11: Differential Equations & Proofs
Logical Foundations of Cyber-Physical Systems

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Logical Foundations of Cyber-Physical Systems

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Outline

1 Learning Objectives
2 Differential Invariants
   • Recap: Ingredients for Differential Equation Proofs
   • Soundness: Derivations Lemma
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1 Learning Objectives

2 Differential Invariants
   - Recap: Ingredients for Differential Equation Proofs
   - Soundness: Derivations Lemma
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Learning Objectives
Differential Equations & Proofs

- discrete vs. continuous analogy
- rigorous reasoning about ODEs
- beyond differential invariant terms
- differential invariant formulas
- cut principles for differential equations
- axiomatization of ODEs
- differential facet of logical trinity

understanding continuous dynamics
relate discrete+continuous

operational CPS effects
state changes along ODE
Differential Facet of Logical Trinity

Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.
How does the semantics of \( e \geq \tilde{e} \) relate to semantics of \( e - \tilde{e} \geq 0 \), syntactically? What about derivatives?
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## Differentials

### Syntax

\[ e ::= x \mid x' \mid c \mid e + k \mid e \cdot k \mid (e)' \]

### Semantics

\[ \omega[(e)'] = \sum_x \omega(x') \frac{\partial \llbracket e \rrbracket}{\partial x}(\omega) \]

### Axioms

- \((e + k)'' = (e)'' + (k)''\)
- \((e \cdot k)'' = (e)'' \cdot k + e \cdot (k)''\)
- \((c())'' = 0\) for constants/numbers \(c()\)
- \((x)'' = x'\) for variables \(x \in V\)

### ODE

\[ \llbracket x' = f(x) \land Q \rrbracket = \{ (\varphi(0)\mid_{x'}^c, \varphi(r)) : \varphi \models x' = f(x) \land Q \} \text{ for some } \varphi : [0, r] \to \mathcal{S}, \text{ some } r \in \mathbb{R} \}

\[ \varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z) \]

...
**Lemma (Differential lemma)**  
(Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)$$

**Lemma (Differential assignment)**  
(Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ then $\varphi \models P \iff [x' := f(x)]P$

**Lemma (Derivations)**  
(Equations of Differentials)

- $(e + k)' = (e)' + (k)'$
- $(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$
- $(c())' = 0$ for constants/numbers $c()$
- $(x)' = x'$ for variables $x \in \mathcal{V}$
**Lemma (Differential lemma)** (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \land Q$ for duration $r > 0$, then for all $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)$

**Lemma (Differential assignment)** (Effect on Differentials)

$DE \ [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q][x' := f(x)]P$

**Lemma (Derivations)** (Equations of Differentials)

$+$ $\ (e + k)' = (e)' + (k)'$

$\cdot$ $\ (e \cdot k)' = (e)' \cdot k + e \cdot (k)'$

$c'$ $\ (c())' = 0$

$x'$ $\ (x)' = x'$
Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

| +'       | (e + k)' = (e)' + (k)' |
| .'      | (e ⋅ k)' = (e)' ⋅ k + e ⋅ (k)' |
| c'      | (c())' = 0 |
| x'      | (x)' = x' |
### Lemma (Derivations)

\[ (e + k)' = (e)' + (k)' \]

### Proof.

\[ \omega[(e + k)'] = \]
Soundness: Proof of Derivations Lemma

**Lemma (Derivations) (Equations of Differentials)**

\[ (e + k)' = (e)' + (k)' \]

**Proof.**

\[ \omega[(e + k)'] = \sum_x \omega(x') \frac{\partial[e + k]}{\partial x}(\omega) \]
Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

\[ (e + k)' = (e)' + (k)' \]

Proof.

\[ \omega[(e + k)'] = \sum_x \omega(x') \frac{\partial [e + k]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial ([e] + [k])}{\partial x}(\omega) \]
### Soundness: Proof of Derivations Lemma

<table>
<thead>
<tr>
<th>Lemma (Derivations)</th>
<th>(Equations of Differentials)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+ \quad (e + k)' = (e)' + (k)'$</td>
<td></td>
</tr>
</tbody>
</table>

#### Proof.

\[
\omega[(e + k)'] = \sum_x \omega(x') \frac{\partial [e + k]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial ([e] + [k])}{\partial x}(\omega) \\
= \sum_x \omega(x') \left( \frac{\partial [e]}{\partial x}(\omega) + \frac{\partial [k]}{\partial x}(\omega) \right)
\]
Lemma (Derivations) (Equations of Differentials)

\[
\frac{d}{dx} (e + k) = \frac{d}{dx} e + \frac{d}{dx} k
\]

Proof.

\[
\omega[(e + k)'] = \sum_x \omega(x') \frac{\partial [e + k]}{\partial x} (\omega) = \sum_x \omega(x') \frac{\partial ([e] + [k])}{\partial x} (\omega)
\]

\[
= \sum_x \omega(x') \left( \frac{\partial [e]}{\partial x} (\omega) + \frac{\partial [k]}{\partial x} (\omega) \right)
\]

\[
= \sum_x \omega(x') \frac{\partial [e]}{\partial x} (\omega) + \sum_x \omega(x') \frac{\partial [k]}{\partial x} (\omega)
\]
Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

\[(e + k)' = (e)' + (k)'

Proof.

\[
\omega[(e + k)'] = \sum_x \omega(x') \frac{\partial [e + k]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial (\llbracket e \rrbracket + \llbracket k \rrbracket)}{\partial x}(\omega)
\]

\[
= \sum_x \omega(x') \left( \frac{\partial \llbracket e \rrbracket}{\partial x}(\omega) + \frac{\partial \llbracket k \rrbracket}{\partial x}(\omega) \right)
\]

\[
= \sum_x \omega(x') \frac{\partial \llbracket e \rrbracket}{\partial x}(\omega) + \sum_x \omega(x') \frac{\partial \llbracket k \rrbracket}{\partial x}(\omega)
\]

\[
= \omega[(e)'] + \omega[(k)']
\]
## Lemma (Derivations) (Equations of Differentials)

\[ (e + k)' = (e)' + (k)' \]

## Proof.

\[
\omega[(e + k)'] = \sum_x \omega(x') \frac{\partial [e + k]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial (e + k)}{\partial x}(\omega)
\]

\[
= \sum_x \omega(x') \left( \frac{\partial e}{\partial x}(\omega) + \frac{\partial k}{\partial x}(\omega) \right)
\]

\[
= \sum_x \omega(x') \frac{\partial e}{\partial x}(\omega) + \sum_x \omega(x') \frac{\partial k}{\partial x}(\omega)
\]

\[
= \omega((e)') + \omega((k)') = \omega((e)' + (k)')
\]
Soundness: Proof of Derivations Lemma

Lemma (Derivations) (Equations of Differentials)

\[ (e + k)' = (e)' + (k)' \]

Proof.

\[
\omega[(e + k)'] = \sum_x \omega(x') \frac{\partial [e + k]}{\partial x}(\omega) = \sum_x \omega(x') \frac{\partial (e + [k])}{\partial x}(\omega)
\]

\[
= \sum_x \omega(x') \left( \frac{\partial [e]}{\partial x}(\omega) + \frac{\partial [k]}{\partial x}(\omega) \right)
\]

\[
= \sum_x \omega(x') \frac{\partial [e]}{\partial x}(\omega) + \sum_x \omega(x') \frac{\partial [k]}{\partial x}(\omega)
\]

\[
= \omega[(e)'] + \omega[(k)'] = \omega[(e)' + (k)'] \quad \text{for all } \omega
\]
### Differential Substitution Lemmas \(\leadsto\) Proofs

#### Lemma (Differential lemma)  
**Differential value vs. Time-derivative**

If \( \varphi \models x' = f(x) \wedge Q \) for duration \( r > 0 \), then for all \( 0 \leq z \leq r \), \( FV(e) \subseteq \{x\} \):

\[
\varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)
\]

#### Lemma (Differential assignment)  
**Effect on Differentials**

\[
DE \ [x' = f(x) \& Q]P \iff [x' = f(x) \& Q][x' := f(x)]P
\]

#### Lemma (Derivations)  
**Equations of Differentials**

- \( +' \) \( (e + k)' = (e)' + (k)' \)
- \( ' \) \( (e \cdot k)' = (e)' \cdot k + e \cdot (k)' \)
- \( c' \) \( (c())' = 0 \)
- \( x' \) \( (x)' = x' \)
Differential equations cannot leave their domains.

\[ [x' = f(x) \& Q] = \{ (\varphi(0)|_{\{x'\}^C}, \varphi(r)) : \varphi \models x' = f(x) \land Q \}
\] for some \( \varphi : [0, r] \rightarrow \mathcal{I} \), some \( r \in \mathbb{R} \}

\[ \varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z) \]
Differential Weakening

\[ \text{DW} \quad [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q](Q \rightarrow P) \]

\[ [x' = f(x) \& Q] = \{ (\varphi(0)|_{x'}^C, \varphi(r)) : \varphi \models x' = f(x) \land Q \]

for some \( \varphi : [0, r] \rightarrow \mathcal{I} \), some \( r \in \mathbb{R} \}\}

\[ \varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z) \]

Differential equations cannot leave their domains.
Differential Weakening

\[ \text{DW } [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q](Q \rightarrow P) \]

Example (Bouncing ball)

\[ \text{DW } \vdash [x' = v, v' = -g \& x \geq 0] 0 \leq x \]

No need to solve any ODEs to prove that bouncing ball is above ground.
Differential Weakening

\[
\text{DW} \ [x' = f(x) \land Q] P \leftrightarrow [x' = f(x) \land Q](Q \rightarrow P)
\]

Example (Bouncing ball)

\[
\begin{align*}
G & \vdash \ [x' = v, \ v' = -g \land x \geq 0] (x \geq 0 \rightarrow 0 \leq x) \\
\text{DW} & \vdash \ [x' = v, \ v' = -g \land x \geq 0] 0 \leq x
\end{align*}
\]

No need to solve any ODEs to prove that bouncing ball is above ground.
Differential Weakening

\[ \text{DW} \ [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q](Q \rightarrow P) \]

**Example (Bouncing ball)**

\[ \begin{align*}
\text{R} & \vdash x \geq 0 \rightarrow 0 \leq x \\
\text{G} & \vdash [x' = v, v' = -g \& x \geq 0](x \geq 0 \rightarrow 0 \leq x) \\
\text{DW} & \vdash [x' = v, v' = -g \& x \geq 0]0 \leq x
\end{align*} \]

No need to solve any ODEs to prove that bouncing ball is above ground.
Differential Weakening

\[ \text{DW } [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q](Q \rightarrow P) \]

**Example (Bouncing ball)**

\[ * \]

\[ \text{R} \]

\[ \vdash x \geq 0 \rightarrow 0 \leq x \]

\[ \text{G} \]

\[ \vdash [x' = v, v' = -g \& x \geq 0](x \geq 0 \rightarrow 0 \leq x) \]

\[ \text{DW} \]

\[ \vdash [x' = v, v' = -g \& x \geq 0]0 \leq x \]

No need to solve any ODEs to prove that bouncing ball is above ground.
**Differential Weakening**

\[ dW \quad \Gamma \vdash [x' = f(x) \& Q]P, \Delta \]

**Differential Weakening**

\[ DW \quad [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q](Q \rightarrow P) \]

**Example (Bouncing ball)**

\[
\begin{align*}
\mathbb{R} & \vdash x \geq 0 \rightarrow 0 \leq x \\
G & \vdash [x' = v, v' = -g \& x \geq 0](x \geq 0 \rightarrow 0 \leq x) \\
DW & \vdash [x' = v, v' = -g \& x \geq 0]0 \leq x
\end{align*}
\]

No need to solve any ODEs to prove that bouncing ball is above ground.
Differential Weakening

\[ \text{dW} \quad \frac{Q \vdash P}{\Gamma \vdash [x' = f(x) \& Q]P, \Delta} \]

\[ \text{DW} \quad [x' = f(x) \& Q]P \iff [x' = f(x) \& Q](Q \rightarrow P) \]

Example (Bouncing ball)

\[
\begin{align*}
\begin{prooftree}
\text{R} \\
\Gamma \vdash x \geq 0 \rightarrow 0 \leq x \\
\Box \\
\end{prooftree}
\end{align*}
\]

\[
\begin{align*}
\begin{prooftree}
\text{G} \\
\Gamma \vdash [x' = v, v' = -g \& x \geq 0](x \geq 0 \rightarrow 0 \leq x) \\
\Box \\
\end{prooftree}
\end{align*}
\]

\[
\begin{align*}
\begin{prooftree}
\text{DW} \\
\Gamma \vdash [x' = v, v' = -g \& x \geq 0]0 \leq x \\
\end{prooftree}
\end{align*}
\]

No need to solve any ODEs to prove that bouncing ball is above ground.
Differential Invariant Terms for Differential Equations

Differential Invariant

\[
\begin{align*}
dl & \quad \vdash [x' := f(x)](e)' = 0 \\
e = 0 & \vdash [x' = f(x) \& Q]e = 0
\end{align*}
\]

\[
\begin{align*}
\text{DI} & \quad ([x' = f(x)] e = 0 \iff e = 0) \iff [x' = f(x)](e)' = 0 \\
\text{DE} & \quad [x' = f(x) \& Q]P \iff [x' = f(x) \& Q][x' := f(x)]P \\
\text{DW} & \quad [x' = f(x) \& Q]P \iff [x' = f(x) \& Q](Q \rightarrow P)
\end{align*}
\]
Differential Invariant Terms for Differential Equations

\[ Q \vdash [x' := f(x)](e)' = 0 \]
\[ e = 0 \vdash [x' = f(x) \& Q]e = 0 \]

\[ \text{DI} \quad ([x' = f(x) \& Q]e = 0 \iff [?Q]e = 0) \iff [x' = f(x) \& Q](e)' = 0 \]

\[ \text{DE} \quad [x' = f(x) \& Q]P \iff [x' = f(x) \& Q][x' := f(x)]P \]

\[ \text{DW} \quad [x' = f(x) \& Q]P \iff [x' = f(x) \& Q](Q \rightarrow P) \]
Differential Invariant Terms for Differential Equations

**Differential Invariant**

\[
\frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) & Q]e = 0}
\]

\[
\text{DI} \quad ([x' = f(x) & Q]e = 0 \iff [?Q]e = 0) \iff [x' = f(x) & Q](e)' = 0
\]

\[
\text{DE} \quad [x' = f(x) & Q]P \leftrightarrow [x' = f(x) & Q][x' := f(x)]P
\]

\[
\text{DW} \quad [x' = f(x) & Q]P \leftrightarrow [x' = f(x) & Q](Q \rightarrow P)
\]

**Proof (dl is a derived rule).**

\[
\frac{e = 0 \vdash [x' = f(x) & Q]e = 0}{Dl}
\]

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Differential Invariant Terms for Differential Equations

**Differential Invariant**

\[
Q \vdash [x' := f(x)](e)' = 0
\]
\[
e = 0 \vdash [x' = f(x) & Q]e = 0
\]

\[
\text{DI} \quad ([x' = f(x) & Q]e = 0 \leftrightarrow [\neg Q]e = 0) \leftrightarrow [x' = f(x) & Q](e)' = 0
\]

\[
\text{DE} \quad [x' = f(x) & Q]P \leftrightarrow [x' = f(x) & Q][x' := f(x)]P
\]

\[
\text{DW} \quad [x' = f(x) & Q]P \leftrightarrow [x' = f(x) & Q](Q \rightarrow P)
\]

**Proof (di is a derived rule).**

\[
\text{DE} \quad \vdash [x' = f(x) & Q](e)' = 0
\]
\[
\text{DI} \quad e = 0 \vdash [x' = f(x) & Q]e = 0
\]
Differential Invariant Terms for Differential Equations

**Differential Invariant**

\[
\frac{Q \vdash [x' := f(x)](e)' = 0}{\therefore e = 0 \vdash [x' = f(x) \& Q]e = 0}
\]

\[
\frac{\text{DI}}{([x' = f(x) \& Q]e = 0 \iff [\neg Q]e = 0) \iff [x' = f(x) \& Q](e)' = 0}
\]

\[
\frac{\text{DE}}{[x' = f(x) \& Q]P \iff [x' = f(x) \& Q][x' := f(x)]P}
\]

\[
\frac{\text{DW}}{[x' = f(x) \& Q]P \iff [x' = f(x) \& Q](Q \rightarrow P)}
\]

**Proof (dl is a derived rule).**

\[
\frac{\text{DW}}{\vdash [x' = f(x) \& Q][x' := f(x)](e)' = 0}
\]

\[
\frac{\text{DE}}{\vdash [x' = f(x) \& Q](e)' = 0}
\]

\[
\frac{\text{DI}}{e = 0 \vdash [x' = f(x) \& Q]e = 0}
\]
Differential Invariant Terms for Differential Equations

**Differential Invariant**

\[
\frac{Q \vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x) \& Q]e = 0}
\]

\[
\text{DI} \quad ([x' = f(x) \& Q]e = 0 \iff [?Q]e = 0) \iff [x' = f(x) \& Q](e)' = 0
\]

\[
\text{DE} \quad [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q][x' := f(x)]P
\]

\[
\text{DW} \quad [x' = f(x) \& Q]P \leftrightarrow [x' = f(x) \& Q](Q \rightarrow P)
\]

**Proof (dl is a derived rule).**

\[
\frac{G, \rightarrow R}{\vdash [x' = f(x) \& Q](Q \rightarrow [x' := f(x)](e)' = 0)}
\]

\[
\frac{\text{DW}}{\vdash [x' = f(x) \& Q][x' := f(x)](e)' = 0}
\]

\[
\frac{\text{DE}}{\vdash [x' = f(x) \& Q](e)' = 0}
\]

\[
\frac{\text{DI}}{e = 0 \vdash [x' = f(x) \& Q]e = 0}
\]
Differential Invariant Terms for Differential Equations

Differential Invariant
dl \[
\begin{align*}
Q \vdash [x' := f(x)](e)' &= 0 \\
e &= 0 \vdash [x' = f(x) & Q]e = 0
\end{align*}
\]

\[
\text{DL} \quad ([x' = f(x) & Q]e = 0 \iff [?Q]e = 0) \iff [x' = f(x) & Q](e)' = 0
\]

\[
\text{DE} \quad [x' = f(x) & Q]P \iff [x' = f(x) & Q][x' := f(x)]P
\]

\[
\text{DW} \quad [x' = f(x) & Q]P \iff [x' = f(x) & Q](Q \rightarrow P)
\]

Proof (dl is a derived rule).

\[
\begin{align*}
Q \vdash [x' := f(x)](e)' &= 0 \\
G, \rightarrow R \quad \vdash [x' = f(x) & Q](Q \rightarrow [x' := f(x)](e)' = 0) \\
\quad \vdash [x' = f(x) & Q][x' := f(x)](e)' = 0 \\
\quad \vdash [x' = f(x) & Q](e)' = 0 \\
e &= 0 \vdash [x' = f(x) & Q]e = 0
\end{align*}
\]

\[
\text{G} \quad \frac{P}{[\alpha]P}
\]
Lemma (Differential lemma) \[ (\text{Differential value vs. Time-derivative}) \]

\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \quad \Rightarrow \quad \forall 0 \leq z \leq r \quad \varphi(z)\[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

Differential Invariant

\[ dI \quad \frac{d}{dt} [x' = f(x)]e = k \]
Lemma (Differential lemma) (Differential value vs. Time-derivative)

\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

**Differential Invariant**

\[ \frac{\vdash \left[ x' := f(x) \right] (e)' = (k)'}{e = k \vdash \left[ x' := f(x) \right] e = k} \]

**Differential Invariant**

\[ (\left[ x' = f(x) \right] e = k \leftrightarrow e = k) \leftrightarrow \left[ x' = f(x) \right] (e)' = (k)' \]
Differential Invariant Equations

**Lemma (Differential lemma) (Differential value vs. Time-derivative)**

\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

**Differential Invariant**

\[ \vdash \begin{array}{c} [x' := f(x)](e)' = (k)' \\ e = k \vdash [x' = f(x)]e = k \end{array} \]

\[ \text{DI } ([x' = f(x)]e = k \leftrightarrow e = k) \leftarrow [x' = f(x)](e)' = (k)' \]

**Proof (\(\equiv\) rate of change from \(\equiv\) initial value. Mean-value theorem).**

\[ \frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] = \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z) \]
Differential Invariant Inequalities

**Lemma (Differential lemma)**

\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \quad \Rightarrow \quad \forall 0 \leq z \leq r \quad \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

**Differential Invariant**

\[ \vdash \frac{d}{dt} [x' := f(x)](e)' \geq (k)' \]

\[ e \geq k \vdash [x' = f(x)]e \geq k \]

**Differential Invariant (DI)**

\( [x' = f(x)] e \geq k \iff e \geq k \iff [x' = f(x)](e)' \geq (k)' \)

**Proof (≥ rate of change from ≥ initial value. Mean-value theorem).**

\[ \frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] \geq \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z) \]
Lemma (Differential lemma)  
\[
\varphi \models x' = f(x) \land Q \text{ for } r > 0 \implies \forall 0 \leq z \leq r \quad \varphi(z)[(e)'] = \frac{\frac{d\varphi(t)[e]}{dt}(z)}{dt}  
\]

(Differential value vs. Time-derivative)

Differential Invariant  
\[
\frac{d[I]}{dt} \models [x' := f(x)](e)' \leq (k)'  
\]

\[
e \leq k \models [x' = f(x)]e \leq k  
\]

\[
\text{DI} \quad ([x' = f(x)]e \leq k \iff e \leq k) \iff [x' = f(x)](e)' \leq (k)'  
\]

Proof (\leq \text{ rate of change from } \leq \text{ initial value. Mean-value theorem}).

\[
\frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] \leq \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z)  
\]
Lemma (Differential lemma) (Differential value vs. Time-derivative)

\[ \varphi |\Rightarrow x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

Differential Invariant

\[ \vdash [x' := f(x)](e)' > (k)' \]

\[ e > k \vdash [x' = f(x)]e > k \]

\[ dI ([x' = f(x)]e > k \iff e > k) \iff [x' = f(x)](e)' > (k)' \]

Proof (> rate of change from > initial value. Mean-value theorem).

\[ \frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] > \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z) \]
Lemma (Differential lemma) (Differential value vs. Time-derivative)

\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

Differential Invariant

\[ \vdash [x' := f(x)](e)' \geq (k)' \]

\[ e > k \vdash [x' = f(x)]e > k \]

DI \ (\ [x' = f(x)]e > k \iff e > k) \leftarrow [x' = f(x)](e)' \geq (k)'

Proof (≥ rate of change from > initial value. Mean-value theorem).

\[ \frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] \geq \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z) \]

André Platzer (CMU)
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

Differential Invariant

\[ \vdash [x' := f(x)][e]', \neq (k)' \]

\[ e \neq k \vdash [x' = f(x)] e \neq k \]

DI \quad (\quad [x' = f(x)] e \neq k \iff e \neq k) \quad \leftarrow [x' = f(x)][e]' \neq (k)'

Proof (\neq \text{ rate of change from } \neq \text{ initial value. Mean-value theorem}).

\[ \frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] \neq \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z) \]
### Differential Invariant Inequalities

**Lemma (Differential lemma) (Differential value vs. Time-derivative):**

\[
\varphi \models x' = f(x) \land Q \text{ for } r > 0 \implies \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z)
\]

#### Differential Invariant

\[
\vdash [x' := f(x)](e)' \neq (k)'
\]

\[
e \neq k \vdash [x' = f(x)] e \neq k
\]

**DI** (\([x' = f(x)] e \neq k \iff e \neq k\) \leftrightarrow [x' = f(x)](e)' \neq (k)')

**Proof (\neq rate of change from \neq initial value. Mean-value theorem).**

\[
\frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] \neq \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z)
\]
Lemma (Differential lemma) (Differential value vs. Time-derivative)

\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

Differential Invariant

\[ \vdash [x' := f(x)](e)' \neq (k)' \quad e \neq k \vdash [x' = f(x)]e \neq k \]

DI (\[x' = f(x)\] e \neq k \iff e \neq k) \iff [x' = f(x)](e)' \neq (k)'

Proof (\neq \text{ rate of change from } \neq \text{ initial value. Mean-value theorem})

\[ \frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] \neq \varphi(z)[(k)'] = \frac{d\varphi(t)[k]}{dt}(z) \]
Differential Invariant Inequalities

Lemma (Differential lemma) (Differential value vs. Time-derivative)

\[ \varphi |\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!}\]
Example: Differential Invariant Inequalities

\[ \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y & \omega \geq 0 \wedge d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \]
$$\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \& \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2$$
Example: Differential Invariant Inequalities: Oscillator

\[
\omega \geq 0 \land d \geq 0 \vdash [x := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 xx' + 2yy' \leq 0
\]

\[
\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2
\]

damped oscillator
Example: Differential Invariant Inequalities: Oscillator

\[\omega \geq 0 \land d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0\]

\[\omega \geq 0 \land d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 xx' + 2yy' \leq 0\]

\[\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2\]
Example: Differential Invariant Inequalities: Oscillator

\[
\omega \geq 0 \land d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0
\]

\[
\omega \geq 0 \land d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 xx' + 2yy' \leq 0
\]

\[
\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2
\]

damped oscillator
Example: Differential Invariant Inequalities: Oscillator

\[ \omega \geq 0 \land d \geq 0 \vdash 2 \omega^2 xy + 2y(-\omega^2 x - 2d \omega y) \leq 0 \]

\[ \omega \geq 0 \land d \geq 0 \vdash \{x' := y, y' := -\omega^2 x - 2d \omega y\} 2 \omega^2 xx' + 2yy' \leq 0 \]

\[ \omega^2 x^2 + y^2 \leq c^2 \vdash \{x = y, y' = -\omega^2 x - 2d \omega y & \omega \geq 0 \land d \geq 0\} \omega^2 x^2 + y^2 \leq c^2 \]

damped oscillator
Differential Invariant Conjunctions

Differential Invariant

\[
\text{dl} \quad A \land B \vdash [x' = f(x)](A \land B)
\]
Differential Invariant Conjunctions

\[
\text{Differential Invariant}
\]

\[
\frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \land B \vdash [x' = f(x)](A \land B)}
\]

\[
\text{DI} \quad ([x' = f(x)](A \land B) \leftrightarrow (A \land B)) \leftrightarrow [x' = f(x)]((A)' \land (B)')
\]

\[
\text{dist}(x, v) \land \text{slow}(v)
\]
Differential Invariant Conjunctions

\[ \vdash [x' := f(x)]((A)' \land (B)') \]

\[ A \land B \vdash [x' = f(x)](A \land B) \]

\( \text{DI} \quad ([x' = f(x)](A \land B) \leftrightarrow (A \land B)) \leftrightarrow [x' = f(x)]((A)' \land (B)') \)

Proof (separately).

\[ \vdash [x' = f(x)](A)' \]

\[ A \vdash [x' = f(x)]A \]

\[ A \land B \vdash [x' = f(x)](A \land B) \]

\[ \vdash [x' = f(x)](B)' \]

\[ B \vdash [x' = f(x)]B \]

\[ A \land B \vdash [x' = f(x)](A \land B) \]

\( [] \land [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q \)
Quantum’s Back for a Differential Invariant Proof

\[ 2gx = 2gH - v^2 \vdash [x'' = -g \& x \geq 0](2gx = 2gH - v^2 \wedge x \geq 0) \]

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.
Quantum’s Back for a Differential Invariant Proof

\[ [] \land [\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q \]

\[
\begin{align*}
2gx = 2gH - v^2 \vdash [x'' = -g \land x \geq 0] & 2gx = 2gH - v^2 \\
2gx = 2gH - v^2 \vdash [x'' = -g \land x \geq 0] & (2gx = 2gH - v^2 \land x \geq 0)
\end{align*}
\]

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.
Quantum’s Back for a Differential Invariant Proof

\[
\begin{align*}
\text{No solutions but still a proof.} \quad & \text{Simple proof with simple arithmetic.} \\
\text{Independent proofs for independent questions.}
\end{align*}
\]
Quantum’s Back for a Differential Invariant Proof

\[
\begin{align*}
\text{dI} & \quad \vdash x \geq 0 \vdash 2gv = -2v(-g) \\
\end{align*}
\]

\[
\begin{align*}
x \geq 0 \vdash [x' := v][v' := -g] & \quad 2gx' = -2vv' \\
\end{align*}
\]

\[
\begin{align*}
2gx = 2gH - v^2 \vdash [x'' = -g \& x \geq 0] & \quad 2gx = 2gH - v^2 \\
\end{align*}
\]

\[
\begin{align*}
\vdash [x'' = -g \& x \geq 0] & \quad x \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\vdash [x'' = -g \& x \geq 0] & \quad (2gx = 2gH - v^2 \& x \geq 0) \\
\end{align*}
\]

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.
Quantum’s Back for a Differential Invariant Proof

\[
\begin{align*}
\mathbb{R} & \quad \forall x \geq 0 \vdash 2gv = -2v(-g) \\
[&:=] \quad x \geq 0 \vdash [x' := v][v' := -g] 2gx' = -2vv' \\
\vdash 2gx = 2gH - v^2 \vdash [x'' = -g & x \geq 0] 2gx = 2gH - v^2 & \vdash [x'' = -g & x \geq 0] x \geq 0 \\
\vdash 2gx = 2gH - v^2 \vdash [x'' = -g & x \geq 0] (2gx = 2gH - v^2 \land x \geq 0)
\end{align*}
\]

No solutions but still a proof.

Simple proof with simple arithmetic.

Independent proofs for independent questions.
Quantum’s Back for a Differential Invariant Proof

\[
\begin{align*}
\mathbb{R} & \vdash x \geq 0 \implies 2gv = -2v(-g) \\
\vdash x \geq 0 \implies [x' := v][v' := -g] 2gx' = -2vv' \\
2gx = 2gH - v^2 & \vdash [x'' = -g & x \geq 0] 2gx = 2gH - v^2 \\
2gx = 2gH - v^2 & \vdash [x'' = -g & x \geq 0] (2gx = 2gH - v^2 \land x \geq 0)
\end{align*}
\]

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.
Quantum’s Back for a Differential Invariant Proof

\[
\begin{align*}
\mathbb{R} & \quad \vdash x \geq 0 \implies 2gv = -2v(-g) \\
[\vdash x : = v][v' := -g] & \quad 2gx' = -2vv' \\
2gx = 2gH - v^2 & \quad \vdash [x'' = -g \land x \geq 0]2gx = 2gH - v^2 \\
2gx = 2gH - v^2 & \quad \vdash [x'' = -g \land x \geq 0](2gx = 2gH - v^2 \land x \geq 0)
\end{align*}
\]

No solutions but still a proof.
Simple proof with simple arithmetic.
Independent proofs for independent questions.
Differential Invariant Conjunctions

\[ \textbf{Differential Invariant} \]

\[ \vdash [x' := f(x)]((A)' \land (B)') \]

\[ A \land B \vdash [x' = f(x)](A \land B) \]

\[ \textbf{DI} \quad ([x' = f(x)](A \land B) \leftrightarrow (A \land B)) \leftarrow [x' = f(x)]((A)' \land (B)') \]

\[ \text{dist}(x, v) \land \text{slow}(v) \]
Differential Invariant Disjunctions

**Differential Invariant**

\[
\text{dl} \quad \frac{\vdash [x' := f(x)]((A)' \lor (B)')}{A \lor B \vdash [x' = f(x)](A \lor B)}
\]

\[
\text{DI} \quad ([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)) \leftarrow [x' = f(x)]((A)' \lor (B)')
\]
Differential Invariant Disjunctions

\[
\frac{\vdash [x' := f(x)]((A)' \lor (B)')}{A \lor B \vdash [x' := f(x)](A' \lor B)}
\]

\[
\text{DI} \quad ([x' = f(x)](A \lor B) \iff (A \lor B)) \quad \iff \quad [x' = f(x)]((A)' \lor (B)')
\]

\[\text{dist}(x, v) \lor \text{slow}(v)\]
Differential Invariant Disjunctions

**Differential Invariant**

\[
\text{dl} \quad \frac{\vdash [x' := f(x)]((A)' \land (B)')}{A \lor B \vdash [x = f(x)](A \lor B)}
\]

\[
\text{DI} \quad ([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)) \leftrightarrow [x' = f(x)]((A)' \land (B)')
\]

\[
\text{dist}(x, v) \lor \text{slow}(v)
\]
Differential Invariant Disjunctions

Differential Invariant

\[ dl \vdash [x' := f(x)]((A)' \land (B)') \]

\[ A \lor B \vdash [x' = f(x)](A \lor B) \]

\[ DI \quad ([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)) \leftrightarrow [x' = f(x)]((A)' \land (B)') \]

Proof (separately).

\[ \begin{array}{c}
\text{*}
\hline
A \vdash A \lor B \\
\text{MR}
\hline
A \vdash [x' = f(x)]A
\end{array} \quad \begin{array}{c}
\text{DI}
\hline
A \vdash [x' = f(x)](A \lor B)
\end{array} \quad \begin{array}{c}
\text{*}
\hline
B \vdash A \lor B \\
\text{MR}
\hline
B \vdash [x' = f(x)]B
\end{array} \quad \begin{array}{c}
\text{DI}
\hline
B \vdash [x' = f(x)](A \lor B)
\end{array} \quad \begin{array}{c}
\text{∨L}
\hline
A \lor B \vdash [x' = f(x)](A \lor B)
\end{array} \]

\[ dist(x, v) \lor \text{slow}(v) \]
Differential Invariant Disjunctions

Differential Invariant

\[\vdash [x' := f(x)]((A)' \land (B)')\]

\[A \lor B \vdash [x' = f(x)](A \lor B)\]

\[
\text{DI} \quad ([x' = f(x)](A \lor B) \leftrightarrow (A \lor B)) \leftarrow [x' = f(x)]((A)' \land (B)')
\]

Proof (separately).

\[
\begin{array}{c}
\ast
\hline
A \vdash A \lor B & A \vdash [x' = f(x)]A \\
\hline
\text{MR} & \text{DI} \\
\hline
A \vdash [x' = f(x)](A \lor B)
\end{array}
\]

\[
\begin{array}{c}
\ast
\hline
B \vdash A \lor B & B \vdash [x' = f(x)]B \\
\hline
\text{MR} & \text{DI} \\
\hline
B \vdash [x' = f(x)](A \lor B)
\end{array}
\]

\[
A \lor B \vdash [x' = f(x)](A \lor B)
\]

\[
[\alpha](P \land Q) \leftrightarrow [\alpha]P \land [\alpha]Q
\]
Assuming Invariants

\[ Q \rightarrow [x' := f(x)](F)' \]

\[ F \vdash [x' = f(x) & Q]F \]

\[ F \land Q \rightarrow [x' := f(x)](F)' \]

\[ F \vdash [x' = f(x) & Q]F \]

Loop

\[ F \vdash [\alpha]F \]

\[ F \vdash [\alpha^*]F \]
Assuming Invariants

\[ \neg F, \neg F \]

\[ Q \rightarrow [x' := f(x)](F)' \]

\[ F \vdash [x' = f(x) \& Q]F \]

\[ F \land Q \rightarrow [x' := f(x)](F)' \]

\[ F \vdash [x' = f(x) \& Q]F \]

Example (Restrictions)

\[ v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v]v^2 - 2v + 1 = 0 \]
Assuming Invariants

\[ Q \rightarrow [x' := f(x)](F)' \]
\[ F \vdash [x' = f(x) & Q]F \]

Example (Restrictions)

\[ v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v] 2vv' - 2v' = 0 \]
\[ v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v] v^2 - 2v + 1 = 0 \]
Assuming Invariants

\[ Q \rightarrow [x' := f(x)](F)' \]
\[ F \models [x' = f(x) & Q]F \]

\[ F \land Q \rightarrow [x' := f(x)](F)' \]
\[ F \models [x' = f(x) & Q]F \]

Example (Restrictions)

\[ v^2 - 2v + 1 = 0 \models 2vw - 2w = 0 \]
\[ v^2 - 2v + 1 = 0 \models [v' := w][w' := -v]2vv' - 2v' = 0 \]
\[ v^2 - 2v + 1 = 0 \models [v' = w, w' = -v]v^2 - 2v + 1 = 0 \]
Assuming Invariants

\[ Q \rightarrow [x' := f(x)](F)' \]
\[ F \vdash [x' = f(x) \land Q]F \]

Example (Restrictions)

\[ v^2 - 2v + 1 = 0 \vdash 2vw - 2w = 0 \]
\[ v^2 - 2v + 1 = 0 \vdash [v' := w][w' := -v] 2vv' - 2v' = 0 \]
\[ v^2 - 2v + 1 = 0 \vdash [v' = w, w' = -v] v^2 - 2v + 1 = 0 \]
Assuming Invariants

\[
Q \rightarrow [x' := f(x)](F)'
\]

\[
\frac{F \vdash [x' = f(x) \& Q]F}{F \vdash [x' = f(x) \& Q]F}
\]

Example (Restrictions are unsound!)

\[
\begin{align*}
\nu^2 - 2\nu + 1 &= 0 \\
\nu^2 - 2\nu + 1 &= 0 \\
\nu^2 - 2\nu + 1 &= 0
\end{align*}
\]

\[
\begin{align*}
\nu^2 - 2\nu + 1 &= 0 \\
\nu^2 - 2\nu + 1 &= 0 \\
\nu^2 - 2\nu + 1 &= 0
\end{align*}
\]

(unsound)

\[
\begin{align*}
\nu^2 - 2\nu + 1 &= 0 \vdash 2\nu w - 2w = 0 \\
\nu^2 - 2\nu + 1 &= 0 \vdash [v' = w][w' = -\nu] 2\nu v' - 2\nu' = 0 \\
\nu^2 - 2\nu + 1 &= 0 \vdash [v' = w, w' = -\nu] \nu^2 - 2\nu + 1 = 0
\end{align*}
\]
Outline

1 Learning Objectives

2 Differential Invariants
   - Recap: Ingredients for Differential Equation Proofs
   - Soundness: Derivations Lemma
   - Differential Weakening
   - Equational Differential Invariants
   - Differential Invariant Inequalities
   - Disequational Differential Invariants
   - Example Proof: Damped Oscillator
   - Conjunctive Differential Invariants
   - Disjunctive Differential Invariants
   - Assuming Invariants

3 Differential Cuts

4 Soundness

5 Summary
Proof (Soundness).
Let \( \phi |\|= x' = f(x) \land Q \) starting in \( \omega \in \[[F]\] \).
Thus, \( \phi |\|= x' = f(x) \land Q \land C \).
Thus, \( \phi (r) \in \[[F]\] \) by second premise.
Proof (Soundness).

Let $\phi \models x' = f(x) \land Q$ starting in $\omega \in \{ \begin{array}{} F \end{array} \}$. Thus, $\phi \models x' = f(x) \land Q \land C$. Thus, $\phi(r) \models \{ \begin{array}{} F \end{array} \}$ by second premise.
Differential Cuts

Differential Cut

\[ F \vdash [x' = f(x)]C \quad F \vdash [x' = f(x) \& C]F \]

\[ F \vdash [x' = f(x)]F \]
Differential Cuts

Differential Cut

\[
F \vdash [x' = f(x) \& Q] \quad C \quad F \vdash [x' = f(x) \& Q \land C] \quad F
\]

\[
F \vdash [x' = f(x) \& Q] \quad F
\]

Proof (Soundness).
Let \( ϕ \models ω \in [\lceil F \rceil] \).
Thus, \( ϕ \models [x' = f(x) \& Q] \land C \).
Thus, \( ϕ(r) \in [\lceil F \rceil] \) by second premise.
Proof (Soundness).

Let \( \phi \models x' = f(x) \land Q \) starting in \( \omega \in \mathbb{F} \).

Thus, \( \phi \models x' = f(x) \land Q \land C \).

Thus, \( \phi(r) \models [x' = f(x) \land Q]F \) by second premise.
Proof (Soundness).

Let $\phi \models [x' = f(x) \land Q]$ starting in $\omega \in \blacksquare F$. Thus, $\phi \models [x' = f(x) \land Q \land C]$. Thus, $\phi(r) \in \blacksquare F$ by second premise.
Differential Cut

\[ F \vdash [x' = f(x) & Q] \quad F \vdash [x' = f(x) & Q \land C] \quad F \vdash [x' = f(x) & Q] \]

Proof (Soundness).

Let \( \phi \models x' = f(x) \land Q \) starting in \( \omega \in \text{[[F]]} \).

Thus, \( \phi \models x' = f(x) \land Q \land C \).

Thus, \( \phi(\rho) \in \text{[[F]]} \) by second premise.
Proof (Soundness).

Let $\phi \models x' = f(x) \land Q$ starting in $\omega \in \llbracket F \rrbracket$.

Thus, $\phi \models x' = f(x) \land Q \land C$.

Thus, $\phi(r) \in \llbracket F \rrbracket$ by second premise.
Differential Cut

\[ F \vdash [x' = f(x) & Q] C \]

\[ F \vdash [x' = f(x) & Q \wedge C] F \]

\[ F \vdash [x' = f(x) & Q] F \]

Proof (Soundness).

Let \( \varphi \models x' = f(x) \wedge Q \) starting in \( \omega \in \llbracket F \rrbracket \).

\( \omega \in \llbracket [x' = f(x) & Q] C \rrbracket \) by left premise.

Thus, \( \varphi \models x' = f(x) \wedge Q \wedge C \).

Thus, \( \varphi(r) \in \llbracket F \rrbracket \) by second premise.

\( \Box \)
Differential Cut Example: Increasingly Damped Oscillator

$$\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 & \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2$$
Increasingly Damped Oscillator

\[ \omega^2 x^2 + y^2 \leq c^2 \models [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2 \]
Increasingly damped oscillator

\[
\begin{align*}
\text{dl} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 & \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \\
\text{dC} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 & \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2
\end{align*}
\]
Differential Cut Example: Increasingly Damped Oscillator

\[
\begin{align*}
\text{dl} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0 \& d \geq 0] \quad \omega^2 x^2 + y^2 \leq c^2 \\
\text{dC} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0] \quad \omega^2 x^2 + y^2 \leq c^2
\end{align*}
\]

dl \quad d \geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0] \quad d \geq 0

increasingly damped oscillator
Differential Cut Example: Increasingly Damped Oscillator

\[\begin{align*}
\text{dl} & : \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \\
\text{dC} & : \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2
\end{align*}\]

\[\begin{align*}
[:=] & : \quad \omega \geq 0 \vdash [d' := 7] d' \geq 0 \\
\text{dl} & : \quad d \geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0] d \geq 0
\end{align*}\]

increasingly damped oscillator
Differential Cut Example: Increasingly Damped Oscillator

\[\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0 \& d \geq 0] \omega^2 x^2 + y^2 \leq c^2\]

\[\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2\]

\[\begin{align*}
\mathbb{R} & \quad \omega \geq 0 \vdash 7 \geq 0 \\
,:=] & \quad \omega \geq 0 \vdash [d' := 7] d' \geq 0 \\
dl & \quad d \geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \& \omega \geq 0] d \geq 0
\end{align*}\]

increasingly damped oscillator
Increasingly Damped Oscillator

\[
\begin{align*}
\omega \geq 0 \land d \geq 0 & \vdash [x' := y][y' := -\omega^2 x - 2d \omega y] \omega^2 x x' + 2 y y' \leq 0 \\
\omega^2 x^2 + y^2 \leq c^2 & \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \\
\omega^2 x^2 + y^2 \leq c^2 & \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2
\end{align*}
\]

\[ * \]

\[
\begin{align*}
\mathbb{R} \quad \omega \geq 0 & \vdash 7 \geq 0 \\
[\vdash] \quad \omega \geq 0 & \vdash [d' := 7] d' \geq 0 \\
dl \quad d \geq 0 & \vdash [x' = y, y' = -\omega^2 x - 2d \omega y, d' = 7 \land \omega \geq 0] d \geq 0
\end{align*}
\]

increasingly damped oscillator
Differential Cut Example: Increasingly Damped Oscillator

\[
\begin{align*}
\mathbb{R} & \quad \omega \geq 0 \land d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0 \\
\vdash & \quad \omega \geq 0 \land d \geq 0 \vdash \left[ x' := y \right] \left[ y' := -\omega^2 x - 2d\omega y \right] 2\omega^2 xx' + 2yy' \leq 0 \\
\text{dl} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash \left[ x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \& \omega \geq 0 \land d \geq 0 \right] \omega^2 x^2 + y^2 \leq c^2 \\
\text{dC} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash \left[ x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \& \omega \geq 0 \right] \omega^2 x^2 + y^2 \leq c^2
\end{align*}
\]

\[
\begin{align*}
\vdash & \quad \omega \geq 0 \vdash 7 \geq 0 \\
\vdash & \quad \omega \geq 0 \vdash \left[ d' := 7 \right] d' \geq 0 \\
\text{dl} & \quad d \geq 0 \vdash \left[ x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \& \omega \geq 0 \right] d \geq 0
\end{align*}
\]

increasingly damped oscillator
Differential Cut Example: Increasingly Damped Oscillator

\[\begin{align*}
\mathbb{R} & \quad \omega \geq 0 \land d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) \leq 0 \\
[::] & \quad \omega \geq 0 \land d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 xx' + 2yy' \leq 0 \\
\text{dl} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 & \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \\
\text{dC} & \quad \omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 & \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2
\end{align*}\]

\[\begin{align*}
\mathbb{R} & \quad \omega \geq 0 \vdash 7 \geq 0 \\
[::] & \quad \omega \geq 0 \vdash [d' := 7] d' \geq 0 \\
\text{dl} & \quad d \geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 & \omega \geq 0] d \geq 0
\end{align*}\]

increasingly damped oscillator
Differential Cut Example: Increasingly Damped Oscillator

\[
\begin{align*}
\omega \geq 0 \land d \geq 0 \vdash 2\omega^2 xy + 2y(-\omega^2 x - 2d\omega y) & \leq 0 \\
\omega \geq 0 \land d \geq 0 \vdash [x' := y][y' := -\omega^2 x - 2d\omega y] 2\omega^2 xx' + 2yy' & \leq 0 \\
\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \land \omega \geq 0 \land d \geq 0] \omega^2 x^2 + y^2 \leq c^2 \\
\omega^2 x^2 + y^2 \leq c^2 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \land \omega \geq 0] \omega^2 x^2 + y^2 \leq c^2
\end{align*}
\]

\[
\begin{align*}
\omega \geq 0 \vdash 7 \geq 0 \\
\omega \geq 0 \vdash [d' := 7] d' \geq 0 \\
d \geq 0 \vdash [x' = y, y' = -\omega^2 x - 2d\omega y, d' = 7 \land \omega \geq 0] d \geq 0
\end{align*}
\]
Could repeatedly diffcut in formulas to help the proof
Ex: Differential Cuts

\[\begin{align*}
\text{dC} & \quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1
\end{align*}\]
\[ \begin{align*}
\text{dC} &: \quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1 \\
\text{dI} &: \quad y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] y^5 \geq 0
\end{align*} \]
\[
\begin{align*}
\text{Ex: Differential Cuts} \\
\vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1 \\
\vdash [x' := (x - 2)^4 + y^5] [y' := y^2] 5y^4 y' \geq 0 \\
\vdash y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] y^5 \geq 0
\end{align*}
\]
Differential Cuts

\[
\begin{align*}
\text{dC} & \quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2]x^3 \geq -1 \\
\text{R} & \quad \vdash 5y^4y^2 \geq 0 \\
[=:] & \quad \vdash [x' := (x - 2)^4 + y^5][y' := y^2]5y^4y' \geq 0 \\
\text{dl} & \quad y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{align*}
\]
Differential Cuts

\[ x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2]x^3 \geq -1 \]

\[ \begin{array}{c}
\top && \star \vDash 5y^4y^2 \geq 0 \\
\top & [\vdash \vdash] & \vdash [x' := (x - 2)^4 + y^5][y' := y^2]5y^4y' \geq 0 \\
\top & \top & \vdash [x' = (x - 2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array} \]
Differential Equations & Proofs

Ex: Differential Cuts

\[ \begin{align*}
\text{dl} & \quad x^3 \geq -1 \vdash [x' = (x - 2)^4 + y^5, y' = y^2 & \land y^5 \geq 0] x^3 \geq -1 \\
\text{dC} & \quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1 \\
\end{align*} \]

\[ \begin{align*}
\therefore & \quad 5y^4 y^2 \geq 0 \\
\therefore & \quad [x' := (x - 2)^4 + y^5] [y' := y^2] 5y^4 y' \geq 0 \\
\therefore & \quad y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] y^5 \geq 0
\end{align*} \]
\[
\begin{align*}
\text{[:=]} & \quad y^5 \geq 0 \vdash [x' := (x - 2)^4 + y^5][y' := y^2] 3x^2 x' \geq 0 \\
\text{dl} & \quad x^3 \geq -1 \vdash [x' = (x - 2)^4 + y^5, y' = y^2 \& y^5 \geq 0] x^3 \geq -1 \\
\text{dC} & \quad x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] x^3 \geq -1
\end{align*}
\]

\[
\begin{align*}
\text{[:=]} & \quad \vdash [x' := (x - 2)^4 + y^5][y' := y^2] 5y^4 y' \geq 0 \\
\text{dl} & \quad y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2] y^5 \geq 0
\end{align*}
\]
\( \mathbb{R} \)

\[ y^5 \geq 0 \vdash 3x^2((x - 2)^4 + y^5) \geq 0 \]

\[
\begin{array}{l}
\vdash [x' := (x - 2)^4 + y^5][y' := y^2]3x^2x' \geq 0
\end{array}
\]

\( [:=] \)

\[ y^5 \geq 0 \vdash [x' := (x - 2)^4 + y^5][y' := y^2]3x^2x' \geq 0 \]

\[
\begin{array}{l}
\vdash [x' := (x - 2)^4 + y^5, y' = y^2 \& y^5 \geq 0]x^3 \geq -1
\end{array}
\]

\( dl \)

\[ x^3 \geq -1 \vdash [x' := (x - 2)^4 + y^5, y' = y^2 \& y^5 \geq 0]x^3 \geq -1 \]

\( dC \)

\[ x^3 \geq -1 \land y^5 \geq 0 \vdash [x' := (x - 2)^4 + y^5, y' = y^2]x^3 \geq -1 \]

\[
\begin{array}{l}
\vdash 5y^4y^2 \geq 0
\end{array}
\]

\( R \)

\[
\begin{array}{l}
\vdash 5y^4y^2 \geq 0
\end{array}
\]

\[
\begin{array}{l}
\vdash [x' := (x - 2)^4 + y^5][y' := y^2]5y^4y' \geq 0
\end{array}
\]

\( [:=] \)

\[
\begin{array}{l}
\vdash [x' := (x - 2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array}
\]

\( dl \)

\[
\begin{array}{l}
\vdash [x' := (x - 2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{array}
\]
Ex: Differential Cuts

\begin{align*}
\mathbb{R} & \\
\vdash y^5 \geq 0 \vdash 3x^2((x - 2)^4 + y^5) \geq 0 \\
[:=] & \\
\vdash y^5 \geq 0 \vdash [x' := (x - 2)^4 + y^5][y' := y^2]3x^2x' \geq 0 \\
dl & \\
\vdash x^3 \geq -1 \vdash [x' = (x - 2)^4 + y^5, y' = y^2 \land y^5 \geq 0]x^3 \geq -1 \ \triangleright \\
dC & \\
\vdash x^3 \geq -1 \land y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2]x^3 \geq -1 \\
\end{align*}

\begin{align*}
\mathbb{R} & \\
\vdash 5y^4y^2 \geq 0 \\
[:=] & \\
\vdash [x' := (x - 2)^4 + y^5][y' := y^2]5y^4y' \geq 0 \\
dl & \\
\vdash y^5 \geq 0 \vdash [x' = (x - 2)^4 + y^5, y' = y^2]y^5 \geq 0
\end{align*}
Soundness Proof: Differential Invariants

**Lemma (Differential lemma)**
\[ \varphi \models x' = f(x) \land Q \text{ for } r > 0 \Rightarrow \forall 0 \leq z \leq r \varphi(z)[(e)'] = \frac{d\varphi(t)[e]}{dt}(z) \]

**Differential Invariant**
\[ \text{DI } ([x' = f(x)]e \geq 0 \iff e \geq 0) \]
\[ \iff [x' = f(x)](e)' \geq 0 \]

**Proof (≥ rate of change from ≥ initial value. Case } r = 0 \text{ is easier.)**

\[ h(t) \overset{\text{def}}{=} \varphi(t)[e] \text{ is differentiable on } [0, r] \text{ if } r > 0 \text{ by diff. lemma.} \]
\[ \frac{dh(t)}{dt}(z) = \frac{d\varphi(t)[e]}{dt}(z) = \varphi(z)[(e)'] \geq 0 \text{ by lemma + assume for all } z. \]
\[ h(r) - h(0) = (r - 0) \frac{dh(t)}{dt}(\xi) \geq 0 \text{ by mean-value theorem for some } \xi. \]
Outline

1. Learning Objectives
2. Differential Invariants
   - Recap: Ingredients for Differential Equation Proofs
   - Soundness: Derivations Lemma
   - Differential Weakening
   - Equational Differential Invariants
   - Differential Invariant Inequalities
   - Disequational Differential Invariants
   - Example Proof: Damped Oscillator
   - Conjunctive Differential Invariants
   - Disjunctive Differential Invariants
   - Assuming Invariants
3. Differential Cuts
4. Soundness
5. Summary
Summary: Differential Invariants for Differential Equations

Differential Weakening

\[ Q \vdash F \]
\[ \Gamma \vdash [x' = f(x) \& Q]F \]

Differential Invariant

\[ Q \vdash [x' := f(x)](F)' \]
\[ F \vdash [x' = f(x) \& Q]F \]

Differential Cut

\[ F \vdash [x' = f(x) \& Q]C \]
\[ F \vdash [x' = f(x) \& Q \land C]F \]
\[ F \vdash [x' = f(x) \& Q]F \]
Summary: Differential Invariants for Differential Equations

Differential Weakening

\[ Q \vdash F \]
\[ \Gamma \vdash [x' = f(x) & Q]F \]

Differential Invariant

\[ Q \vdash [x' := f(x)](F)' \]
\[ F \vdash [x' = f(x) & Q]F \]

Differential Cut

\[ F \vdash [x' = f(x) & Q]C \]
\[ F \vdash [x' = f(x) & Q \land C]F \]
\[ F \vdash [x' = f(x) & Q]F \]

DW  \[ [x' = f(x) & Q]F \leftrightarrow [x' = f(x) & Q](Q \rightarrow F) \]

DI  \[ ([x' = f(x) & Q]F \leftrightarrow [?Q]F) \leftrightarrow (Q \rightarrow [x' = f(x) & Q](F)') \]

DC  \[ ([x' = f(x) & Q]F \leftrightarrow [x' = f(x) & Q \land C]F) \leftrightarrow [x' = f(x) & Q]C \]
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