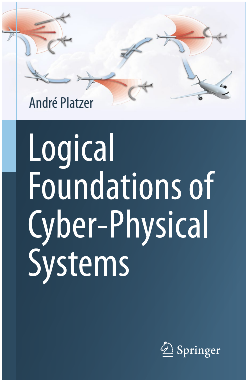
03: Choice & Control

Logical Foundations of Cyber-Physical Systems



André Platzer

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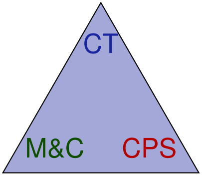
Outline

Outline

nondeterminism
abstraction
programming languages for CPS
semantics
compositionality



models
core principles
discrete+
continuous

operational effect
operational precision

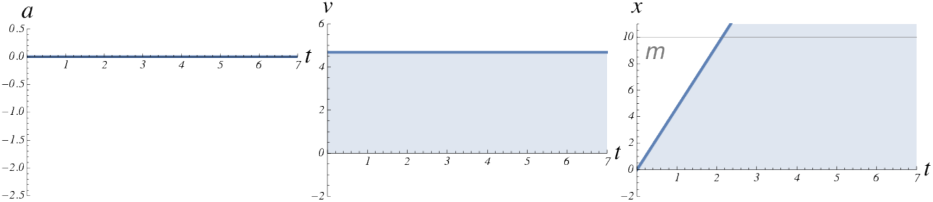
Outline

Example (Speedy the point)

$$\{x' = v, v' = a\}$$

Purely continuous dynamics

What about the cyber?

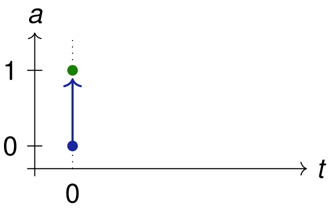


Example (Speedy the point)

$$a := a + 1$$

Purely discrete dynamics

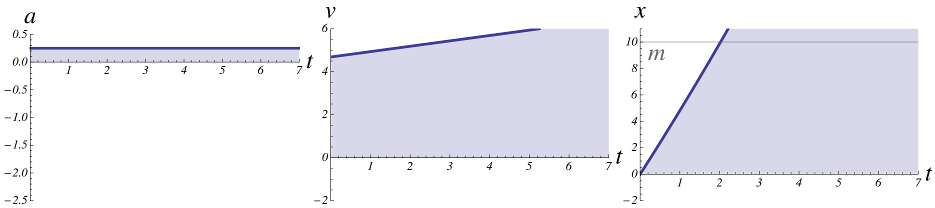
How do both meet?



Example (Speedy the point)

$$a := a + 1;\ \{x' = v, v' = a\}$$

Hybrid dynamics, i.e., composition of continuous and discrete dynamics
Here: sequential composition first;second



Playing with Acceleration and Braking

Example (Speedy the point)

$$\begin{aligned}
&a := -2; \quad \{x' = v, v' = a\};\\
&a := 0.25; \{x' = v, v' = a\};\\
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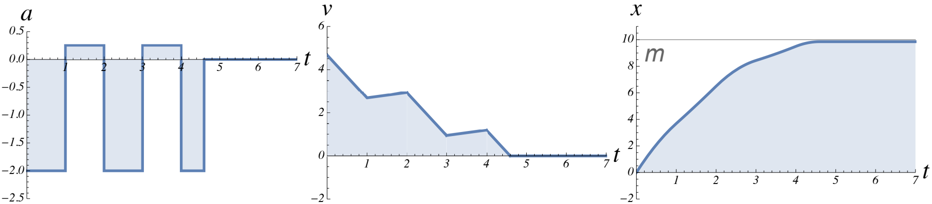

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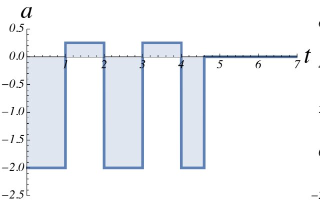
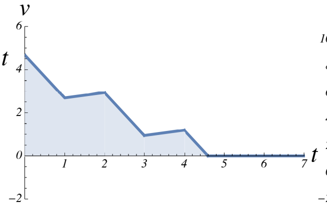
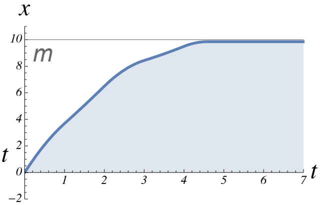
How long to follow an ODE?







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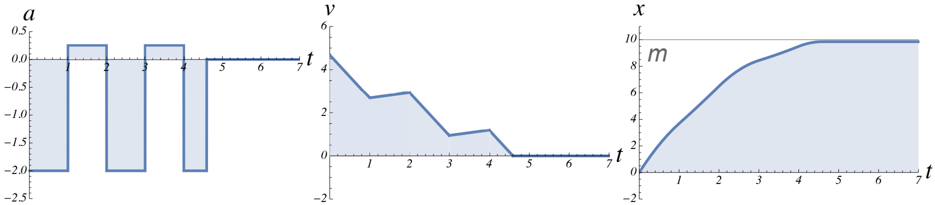
$$

How to check conditions before actions?



Example (Speedy the point)

$$\text{if}(v < 4)\, a := a + 1 \text{ else } a := -b;$$
$$\{x' = v, v' = a\}$$

Velocity-dependent control

Playing with Acceleration and Braking

Example (Speedy the point)

$$\text{if}(x - m > s)\, a := a + 1 \text{ else } a := -b;$$
$$\{x' = v, v' = a\}$$

Distance-dependent control for obstacle m

Playing with Acceleration and Braking

Example (Speedy the point)

$$\text{if}(x - m > s \land v < 4)\, a := a + 1 \;\text{else}\; a := -b;$$
$$\{x' = v, v' = a\}$$

Velocity and distance-dependent control

Iterative Design

Start as simple as possible, then add challenges once basics are correct.

Example (Speedy the point)

$$\text{if}(x - m > s \land v < 4 \land \text{efficiency})\, a := a + 1 \,\text{else}\, a := -b;$$
$$\{x' = v, v' = a\}$$

Also only accelerate if it's efficient to do so

Example (Speedy the point)

$$\text{if}(x - m > s \land v < 4 \land \text{efficiency})\, a := a + 1 \,\text{else}\, a := -b;$$
$$\{x' = v, v' = a\}$$

Exact models are unnecessarily complex. Not all features are safety-critical.

Example (Speedy the point)

$$(a := a+1 \cup a := -b);$$
$$\{x' = v, v' = a\}$$

Nondeterministic choice $\cup$ allows either side to be run, arbitrarily

Power of Abstraction

Only include relevant aspects, elide irrelevant detail.
The model and its analysis become simpler. And apply to more systems.

Example (Speedy the point)

$$(a := a + 1 \;\cup\; a := -b);$$
$$\{x' = v, v' = a\}$$

Nondeterministic choice $\cup$ allows either side to be run, arbitrarily

Oops, now it got too simple! Not every choice is always acceptable.

Example (Speedy the point)

$$(?v < 4; a := a + 1 \cup a := -b);$$
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Test $?Q$ checks if formula Q is true in current state

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$$(?v < 4; a := a + 1 \cup a := -b);$$
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Test $?Q$ checks if formula Q is true in current state, otherwise run fails.

Discarding failed runs and backtracking

System runs that fail tests are discarded and not considered further.

$?v < 4;\ v := v + 1$ only runs if

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Broader significance of nondeterminism

Nondeterminism is a tool for abstraction to focus on critical aspects.
Nondeterminism is essential to describe imperfectly known environment.

Playing with Acceleration and Braking

Example (Speedy the point)

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Test $?Q$ checks if formula Q is true in current state, otherwise run fails.

Discarding failed runs and backtracking

System runs that fail tests are discarded and not considered further.

$?v < 4;\ v := v + 1$	only runs if $v < 4$ initially true
$v := v + 1;\ ?v < 4$	only runs if $v < 3$ initially true

Broader significance of nondeterminism

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Nondeterminism is essential to describe imperfectly known environment.

Example (Speedy the point)

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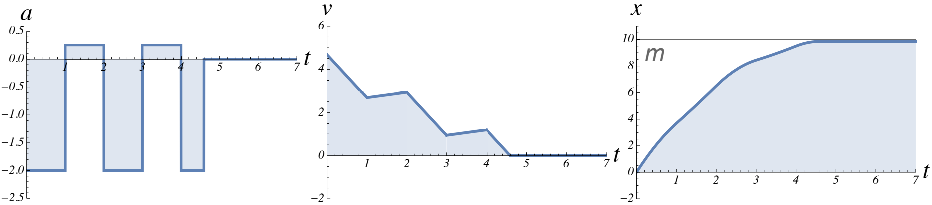
&\{x' = v, v' = a\}
\end{aligned}
$$

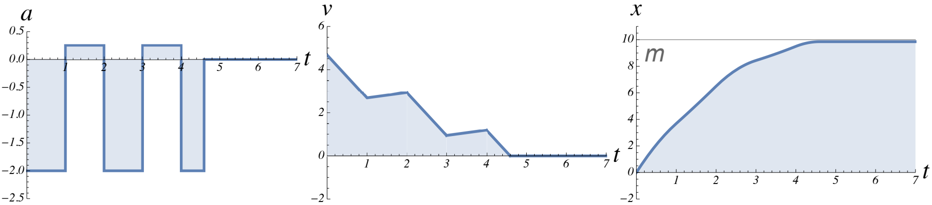
Repeated control needs longer programs, e.g., by copy&paste



Example (Speedy the point)

$$\big((?v < 4; a := a+1 \cup a := -b);\\ \{x' = v, v' = a\}\big)^*$$

Nondeterministic repetition * repeats *any* arbitrary number of times



Outline

Hybrid Programs: Syntax

Definition (Syntax of hybrid program α)

$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

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Discrete Assign

Test Condition

Differential Equation

Nondet. Choice

Seq. Compose

Nondet. Repeat

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Discrete Assign | Test Condition | Differential Equation | Nondet. Choice | Seq. Compose | Nondet. Repeat

Like regular expressions. Everything nondeterministic

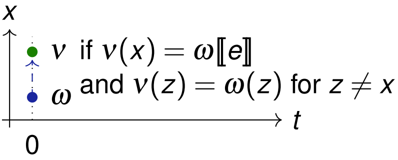
Hybrid Programs: Semantics

$$\omega \xrightarrow{x := e} \nu$$

$$\omega \xrightarrow{x' = f(x) \,\&\, Q} \nu$$

$$\omega \xrightarrow{?Q} \nu$$

$$\omega \xrightarrow{x := e} \nu$$



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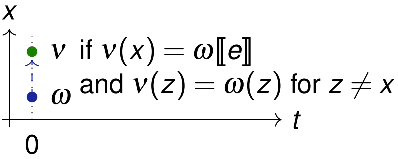

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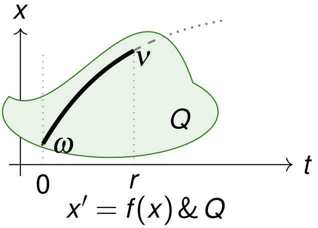

$$\omega \xrightarrow{?Q} \nu$$

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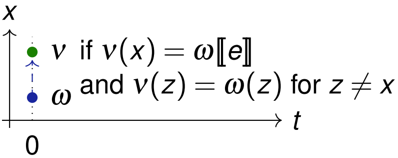

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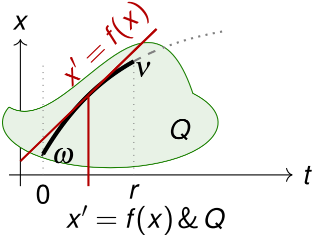

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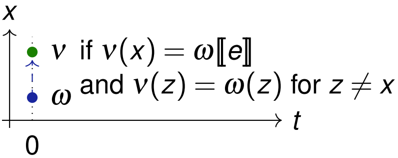

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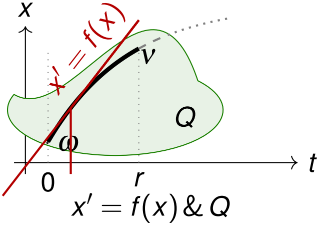
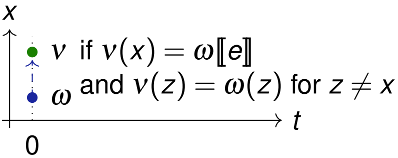


$$\omega \xrightarrow{?Q} \nu$$

Hybrid Programs: Semantics

$$\omega \xrightarrow{x := e} \nu$$



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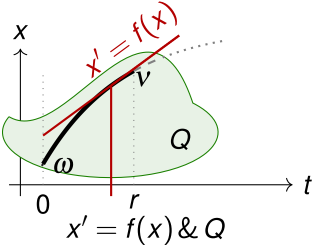

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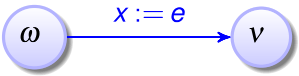
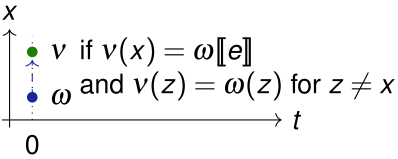
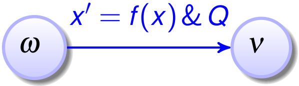
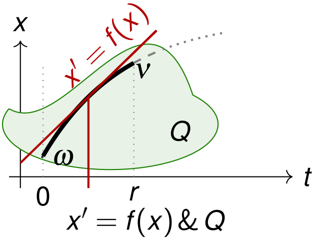
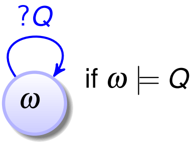
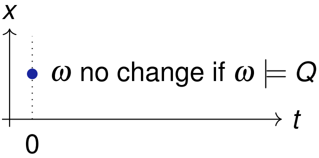

ω
$x := e$
ν


x
ν if $\nu(x) = \omega[\![e]\!]$
ω and $\nu(z) = \omega(z)$ for $z \neq x$
t
0

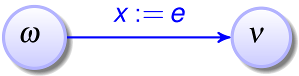

ω
$x' = f(x) \,\&\, Q$
ν

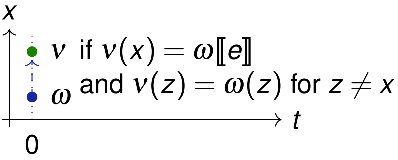

x
$x' = f(x)$
ν
Q
ω
t
0
r
$x' = f(x) \,\&\, Q$

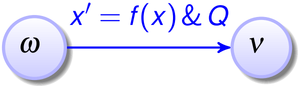

$?Q$
ω
if $\omega \models Q$

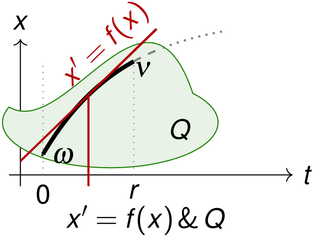

x
ω no change if $\omega \models Q$
t
0

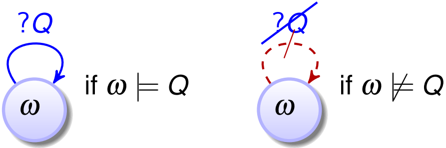
Hybrid Programs: Semantics

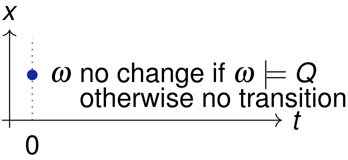












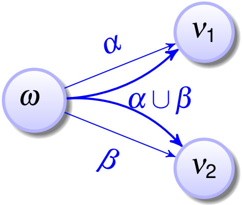
Hybrid Programs: Semantics

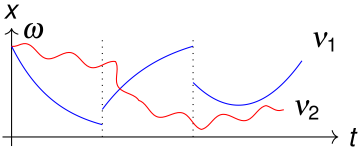
$$\omega \xrightarrow{\alpha \cup \beta} \nu$$

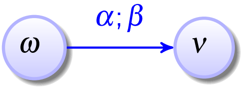
$$\omega \xrightarrow{\alpha;\beta} \nu$$

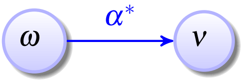
$$\omega \xrightarrow{\alpha^*} \nu$$

Hybrid Programs: Semantics

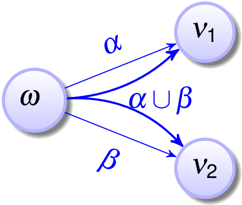


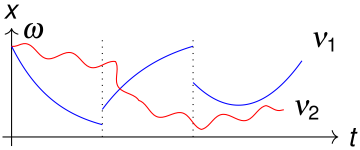


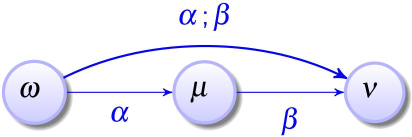


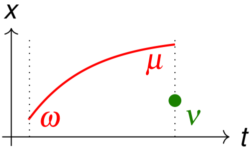


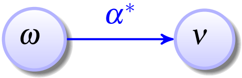
Hybrid Programs: Semantics











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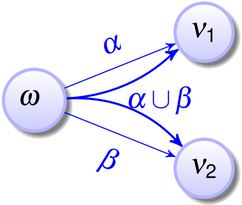

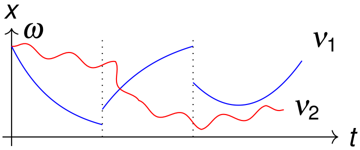

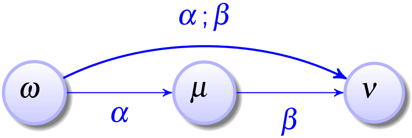

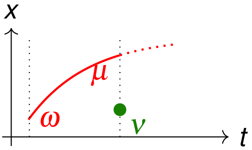

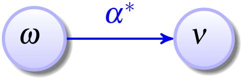

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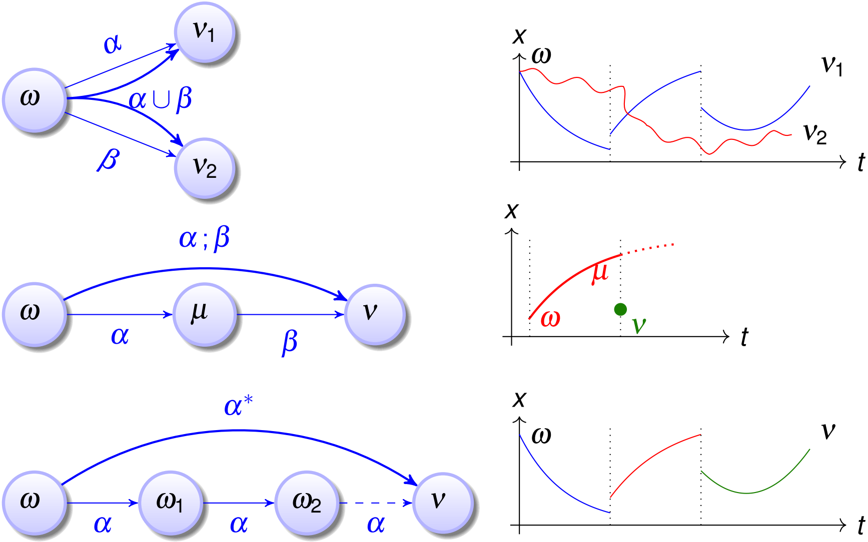
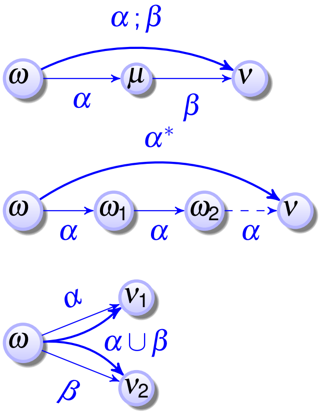
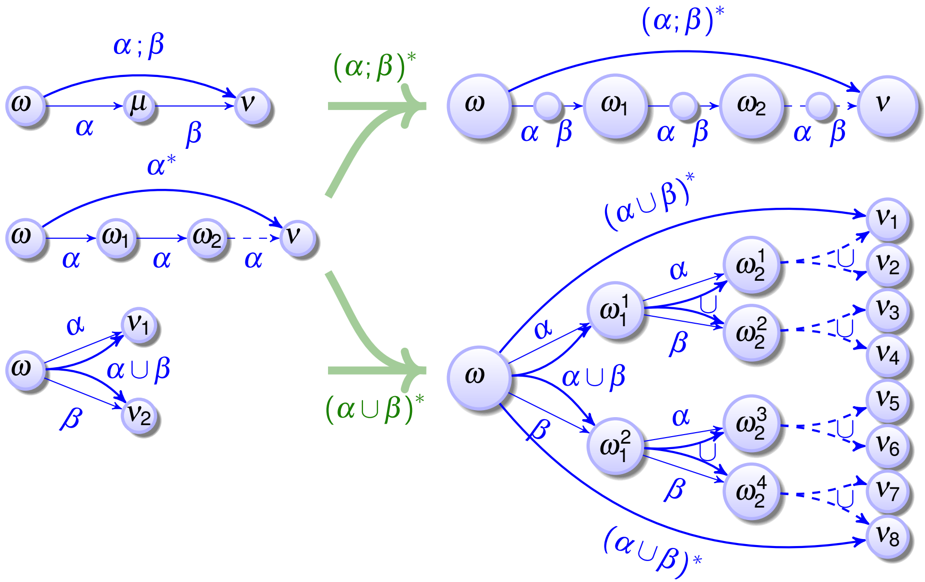


Plug-in for Semantics of Composed Hybrid Programs



Plug-in for Semantics of Composed Hybrid Programs



Hybrid Programs: Syntax & Semantics

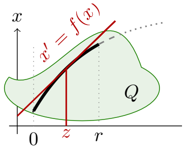
Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) ($[\![\cdot]\!] : \mathrm{HP} \to \wp(\mathscr{S} \times \mathscr{S})$)

$$\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\} \\
[\![?Q]\!] &= \{(\omega, \omega) \ : \ \omega \models Q\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha; \beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] = \{(\omega, \nu) \ : \ (\omega, \mu) \in [\![\alpha]\!] \text{ and } (\mu, \nu) \in [\![\beta]\!]\} \\
[\![\alpha^*]\!] &= [\![\alpha]\!]^* = \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!] \qquad \alpha^n \equiv \underbrace{\alpha; \alpha; \alpha; \ldots; \alpha}_{n \text{ times}}
\end{aligned}$$

compositional



Hybrid Programs: Syntax & Semantics

Definition (Syntax of hybrid program α)

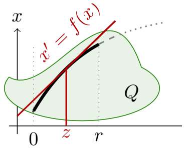
$$\alpha,\beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) ($[\![\cdot]\!] : \mathrm{HP} \to \wp(\mathscr{S} \times \mathscr{S})$)

$$\begin{aligned}
[\![x := e]\!] &= \{(\omega,\nu) \;:\; \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\}\\
[\![?Q]\!] &= \{(\omega,\omega) \;:\; \omega \models Q\}\\
[\![x' = f(x)]\!] &= \{(\varphi(0),\varphi(r)) \;:\; \varphi \models x' = f(x) \text{ for some duration } r \geq 0\}\\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!]\\
[\![\alpha;\beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!]\\
[\![\alpha^*]\!] &= [\![\alpha]\!]^* = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!]
\end{aligned}$$

compositional

1. $\varphi(z)(x') = \frac{d\varphi(t)(x)}{dt}(z)$ exists at all times $0 \leq z \leq r$
2. $\varphi(z) \models x' = f(x) \wedge Q$ for all times $0 \leq z \leq r$
3. $\varphi(z) = \varphi(0)$ except at x, x'



Notational Conventions: Names

Example (Naming Conventions)

Letters	Convention
x, y, z	variables
$e, \tilde{e}$	terms
P, Q	formulas
α, β	programs
c	constant symbols
f, g, h	function symbols
p, q, r	predicate symbols

In CPS applications, all bets are off because names follow application:
x position v velocity and a acceleration variables

Notational Conventions: Precedence

Convention (Operator Precedence)

1. Unary operators (including *, $\neg$ and $\forall x, \exists x$) bind stronger than binary.
2. $\wedge$ binds stronger than $\vee$, which binds stronger than $\rightarrow, \leftrightarrow$
3. ; binds stronger than $\cup$
4. Arithmetic operators $+, -, \cdot$ associate to the left
5. Logical and program operators associate to the right

Example (Operator Precedence)

$\forall x\, P \wedge Q \equiv (\forall x\, P) \wedge Q$ $\qquad$ $\forall x\, P \rightarrow Q \equiv (\forall x\, P) \rightarrow Q.$

$\alpha;\beta \cup \gamma \equiv (\alpha;\beta) \cup \gamma$ $\qquad$ $\alpha \cup \beta;\gamma \equiv \alpha \cup (\beta;\gamma)$ $\qquad$ $\alpha;\beta^* \equiv \alpha;(\beta^*)$

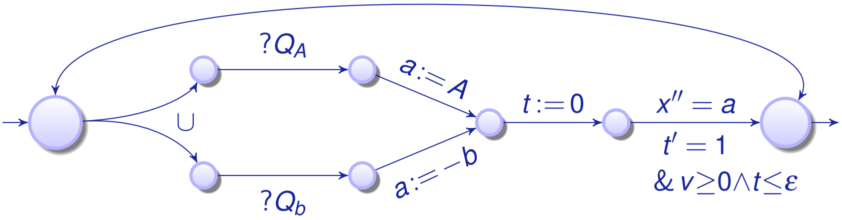
$P \rightarrow Q \rightarrow R \equiv P \rightarrow (Q \rightarrow R).$

But $\rightarrow, \leftrightarrow$ expect explicit parentheses. Illegal: $P \rightarrow Q \leftrightarrow R$ $\qquad$ $P \leftrightarrow Q \rightarrow R$

Outline

Branching Transition Structure in Hybrid Programs

$$
\begin{aligned}
\mathrm{Robot} &\equiv (\mathrm{ctrl}\,;\mathrm{drive})^* \\
\mathrm{ctrl} &\equiv (?Q_A; a := A) \\
&\quad \cup (?Q_b; a := -b) \\
\mathrm{drive} &\equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}
\end{aligned}
$$



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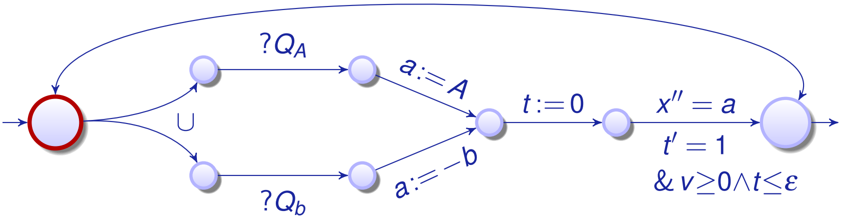

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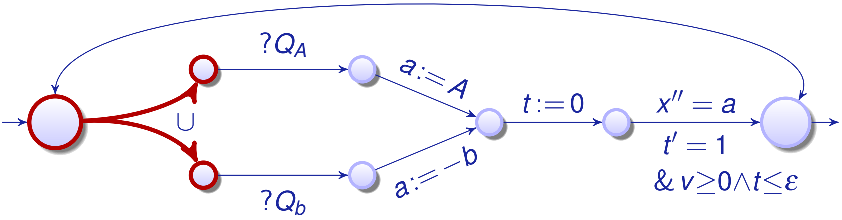
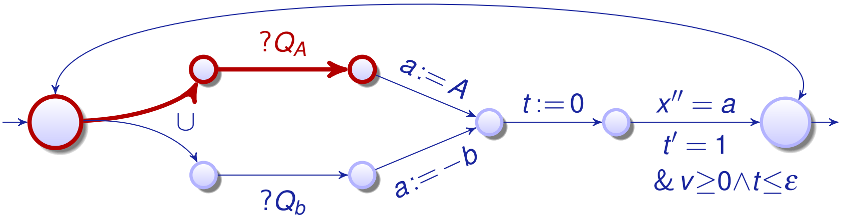
Branching Transition Structure in Hybrid Programs



$$\begin{aligned} \text{Robot} &\equiv (\text{ctrl}\,;\text{drive})^* \\ \text{ctrl} &\equiv (?Q_A; a := A) \\ &\quad \cup (?Q_b; a := -b) \\ \text{drive} &\equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\} \end{aligned}$$



$$\mathrm{Robot} \equiv (\mathrm{ctrl}\,;\mathrm{drive})^*$$

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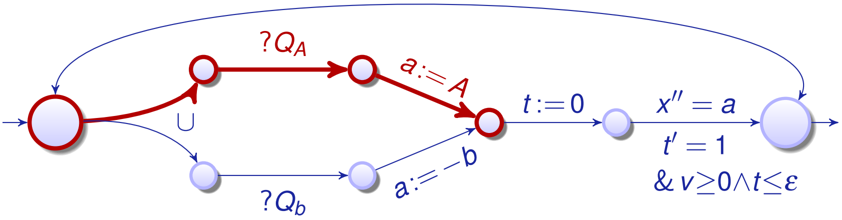

$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

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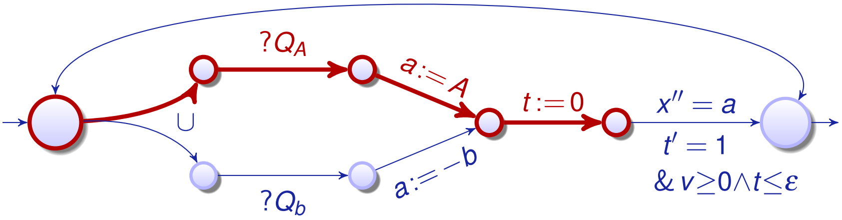
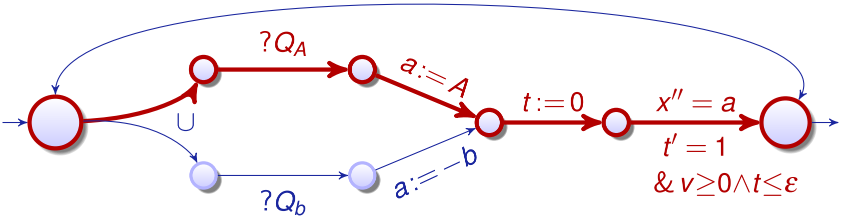


$$\begin{aligned}
\text{Robot} &\equiv (\text{ctrl}\,;\text{drive})^* \\
\text{ctrl} &\equiv (?Q_A;\, a:=A) \\
&\quad \cup (?Q_b;\, a:=-b) \\
\text{drive} &\equiv t:=0;\, \{x'=v, v'=a, t'=1 \,\&\, v\geq 0\land t\leq\varepsilon\}
\end{aligned}$$



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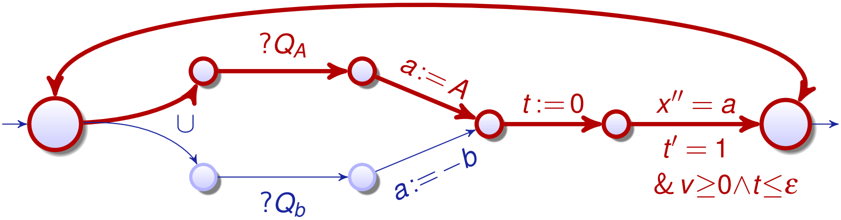

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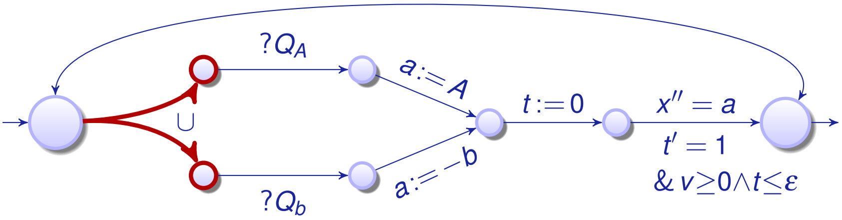
\end{aligned}
$$

Branching Transition Structure in Hybrid Programs



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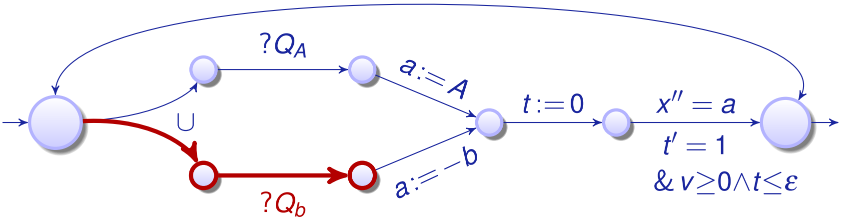
\end{aligned}$$



$$\begin{aligned}
\text{Robot} &\equiv (\text{ctrl} \,;\, \text{drive})^* \\
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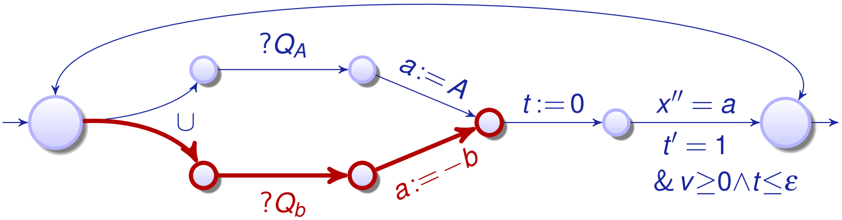
\end{aligned}$$

Branching Transition Structure in Hybrid Programs



$$\mathrm{Robot} \equiv (\mathrm{ctrl}\,;\mathrm{drive})^*$$

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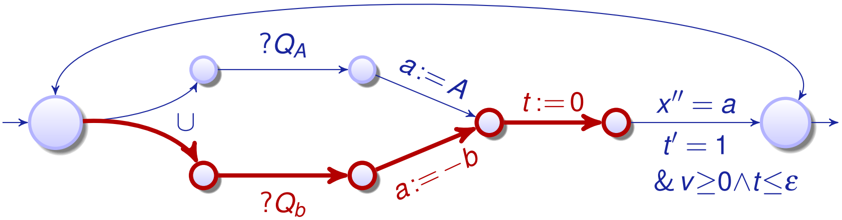

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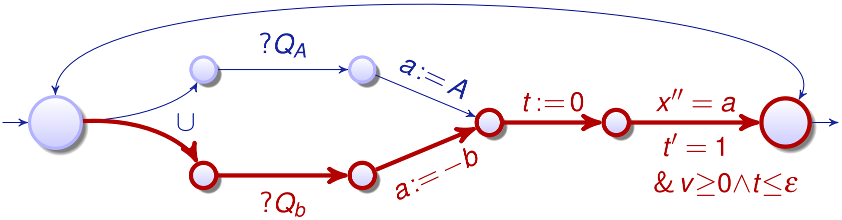

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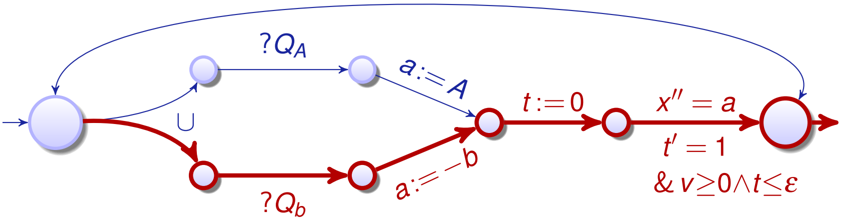

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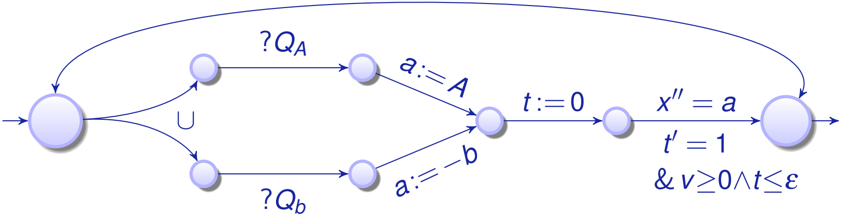


$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv$$

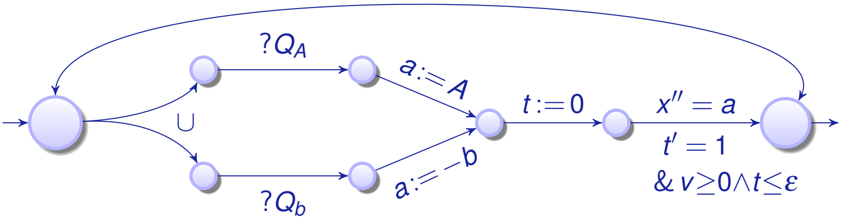
$$\text{while}(Q)\,\alpha \equiv$$

$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A)$$

$$\cup (?Q_b; a := -b)$$

$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}$$



$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha)\cup(?\neg Q;\beta)$$

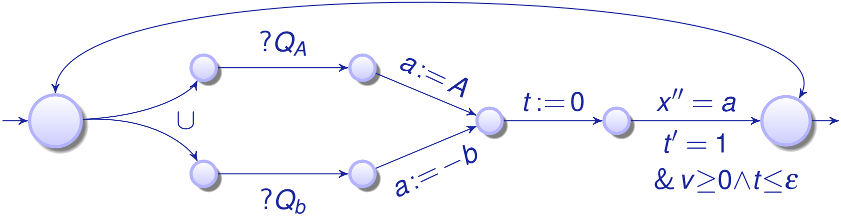
$$\text{while}(Q)\,\alpha \equiv$$

$$\text{Robot} \equiv (\text{ctrl}\,;\text{drive})^*$$

$$\text{ctrl} \equiv (?Q_A; a := A)$$

$$\cup\,(?Q_b; a := -b)$$

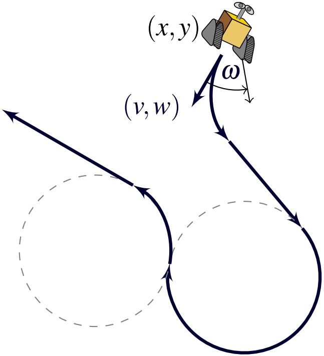
$$\text{drive} \equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}$$



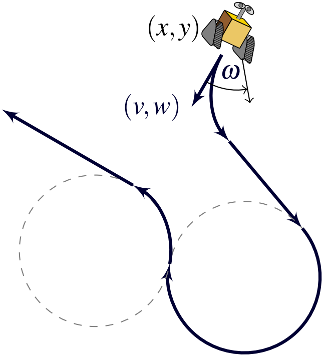
$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\begin{aligned}
\text{Robot} &\equiv (\text{ctrl};\text{drive})^* \\
\text{ctrl} &\equiv (?Q_A; a := A) \\
&\quad \cup (?Q_b; a := -b) \\
\text{drive} &\equiv t := 0; \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \land t \leq \varepsilon\}
\end{aligned}$$

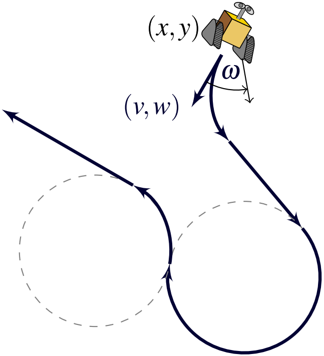

(x, y)
ω
(v, w)

Runaround Robot with Dubins Paths



Example (Runaround Robot)

$$\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);\\ \{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^*$$



Example (Runaround Robot)

$$\big((?Q_{-1};\omega:=-1 \cup ?Q_1;\omega:=1 \cup ?Q_0;\omega:=0);$$
$$\{x'=v, y'=w, v'=\omega w, w'=-\omega v\}\big)^*$$

Example (Speedy the point)

$$
\begin{aligned}
&(?v < 4; a := a+1 \cup a := -b);\\
&\{x' = v, v' = a\};\\
&(?v < 4; a := a+1 \cup a := -b);\\
&\{x' = v, v' = a\};\\
&(?v < 4; a := a+1 \cup a := -b);\\
&\{x' = v, v' = a\}
\end{aligned}
$$

Example (Speedy the point)

$$?v < 4; a := a + 1;$$
$$\{x' = v, v' = a\};$$
$$?v < 4; a := a + 1;$$
$$\{x' = v, v' = a\};$$
$$?v < 4; a := a + 1;$$
$$\{x' = v, v' = a\}$$

A Matter of Choice: Recall Speedy, Remove Stuff

Example (Speedy the point)

$$
\begin{aligned}
&?v < 4; a := a+1\,;\\
&\{x' = v, v' = a\};\\
&?v < 4; a := a+1\,;\\
&\{x' = v, v' = a\};\\
&?v < 4; a := a+1\,;\\
&\{x' = v, v' = a\}
\end{aligned}
$$

No wait, now it's a bad model! The HP assumes the test $v < 4$ passes after each ODE. No other choices are available.

Don't let your controller discard important cases!

Outline

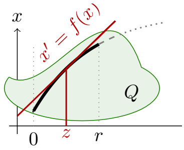
Definition (Syntax of hybrid program α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (Semantics of hybrid programs) $([\![\cdot]\!] : \mathrm{HP} \to \wp(\mathscr{S} \times \mathscr{S}))$

$$\begin{aligned}
[\![x := e]\!] &= \{(\omega, \nu) \ : \ \nu = \omega \text{ except } \nu[\![x]\!] = \omega[\![e]\!]\} \\
[\![?Q]\!] &= \{(\omega, \omega) \ : \ \omega \models Q\} \\
[\![x' = f(x)]\!] &= \{(\varphi(0), \varphi(r)) \ : \ \varphi \models x' = f(x) \text{ for some duration } r \geq 0\} \\
[\![\alpha \cup \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha;\beta]\!] &= [\![\alpha]\!] \circ [\![\beta]\!] \\
[\![\alpha^*]\!] &= [\![\alpha]\!]^* = \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!]
\end{aligned}$$

compositional



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