15-251 : Great Theoretical Ideas In Computer Science

Fall 2013

Assignment 1

Due: Thursday, Sep. 05, 2013 11:59 PM

Name: ______________________________

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Question:	1	2	3	4	Total
Points:	40	20	40	10	110
Score:					

1. Pancakes with Problems

(20) (a) Suppose we have a special spatula which only lets us flip an even number of pancakes (from the top of the stack) each time. Show that the minimum number of flips needed using this spatula to sort any sortable stack of $2n$ pancakes is at least P_n and at most $3n - 2$ for all $n \geq 1$. (By "sortable" we mean that it is possible to sort the stack using the special spatula)

Solution:

(20) (b) Suppose now we are allowed to take any contiguous set of pancakes and flip them in place (they need not be on the top of the stack). Let Q_n be the maximum over stacks of size n of the minimum number of flips required to sort that stack, using this new flipping operation. Show that $n/2 \leq Q_n \leq n - 1$ for all $n \geq 2$.

Solution:

2. Bad Induction Proofs

For each of the proposed claims below, examine the proposed proof and point out the flaw in it. Do not just explain why the claim is wrong; rather you should explain how the argument violates the notion of a valid proof.

(10) (a) **Claim:** $\log_3 n = \log_5 n$ for all natural numbers $n \geq 1$.

Proof (by strong induction)

The inductive hypothesis statement P_n will be "$\log_3 n = \log_5 n$".

- *Base case:* When $n = 1$, $\log_3 1 = 0 = \log_5 1$.
- *Inductive hypothesis:* Assume P_k for all $k \leq n$, and use this to show P_{n+1}.
- *Inductive step:* Write $n+1$ as a product of two natural number p, q, so that we have

$$\log_3(n+1) = \log_3(pq) = \log_3 p + \log_3 q = \log_5 p + \log_5 q = \log_5(pq) = \log_5(n+1)$$

where we used the IH when switching the base of the logarithm. Thus, we have proven the claim.

Solution:

(10) (b) **Claim:** For all positive integers n, we have $251^{n-1} = 1$.

Proof (by strong induction on n)

- *Base case:* When $n = 1$, the claim is true since $251^{n-1} = 251^0 = 1$
- *Inductive hypothesis:* Suppose the claim holds for any number less or equal than n.
- *Inductive step:* We must now show that the claim is true for $n + 1$. We have

$$251^{(n+1)-1} = 251^n = \frac{251^{n-1} \times 251^{n-1}}{251^{n-2}} = \frac{1 \times 1}{1} = 1;$$

so the claim is true for $n + 1$ as well.

Solution:

3. Induction Problems

(a) **Features of Fibonacci** For these problems, use $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

(10) i. Prove that $\forall n, k > 0, F_{n+k} = F_{n+1}F_k + F_nF_{k-1}$

Solution:

(10) ii. Prove that $\forall n \geq 0$, $\sum_{i=0}^{n} \binom{n-i}{i} = F_{n+1}$.

Solution:

(20) (b) **Circles dividing the plane**

Suppose we want to divide the plane into regions by drawing circles. Without any circles, the plane is undivided so there is one region. With one circle, you can divide the plane into two regions. With two we can create four regions, and with three we can create eight regions. We claim that you can't divide the plane into more than $n^2 - n + 2$ regions by drawing n circles. Prove the correctness of this claim by induction.

Solution:

4. Bonus Question - More Pancake Flipping

In Question 1(b), you showed that at least $n/2$ flips of contiguous sets of pancakes are needed to sort the *worst* stack of n pancakes. In this problem, you will derive bounds on the number of flips for *every* stack of n pancakes (still using the flipping operation from Question 1(b)).

Bad Pairs: Recall from the lecture that a pair of adjacent pancakes is "bad" if their values differ by more than 1, as such a pair needs to be broken in the end. Put an imaginary 0 at the top and an imaginary $n + 1$ at the bottom of the stack (you cannot move them around), so the first pancake contributes a "bad pair" if its value is not 1, and the last pancake also contributes a bad pair if its value is not n. Imagine a separator between the pancakes in every bad pair.

Patches: A "patch" is the set of pancakes between any two consecutive separators. A patch is called decreasing if the values decrease from top to bottom, and called increasing otherwise. For example, (4 3 2) is a decreasing patch, and (6 7 8 9) is an increasing patch. Singleton patches are decreasing, except for the first and the last pancake.

(5) (a) Show that every stack of pancakes that has a decreasing patch also has a flip that reduces the number of bad pairs by at least 1.

Solution:

(5) (b) Use part (a) to show the following: For any stack of n pancakes with $b \geq 2$ bad pairs, if t is the number of flips required to sort it, then $b/2 \leq t \leq 2b - 3$.

[Actually, it is possible to improve the upper bound to $b - 1$, but it is beyond the scope of this course.]

Solution: