2008 International Conference on Emerging Technologies
IEEE-ICET 2008
Rawalpindi, Pakistan, 18-19 October, 2008

Interpolation with rational cubic spirals

Agha Ali Raza[a], Zulfiqar Habib[b] and Manabu Sakai[c]
[a,b] *Department of Computer Science, National University of Computer & Emerging Sciences (FAST-NU), Lahore, Pakistan*
[c] *Department of Mathematics and Computer Science, University of Kagoshima, Kagoshima, Japan 890-0065*
[a]ali.raza@nu.edu.pk, [b]zulfiqar.habib@nu.edu.pk, [c]msakai@sci.kagoshima-u.ac.jp

Abstract

Spiral curves of one-sided, monotone increasing or decreasing curvature have the advantage that the minimum and maximum curvature is at their endpoints and they contain no inflection points or local curvature extrema. This paper derives a spiral condition for a rational cubic spline on the endpoint curvatures under the fixed positional and tangential conditions and provides more flexible spiral regions.

Keywords

Rational cubic curve, Curvature, Spiral, Path planning.

1. Introduction

In visually pleasing curve and surface design, it is often desirable to have a spiral transition curve without extraneous curvature extrema. The purpose may be practical [Gibreel et al.(1999)], e.g., in highway designs, railway routes and aesthetic applications [Burchard et al. (1993)] [1]. Since it is not easy to locate the zeros of the derivative of the curvature of a high degree curve, therefore simplified cases have been developed to join the following pairs of objects: (i) straight line to circle, (ii) circle to circle with a broken back C transition, (iii) circle to circle with an S transition, (iv) straight line to straight line and (v) circle to circle with one circle inside the other ([3] - [6]). For other works on spirals that are rational and polynomial functions and match end conditions please see [1]. Dietz and Piper have numerically computed viable regions for the cubic spirals to aid in adjusting the selection of control points without zero curvature restriction [1]. These viable regions allow the construction of spirals joining one circle to another circle where one lies inside the other by *Kneser's theorem* (case (v)) [2]. Their results have enabled us to use a single cubic curve rather than two segments for the benefit that designers and implementers have fewer entities to be concerned with. However, their numerically obtained spiral regions would become smaller or even empty as the tangent angles become relatively equal.

This paper considers more flexible *rational* cubic spirals to overcome the above mentioned problem. Since there is no closed form for the roots of the derivative of the curvature (of degree 12 for this rational cubic case), we first transform the unit interval [0, 1] to [0, ∞) and then derive a sufficient spiral region with respect to the parameters introduced under the fixed tangent angles at the endpoints [1]. Section 3 represents the parameters in terms of the endpoint curvatures and next derives the spiral regions with respect to the curvatures under the fixed tangent directions. Section 4 gives the numerically determined spiral regions which help us adjust the endpoint curvatures and illustrative numerical examples to demonstrate the flexibility of our rational cubic spline method, especially for the case when the tangent angles are relatively equal.

2. Description of method

Let $\boldsymbol{p}_i$, $i = 0, 1, 2, 3$ be four given control points. The cubic *Bézier curve* defined by them has nine degrees of freedom and is represented in form as

$$\boldsymbol{z}(t) = \frac{(1-t)^3\boldsymbol{p}_0+3t(1-t)^2\boldsymbol{p}_1+3t^2(1-t)\boldsymbol{p}_2+t^3\boldsymbol{p}_3}{(1-t)^3+3w(1-t)^2t+3w(1-t)t^2+t^3}, 0 \le t \le 1 \quad (2.1)$$

Note that the above rational cubic curve reduces to a usual cubic one for a choice of $w = 1$. Its signed curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{z'(t)\times z''(t)}{\|z'(t)^3\|} \quad (2.2)$$

where "×", "." and $\|\cdot\|$ mean the *cross*, *inner product* of two vectors and the *Euclidean norm*, respectively. Without loss of generality, we define spirals to be planar arcs with non-negative curvature and continuous non-zero derivative of the curvature. This paper assumes that (i) $\boldsymbol{p}_0 = (-1, 0)$ and $\boldsymbol{p}_3 = (1, 0)$, (ii) the tangent angle between the tangent vector and the vector (1, 0) at $t = 0$ is denoted by ϕ_0, while the tangent angle at $t = 1$ is denoted by ϕ_1 as shown in Figure 1 where the values for ϕ_0 and ϕ_1 are constrained so that $0 < \phi_0 < \phi_1 < \pi/2$ [2]. These constraints restrict our curves to spiral arcs of increasing curvature. Then, for $\boldsymbol{p}_1 = (u_1, v_1)$ and $\boldsymbol{p}_2 = (u_2, v_2)$, note that

$$v_1 = -(u_1 + w) \tan \phi_0, \; v_2 = (u_2 - w) \tan \phi_1 \tag{2.3}$$

As in [3], the tangent lines for the parametric cubic at $t = 0$ and $t = 1$ and the horizontal axis form a triangle where the lengths of the lower two sides are

$d_0 = 2sin\phi_1/sin(\phi_0+\phi_1)$ (for the side touching p_0),
and
$d_1 = 2sin\phi_0/sin(\phi_1+\phi_1)$ (for the side touching p_1).

Note that

$$||z'(0)|| = 3(u_1 + w)/cos\phi_0,$$

and

$$||z'(1)|| = 3(w - u_2)/cos\phi_0.$$

This paper extensively uses the ratio parameters (f_0, f_1):

$$f_0 = \frac{\|\boldsymbol{z}'(0)\|}{d_0}, f_1 = \frac{\|\boldsymbol{z}'(1)\|}{d_1} \tag{2.4}$$

Therefore, we have

$$(w + u_1, w - u_2) = m\left(\frac{f_0}{tan\ \phi_0}, \frac{f_1}{tan\phi_1}\right),$$

$$(v_1, v_2) = -m(f_0, f_1)$$

where

$$m = 2sin\phi_0 sin\phi_1/\{3sin(\phi_0+\phi_1)\},$$

the control points $\boldsymbol{p}_i$, $i = 1, 2$ are given by

$$\boldsymbol{p}_1 (= (u_1, v_1)) = -w(1, 0) + mf_0(cot\ \phi_0, -1),$$
$$\boldsymbol{p}_2 (= (u_2, v_2)) = w(1, 0) - mf_1(cot\ \phi_1, 1). \tag{2.5}$$

The following figure (Figure-1) shows the transition curve, two tangents and two osculating circles at $\boldsymbol{p}_i$, $i = 0, 3$ where the points $\boldsymbol{p}_i$, $0 \le i \le 3$ are denoted counterclockwise by four small discs for $(\phi_0, \phi_1) = (0.7, 1)$ with $(w, f_0, f_1) = (0.8, 1.6, 1.2)$.

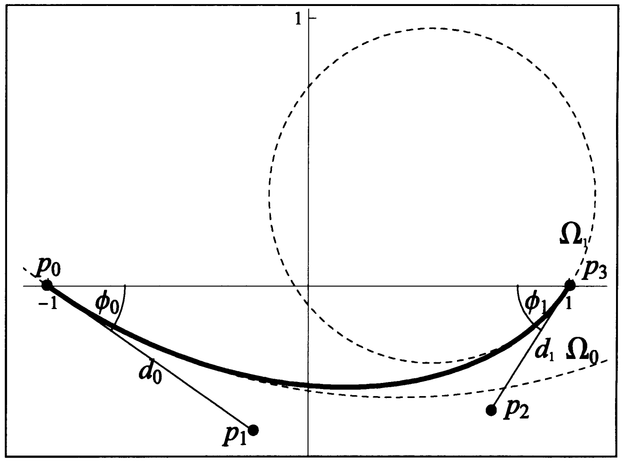


Figure 1 Spiral transition between two circles with given end point p_0 and p_3, tangents, and angles.

In order to convert the parameterization from the strict condition $(0 \le t \le 1)$ to a relatively relaxed condition $(s \ge 0)$ the parameter t is substituted as $t = 1/(1 + s)$ to get,

$$\|\boldsymbol{z}'(t)\|^5 \kappa'(t) =$$

$$\frac{94(1+s)^4 sin\phi_0 sin\phi_1}{\{1-(1-3w)s+s^2\}^8 sin^3(\phi_0+\phi_1)} \sum_{i=0}^{12} m_i H_i(f_0, f_1; \phi_0, \phi_1) s^{12-i} \tag{2.6}$$

Where $H_i(f_0, f_1; \phi_0, \phi_1)$, $0 \le i \le 12$ are functions of f_0, f_1 and ϕ_0, ϕ_1.

The problem of designing cubic spirals could be formulated analytically as finding conditions on the coefficients of the above 12th degree polynomial to ensure the non-positive zeros (for $w = 1$, the polynomial reduces to the quintic one). As the necessary and sufficient solution to this problem is difficult to be found, we give a sufficient one from (2.6):

The rational transition cubic curve of the form (2.1) is a spiral if

$$H_i(f_0, f_1; \phi_0, \phi_1) \ge 0,$$
$$H_{12-i}(f_1, f_0, \phi_1; \phi_0) \le 0,$$
$$\text{for, } 0 \le i \le 6. \tag{2.7}$$

By *Descartes Rule of Signs*, the rational cubic curve of the form (2.1) is a spiral if 2.7 holds.

3. Spiral conditions on the endpoint curvatures

The algebraically simple parameters (f_0, f_1) do not give direct information on how to match or approximate given curvature values at the endpoints [1]. This section considers the problem of finding useful approximation to the admissible set of endpoint curvatures (κ_0, κ_1) $(= (\kappa(0), \kappa(1)))$ of a cubic rational spiral under the fixed positional and tangential end conditions. For this end, we analytically present the parameters (f_0, f_1) in terms of (κ_0, κ_1) with (ϕ_0, ϕ_1) fixed. The endpoint curvatures are given as

$$\kappa_0 = \frac{(3w-f_1)sin\phi_0 sin^2(\phi_0+\phi_1)}{f_0^2 sin^2\phi_1}$$

$$\kappa_1 = \frac{(3w-f_0)sin\phi_1 sin^2(\phi_0+\phi_1)}{f_1^2 sin^2\phi_0} \tag{3.1}$$

From 3.1, we have a quartic equation of f_0:

$$g(f_0)(= f_0^4 - c_2 f_0^2 + c_1 f_0 + c_0) = 0 \tag{3.2}$$

where,

$$d = \frac{sin\phi_1 sin^2(\phi_0+\phi_1)}{w sin^2\phi_0}, d_i = \left(\frac{sin\phi_0}{sin\phi_1}\right)^{3i}$$

$$c_2 = \frac{6d_1 d w^2}{\kappa_0}, c_1 = \frac{d_2 d^3 w^3}{\kappa_0^2 \kappa_1}, c_0 = \frac{9d_2 d^2 w^4}{\kappa_0^2 \kappa_1}\left(\kappa_1 - \frac{d}{3}\right)$$

$$f_1 = \frac{\kappa_0}{d_1 d w}(u^2 - f_0^2), u = w\sqrt{\frac{3d_1 d}{\kappa_0}} \tag{3.3}$$

The system (3.1) has a solution (f_0, f_1) of positive pair if $\kappa_1 > d/3$, and $D(=256\kappa_0^2\kappa_1(3\kappa_1-d)+32d_1 d\kappa_0\kappa_1(3d-8\kappa_1)-d_2 d^4) \geq 0$.

We need to factorize (3.2) into the quadratics:

$$g(f_0) = (f_0^2 + \sqrt{p}f_0 + q)(f_0^2 - \sqrt{p}f_0 + r) \tag{3.4}$$

where $q+r = p-c_2$, $r-q = c_1/\sqrt{p}$ and $qr = c_0$. Hence we have

(i) $q = \frac{1}{2}\left(p - c_2 - \frac{c_1}{\sqrt{p}}\right)$,

(ii) $r = \frac{1}{2}(p - c_2 - \frac{c_1}{\sqrt{p}})$,

(iii) $p(p-c_2)^2 - 4c_0 p - c_1^2$. (3.5)

The cubic equation 3.8 (iii) has at least one positive root and its positive one is given by

$$p = \frac{1}{3}\left(2c_2 + \eta^{\frac{1}{3}} + m\eta^{-\frac{1}{3}}\right), \eta = \frac{1}{2}\left(\mu + \sqrt{\mu^2 - 4\lambda^3}\right) \tag{3.6}$$

with $(\lambda, \mu) = (12c_0 + c_2^2, 27c_1^2 + 72c_0c_2 - 2c_2^3)$.

Since $\kappa_0^8\kappa_1^4(\mu^2 - 4\lambda^3) = -27^2 d_6 d^8 w^{12} D\ (\leq 0)$ (3.7)
let $\eta = re^{i\theta}\left(r = \lambda^{\frac{3}{2}}, 0 \leq \theta \leq \pi\right)$ *to obtain*

$$p = \frac{2}{3}\left(c_2 + \sqrt{\lambda}cos\frac{\theta}{3}\right)(\geq c_2) > 0 \tag{3.8}$$

Thus, q and r being both positive from (3.5), the two positive roots are from $f_0^2 - \sqrt{p}f_0 + r = 0$, i.e.

$$f_0 = \frac{1}{2}\left(\sqrt{p} \pm \sqrt{p-4r}\right) \tag{3.9}$$

For $f_0 = \left(\sqrt{p} - \sqrt{p-4r}\right)/2$ and f_1 from (3.1), the spiral region on the endpoint curvatures (κ_0, κ_1) is given by

$$\frac{d_1 d}{16(3\kappa_1 - d)}\left\{8\kappa_1 - 3d + \sqrt{\frac{(4\kappa_1 - d)^3}{\kappa_1}}\right\} \leq \kappa_0,$$

with the conditions $D \geq 0$ and $\kappa_1 > d/3$.

4. Numerically determined spiral regions

The *"RegionPlot"* command of *Mathematica*® draws the spiral regions of (κ_0, κ_1) denoted by the insides of the closed curves in Figs 2a, 3a and 4a where using a well known result that the osculating circle at the endpoint p_0 is completely included in the osculating one at the other endpoint p_3, we can restrict the possible region as $\kappa_0 \leq sin\phi_0$ and $\kappa_1 \geq (\kappa_0 sin\phi_1 - sin^2\{(\phi_0+\phi_1)/2\})/(\kappa_0 - sin\phi_0)$. The spiral regions with respect to (κ_0, κ_1) move from darker to lighter regions as the parameter w increases. Our *rational* cubic method gives a spiral even for $(\phi_0, \phi_1) = (0.99, 1)$ where "NVR" (*no viable region*) is given in Table 1 [1]. The figures 2b, 3b and 4b show the plots of the spiral regions for some values chosen from within the spiral regions depicted in the part-a of the respective figures. Finally the figures 2c, 3c and 4c are the plots of the curvature κ showing its monotonically increasing trend with parameter t.

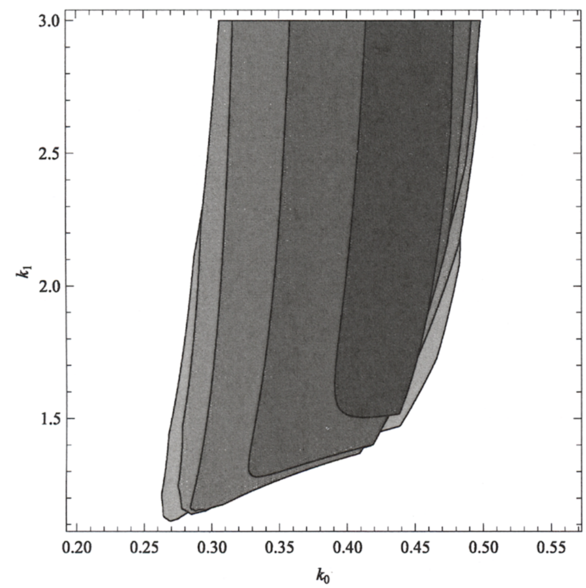
The three numerical cases are presented below:

Case 1: $(\phi_0, \phi_1) = (0.6, 1)$ (*Figures 2a, 2b, 2c*)

Case 2: $(\phi_0, \phi_1) = (0.99, 1)$ (*Figures 3a, 3b, 3c*)

Finally we give one more spiral region of (κ_0, κ_1) to demonstrate usefulness of our rational spline method for $(\phi_0, \phi_1) = (1, 1.01)$ while in [1] "NVR" is denoted even for $(\phi_0, \phi_1) = (1, 1.2)$.

Case 3: $(\phi_0, \phi_1) = (1, 1.01)$ (*Figures 4a, 4b, 4c*)



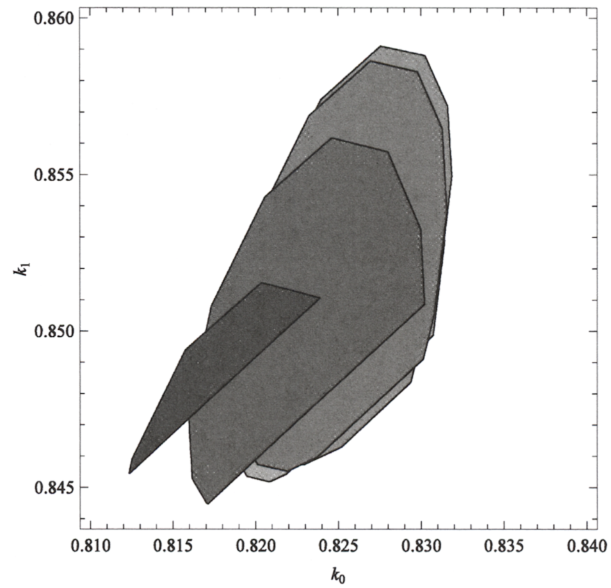


Figure-2a Spiral regions from darker to lighter as w=0.6(0.1)1.0

Figure-3a Spiral regions from darker to lighter as w=0.64(0.02)0.72

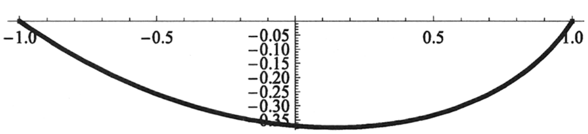


Figure-2b Plot of the Spiral with $(\kappa_0, \kappa_1, w) = (0.42, 1.6, 0.6)$

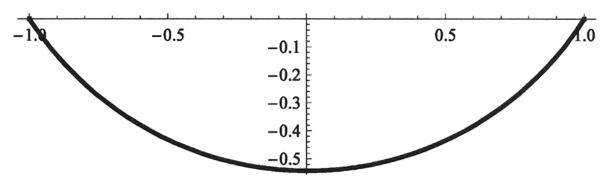


Figure-3b Plot of the Spiral with $(\kappa_0, \kappa_1, w) = (0.825, 0.85, 0.72)$

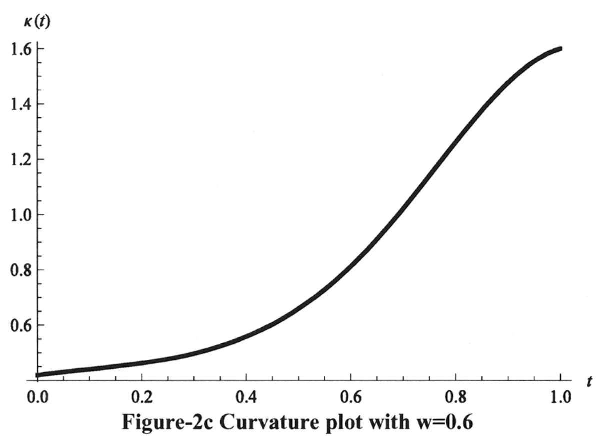


Figure-2c Curvature plot with w=0.6

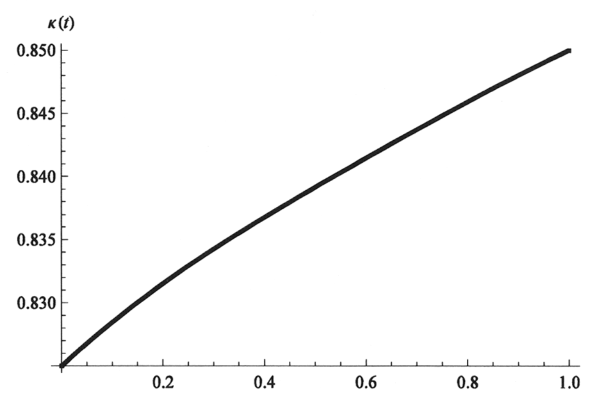


Figure-3c Curvature plot with w=0.72

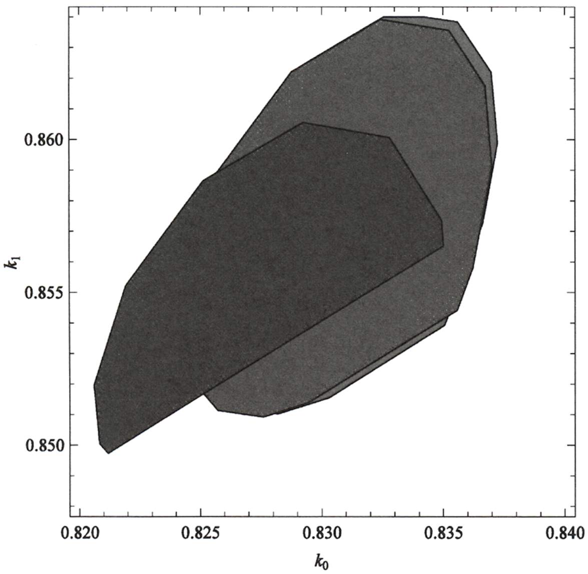


Figure-4a Spiral regions from darker to lighter as w=0.65(0.025)0.7

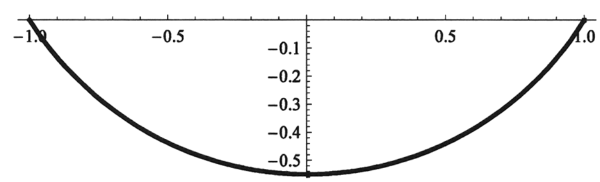


Figure-4b Plot of the Spiral with (κ_0, κ_1, w) = (0.83, 0.855, 0.7)

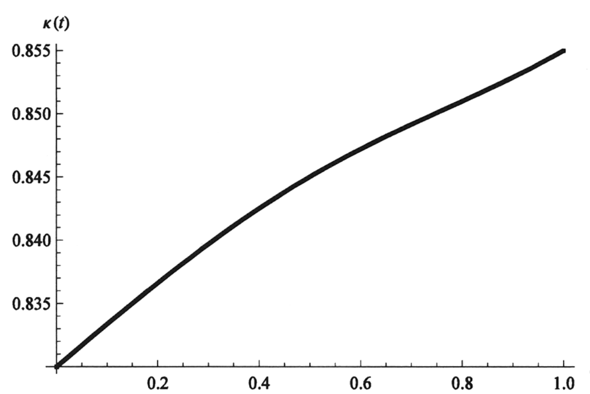


Figure-4c Curvature plot with w=0.7

5. Conclusion

The work presented in this paper is an attempt to improve the flexibility currently available in numerical spiral design techniques as spirals are a useful part of design and path planning applications. The flexibility provided by the rational cubic spirals is far greater than the normal cubic spiral which is used in [1]. As a result, this method finds satisfactory solutions in many cases where the method of [1] reports *no viable region* (*NVR*), especially as $\phi_0 \rightarrow \phi_1$. Case 3 presented in section 4 is such an example.

However, there is still a lot of room for future work in this area. For example, the behavior of spiral design when $\phi_0 = \phi_1$ should be investigated. The possibility of circular arcs using this approach should be scrutinized as rational curves allow circular arcs. Besides, the question of single spiral segment joining concentric circles still stands as a major challenge (previous works allow this using multiple spiral segments, or non-spiral curves ([9], [10])). And finally, the formulation of necessary and sufficient conditions for spiral arcs conforming to given spatial and tangential end conditions remain as a significant milestone.

6. References

[1] D. Dietz & B. Pier (2004), Interpolation with cubic spirals, Computer Aided Geometric Design 21, 165-180.

[2] H. W. Guggenheimer (1963), Differential geometry, New York: MacGraw-Hill.

[3] D. Walton & D. Meek (1996), A planar cubic Bezier spiral,Journal of Computational and Applied Mathematics, 72, 85-100.

[4] D. Meek & D. Walton (1998), Planar spirals that match G2 hermite data, Computer Aided Geometric Deometric Design 15, 103-126.

[5] D. Walton & D. Meek (1999), Planar G2 transition between two circles with a fair cubic Bézier curve, Computer Aided Design, 31, 857-866.

[6] D. Walton & D. Meek (2002), Planar G2 transition with a fair Pythagorean hodograph quintic curve, Journal of Computational and Applied Mathematics, 138, 109-126.

[7] M. D. Walton & D. Meek (2007), G2 curve design with a pair of Pythagorean Hodograph quintic spiral segments, Computer Aided Geometric Design, 24, 267- 285.

[8] T.N.T. Goodman & D. Meek (2007), Planar interpolation with a pair of rational spirals, Journal of Computational and Applied Mathematics, 201, 112-127.

[9] Habib, Z., Sakai, M., 2005. Spiral transition curves and their applications. Scientiae Mathematicae Japonicae 61 (2), 195–206. e2004, 251–262.
http://www.jams.or.jp/scmjol/2004.html.

[10] Habib, Z., Sakai, M., 2007. On PH quintic spirals joining two circles with one circle inside the other. Computer-Aided Design 39 (2), 125–132.
http://dx.doi.org/10.1016/j.cad.2006.10.006.

[11] Habib, Z., Sakai, M., 2007. *G*2 Pythagorean hodograph quintic transition between circles with shape control. Computer Aided Geometric Design 24, 252–266.
http://dx.doi.org/10.1016/j.cagd.2007.03.004.

[12] Habib, Z., Sakai, M., Sarfraz, M., 2004. Interactive shape control with rational cubic splines. Computer-Aided Design & Applications 1 (1–4),709–718.

[13] Habib, Z., Sarfraz, M., Sakai, M., 2005. Rational cubic spline interpolation with shape control. Computers & Graphics 29 (4), 594–605.
http://dx.doi.org/10.1016/j.cag.2005.05.010.

[14] Sakai, M., 1999. Inflection points and singularities on planar rational cubic curve segments. Computer Aided Geometric Design 16, 149–156.

[15] Sarfraz, M., 2005. A rational cubic spline for the visualization of monotone data: An alternate approach. Computer & Graphics 27, 107–121.