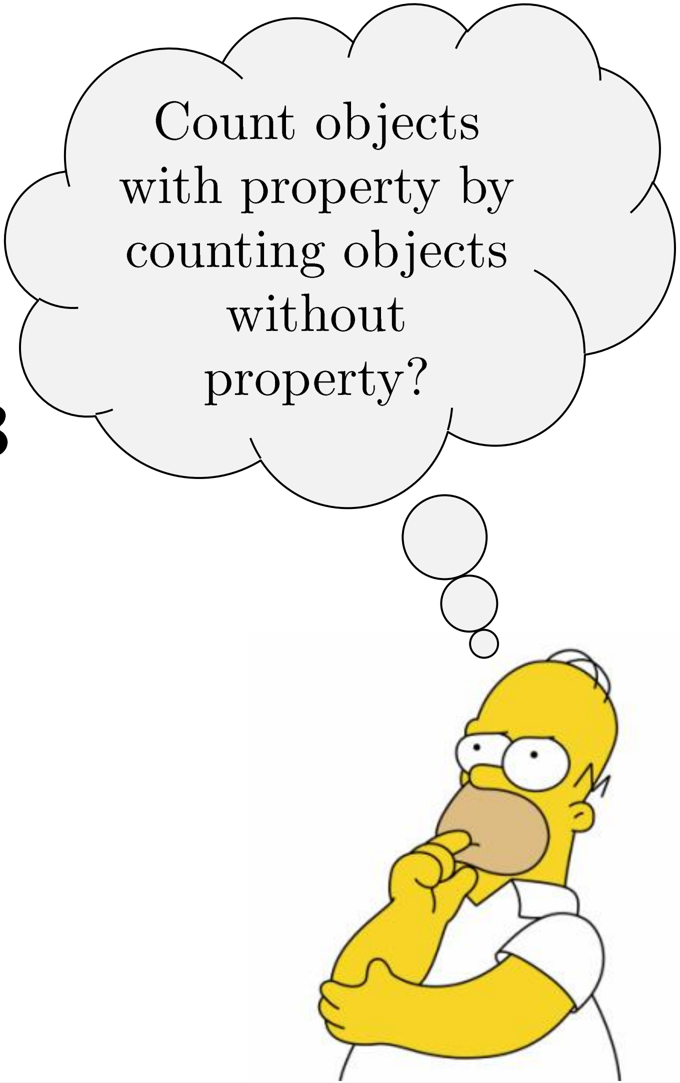
PERMUTING k OF n

- How many sequences of 3 letters?

$$26^3 = 17{,}576$$

- **Vote:** How many sequences of 3 letters where at least one letter appears twice?

1. 1949 
[✓ 2.] 1976
3. 1998
4. 2008 



Count objects with property by counting objects without property?

PERMUTING k OF n

The number of ways of choosing k ordered objects out of n is

$$n \times (n-1) \times \cdots \big(n-(k-1)\big) = \frac{n!}{(n-k)!}$$

ORDERED VS. UNORDERED

- How many ordered pairs from 52 cards?

$$52 \times 51 = 2652$$

- How many unordered pairs?

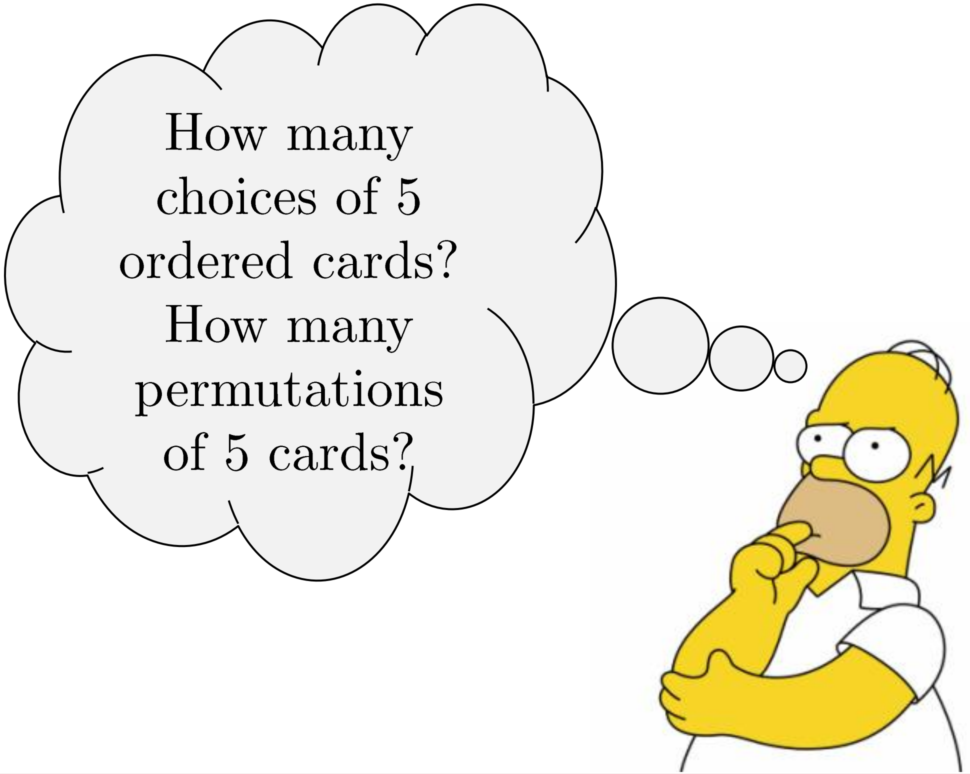
$$\frac{52 \times 51}{2} = 1326$$

- Each unordered pair is listed twice on the list of ordered pairs

ORDERED VS. UNORDERED

- Vote: How many unordered 5 card hands out of deck of 52 cards?

1. $\frac{52\times51\times50\times49\times48}{5}$
[✓ 2.] $\frac{52\times51\times50\times49\times48}{5!}$
3. $\frac{52!}{5!}$
4. $52! \times 5!$



How many choices of 5 ordered cards? How many permutations of 5 cards?

Ordered vs. unordered

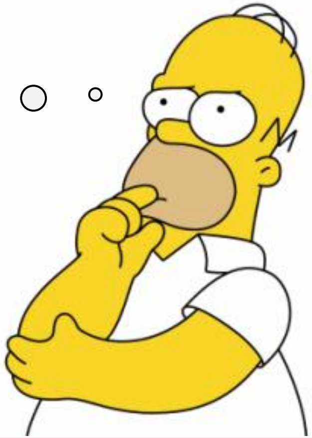
The number of ways of choosing k unordered objects out of n is

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

BACK TO BITS

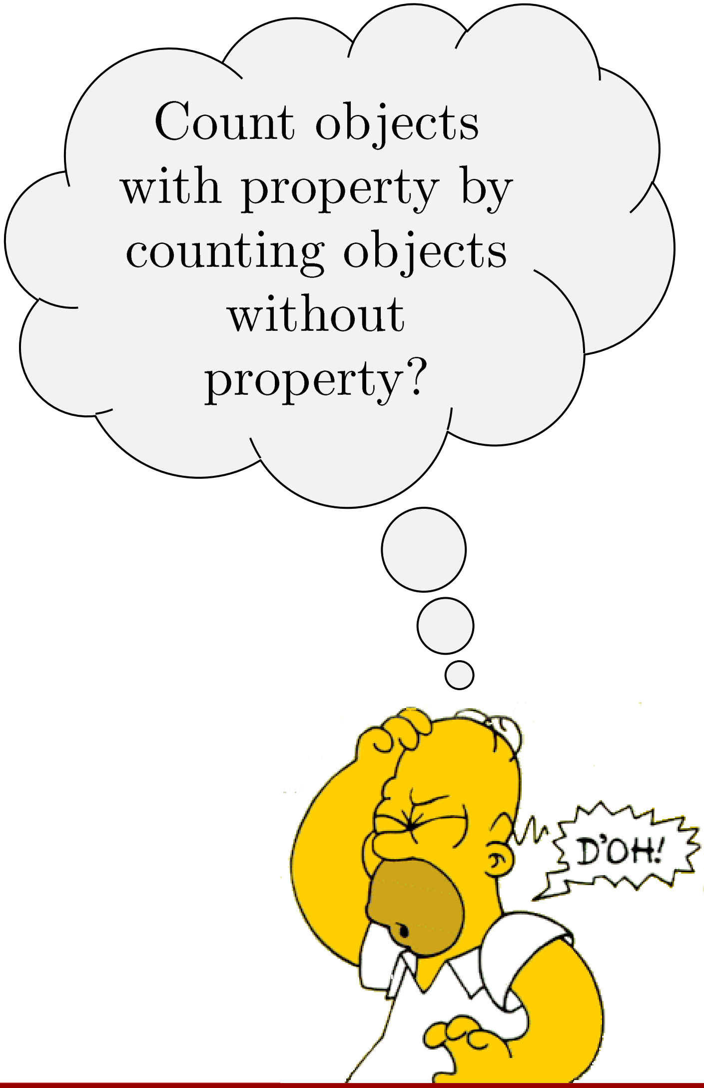
- How many 8-bit sequences with two 0s?
  - Choose two positions for the 0s; the 1s are forced: $\binom{8}{2}$
  - Choose six positions for the 1s; the 0s are forced: $\binom{8}{6}$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{k!\,(n-k!)} = \binom{n}{n-k}$$



BACK TO BITS

- Vote: How many sequences of eight 1s and four 0s s.t. no two 0s are adjacent?
    1. 70
    2. 72
    3. [✓ 126]
    4. 128
- How many sequences of eight 1s and four 0s s.t. no two 0s and no two 1s are adjacent?



COUNTING TIPS

- How many ways to choose 5 cards with at least three aces?
- Method 1:
    - $\binom{4}{3}$ ways of choosing three aces
    - $\binom{49}{2}$ ways of choosing remaining two cards
    - Overall $4 \times 1176 = 4704$

Counting tips

- Method 2:
  - Choose aces and then non-ace cards
  - $\binom{4}{3} \times \binom{48}{2} = 4512$ hands with exactly three aces
  - $\binom{4}{4} \times \binom{48}{1} = 48$ hands with exactly four aces
  - Overall $4512 + 48 = 4560$

$4560 \neq 4704$, doh!

COUNTING TIPS

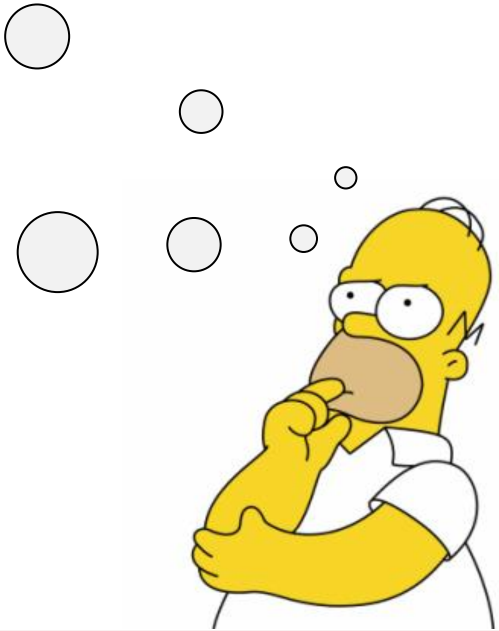
- Method 1 is wrong
- Example: Four different sequences of choices produce the same hand

A♣ A♦ A♥	**A♠ K♦**
A♣ A♦ A♠	**A♥ K♦**
A♣ A♠ A♥	**A♦ K♦**
A♠ A♦ A♥	**A♣ K♦**
3 out of 4 aces	2 out of 49 cards

COUNTING TIPS

How do I know that the other argument is correct? Down with counting!

Reversibility check: For each object, can I reverse-engineer the sequence that constructed it?



REVERSIBILITY TEST

- Method 1: Choose 3 of 4 aces, then 2 of remaining 49 cards

A♣ A♦ A♥A♠ K♦

- Cannot reverse to a unique choice sequence

A♣ A♦ A♥ A♠ K♦

A♣ A♦ A♠ A♥ K♦

A♣ A♠ A♥ A♦ K♦

A♠ A♦ A♥ A♣ K♦

REVERSIBILITY TEST

- Method 2: Choose 3 of 4 aces, then 2 of 48 non-ace cards

A♣ A♦ Q♦ A♠ K♦

- Reverse test: Aces came from choice of aces, others came from choice of non-aces

Think choice trees!

WHAT WE HAVE LEARNED

- Definitions / facts
  - Injection, surjection, bijection
  - Ordered vs. unordered
- Principles / problem solving
  - Addition rule
  - Partition method
  - Correspondence principle
  - Product rule
  - Counting by removing
  - Reversibility test

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