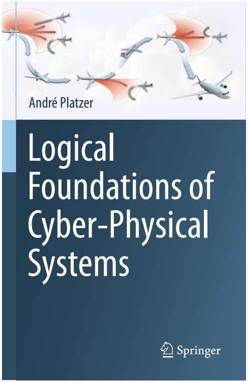
14: Hybrid Systems & Games

Logical Foundations of Cyber-Physical Systems



André Platzer

Carnegie Mellon University
Computer Science Department

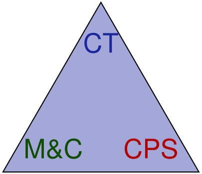
Outline

Outline

Learning Objectives

Hybrid Systems & Games

fundamental principles of computational thinking
logical extensions
PL modularity principles
compositional extensions
differential game logic
best/worst-case analysis
models of alternating computation



adversarial dynamics
conflicting actions
multi-agent systems
angelic/demonic choice

multi-agent state change
CPS semantics
reflections on choices

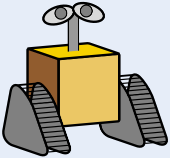
Outline

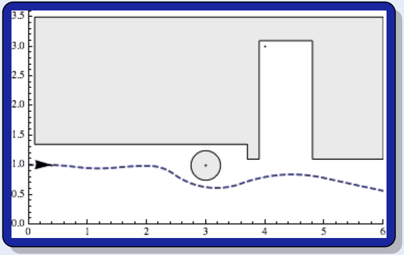
CPS Analysis: Robot Control

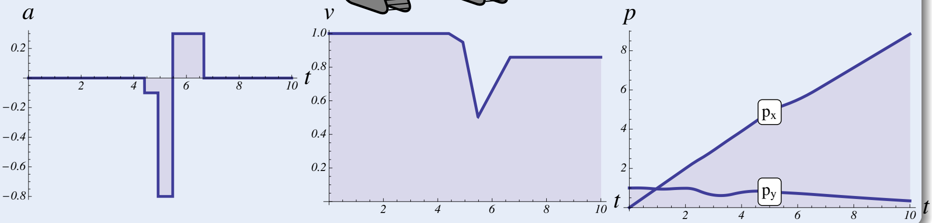
Challenge (Hybrid Systems)

Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)



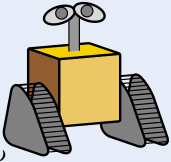


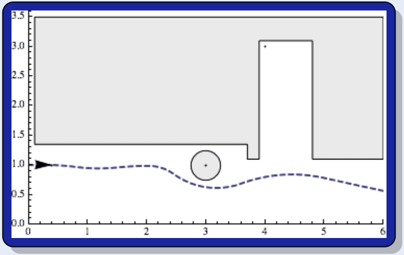


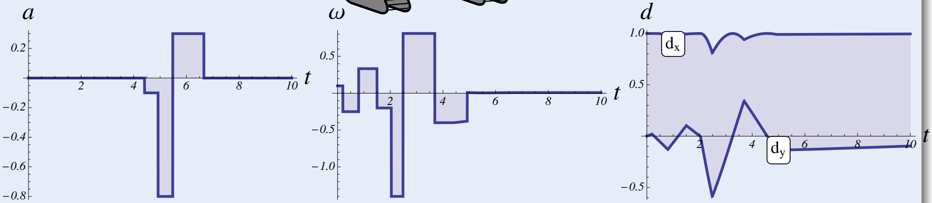
Challenge (Hybrid Systems)

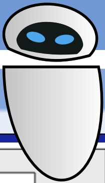
Fixed rule describing state evolution with both

- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)





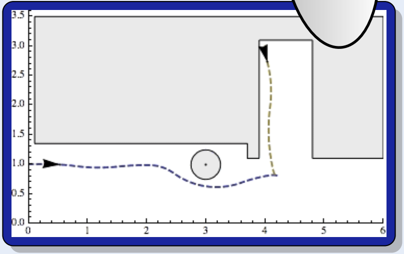


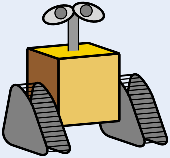


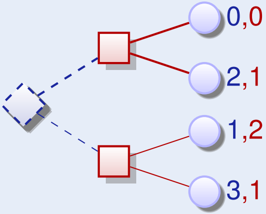
Challenge (Games)

Game rules describing play evolution with both

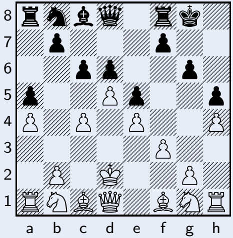
- Angelic choices (player ◇ Angel)
- Demonic choices (player □ Demon)

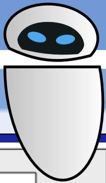






◇\□	Tr	Pl
Trash	1,2	0,0
Plant	0,0	2,1

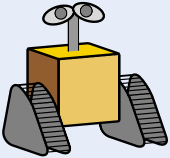


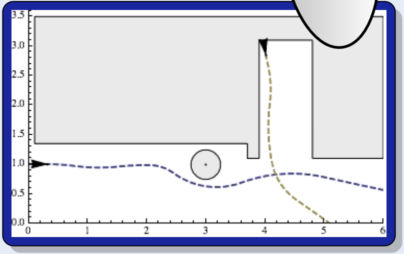


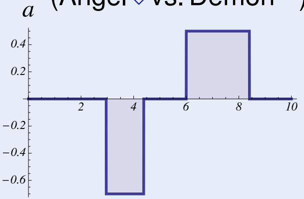
Challenge (Hybrid Games)

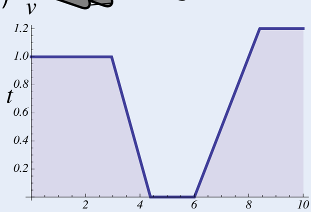
Game rules describing play evolution with

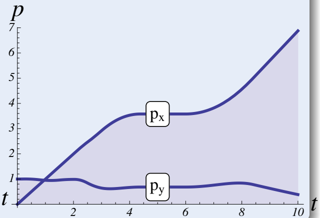
- Discrete dynamics (control decisions)
- Continuous dynamics (differential equations)
- Adversarial dynamics (Angel $\diamond$ vs. Demon $\square$)

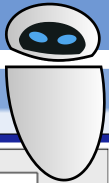












Challenge (Hybrid Games)

Game rules describing play evolution with

- Discrete dynamics (control decisions)
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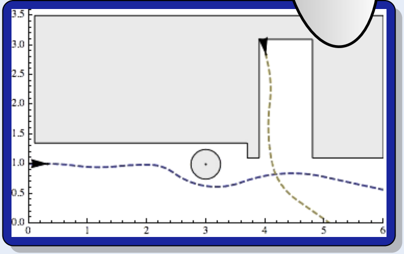

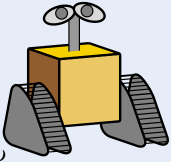

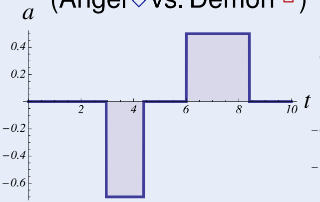

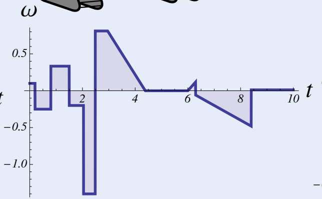

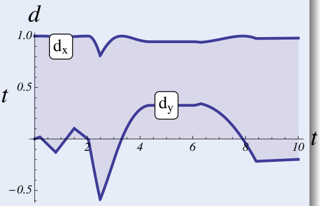

Challenge (Hybrid Games)

Game rules describing play evolution with

- Discrete dynamics (control decisions)
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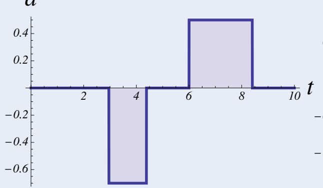
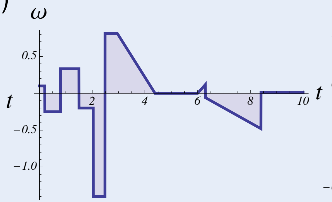
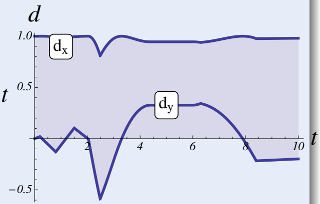
- Adversarial dynamics (Angel ◇ vs. Demon □)



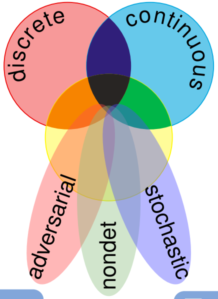




CPSs are Multi-Dynamical Systems

CPS Dynamics

CPS are characterized by multiple facets of dynamical systems.



CPS Compositions

CPS combines multiple simple dynamical effects.

Descriptive simplification

Tame Parts

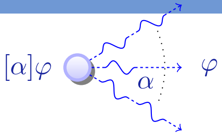
Exploiting compositionality tames CPS complexity.

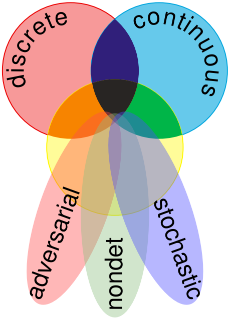
Analytic simplification

Dynamic Logics for Dynamical Systems

differential dynamic logic

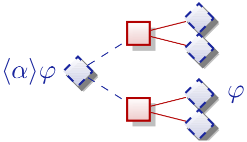
$\mathrm{dL} = \mathrm{DL} + \mathrm{HP}$





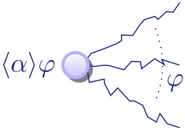
differential game logic

$\mathrm{dGL} = \mathrm{GL} + \mathrm{HG}$



stochastic differential DL

$\mathrm{SdL} = \mathrm{DL} + \mathrm{SHP}$



quantified differential DL

$\mathrm{QdL} = \mathrm{FOL} + \mathrm{DL} + \mathrm{QHP}$

Outline

Differential Dynamic Logic dL: Syntax

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

Differential Dynamic Logic dL: Syntax

Discrete Assign

Test Condition

Differential Equation

Nondet. Choice

Seq. Compose

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle\alpha\rangle P$$

All Reals

Some Reals

All Runs

Some Runs

Nondet. Choice

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

Definition (dL Formula P)

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Nondeterminism during HP runs

Differential Dynamic Logic dL: Nondeterminism

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

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Nondeterminism during HP runs

Differential Equation

Nondet. Choice

Nondet. Repeat

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All Choices

Some Choice

All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^*$$

Definition (dL Formula P)

$$e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid [\alpha]P \mid \langle \alpha \rangle P$$

Modality decides the mode: help/hurt

All Choices

Some Choice

Differential Dynamic Logic dL: Nondeterminism

All choices resolved in one way

Differential Equation

Nondet. Choice

Nondet. Repeat

Definition (Hybrid program α)

$$x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^*$$

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Modality decides the mode: help/hurt

All Choices

Some Choice

$[\alpha_1]\langle\alpha_2\rangle[\alpha_3]\langle\alpha_4\rangle P$ only fixed interaction depth

$\diamond$ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

Let Angel be one player

Control & Dual Control Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

Let Angel be one player

Let Demon be another player

◇ **Angel Ops**

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ **Demon Ops**

$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

Duality operator ^{d} passes control between players

Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

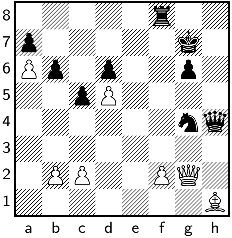
d

d

◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

Duality operator ^{d} passes control between players



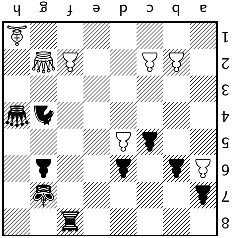
Game Operators

◇ Angel Ops	
$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$$\text{Angel Ops} \underset{d}{\overset{d}{\rightleftarrows}} \text{Demon Ops}$$

□ Demon Ops	
$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

Duality operator ^{d} passes control between players



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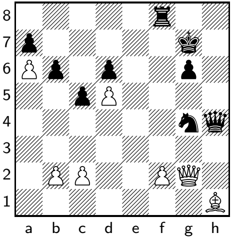
Duality operator ^{d} passes control between players



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$?Q$	challenge

d

d

◻ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\begin{aligned}
\text{if}(Q)\,\alpha\,\text{else}\,\beta &\equiv \\
\text{while}(Q)\,\alpha &\equiv \\
\alpha \cap \beta &\equiv \\
\alpha^{\times} &\equiv \\
(x' = f(x)\,\&\,Q)^d &\quad x' = f(x)\,\&\,Q \\
(x := e)^d &\quad x := e \\
?Q^d &\quad ?Q
\end{aligned}$$

Definable Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$\xrightarrow{d}$ / $\xleftarrow{d}$

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{d}$	evolve
$?Q^{d}$	challenge

$$\text{if}(Q)\,\alpha\,\text{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\text{while}(Q)\,\alpha \equiv$$

$$\alpha \cap \beta \equiv$$

$$\alpha^{\times} \equiv$$

$$(x' = f(x) \,\&\, Q)^{d} \qquad x' = f(x) \,\&\, Q$$

$$(x := e)^{d} \qquad x := e$$

$$?Q^{d} \qquad ?Q$$

Definable Game Operators

◇ Angel Ops	
$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$\xrightarrow{d}$ / $\xleftarrow{d}$

□ Demon Ops	
$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv$$

$$\alpha^\times \equiv$$

$$(x' = f(x)\,\&\,Q)^d \qquad x' = f(x)\,\&\,Q$$

$$(x := e)^d \qquad x := e$$

$$?Q^d \qquad ?Q$$

Definable Game Operators

◇ Angel Ops

$\cup$	choice
*	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$^\times$	repeat
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$?Q^d$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

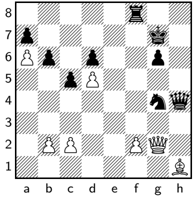
$$\alpha \cap \beta \equiv$$

$$\alpha^\times \equiv$$

$$(x' = f(x) \,\&\, Q)^d \qquad x' = f(x) \,\&\, Q$$

$$(x := e)^d \qquad x := e$$

$$?Q^d \qquad ?Q$$



Definable Game Operators

◇ Angel Ops	
$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$\xrightarrow{\mathrm{d}}$

$\xleftarrow{\mathrm{d}}$

□ Demon Ops	
$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

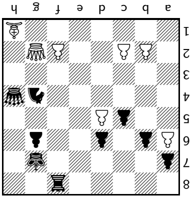
$$\alpha \cap \beta \equiv (\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}$$

$$\alpha^{\times} \equiv$$

$$(x' = f(x) \,\&\, Q)^{\mathrm{d}} \qquad x' = f(x) \,\&\, Q$$

$$(x := e)^{\mathrm{d}} \qquad x := e$$

$$?Q^{\mathrm{d}} \qquad ?Q$$



Definable Game Operators

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□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$

$$\alpha^\times \equiv ((\alpha^d)^*)^d$$

$$(x' = f(x) \,\&\, Q)^d \qquad x' = f(x) \,\&\, Q$$

$$(x := e)^d \qquad x := e$$

$$?Q^d \qquad ?Q$$

Definable Game Operators

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}$$

$$\alpha^{\times} \equiv ((\alpha^{\mathrm{d}})^*)^{\mathrm{d}}$$

$$(x' = f(x)\,\&\,Q)^{\mathrm{d}} \not\equiv x' = f(x)\,\&\,Q$$

$$(x := e)^{\mathrm{d}} \qquad x := e$$

$$?Q^{\mathrm{d}} \qquad ?Q$$

◇ Angel Ops

$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

d

d

□ Demon Ops

$\cap$	choice
$\times$	repeat
$x' = f(x)^{\mathrm{d}}$	evolve
$?Q^{\mathrm{d}}$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$

$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$

$$\alpha \cap \beta \equiv (\alpha^{\mathrm{d}} \cup \beta^{\mathrm{d}})^{\mathrm{d}}$$

$$\alpha^{\times} \equiv ((\alpha^{\mathrm{d}})^*)^{\mathrm{d}}$$

$$(x' = f(x)\,\&\,Q)^{\mathrm{d}} \not\equiv x' = f(x)\,\&\,Q$$

$$(x := e)^{\mathrm{d}} \equiv x := e$$

$$?Q^{\mathrm{d}} \qquad ?Q$$

◇ Angel Ops	
$\cup$	choice
$*$	repeat
$x' = f(x)$	evolve
$?Q$	challenge

$\xrightarrow{d}$ / $\xleftarrow{d}$

□ Demon Ops	
$\cap$	choice
$\times$	repeat
$x' = f(x)^d$	evolve
$?Q^d$	challenge

$$\mathrm{if}(Q)\,\alpha\,\mathrm{else}\,\beta \equiv (?Q;\alpha) \cup (?\neg Q;\beta)$$
$$\mathrm{while}(Q)\,\alpha \equiv (?Q;\alpha)^*;?\neg Q$$
$$\alpha \cap \beta \equiv (\alpha^d \cup \beta^d)^d$$
$$\alpha^\times \equiv ((\alpha^d)^*)^d$$
$$(x' = f(x)\,\&\,Q)^d \not\equiv x' = f(x)\,\&\,Q$$
$$(x := e)^d \equiv x := e$$
$$?Q^d \not\equiv ?Q$$

Outline

Hybrid Games: Syntax

Definition (Hybrid game α)

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

Definition (Hybrid game α)

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Hybrid Games: Syntax

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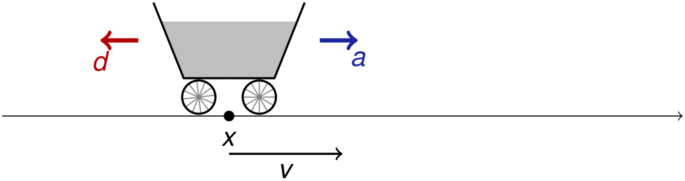
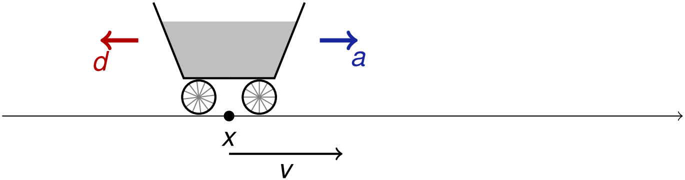
$$\alpha, \beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

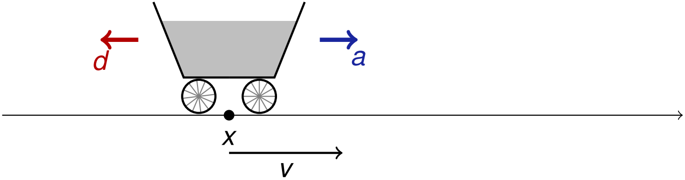
Example: Push-around Cart





$$\big((a:=1 \cup a:=-1);(d:=1 \cup d:=-1)^{\mathrm{d}};\{x'=v,v'=a+d\}\big)^*$$

Example: Push-around Cart



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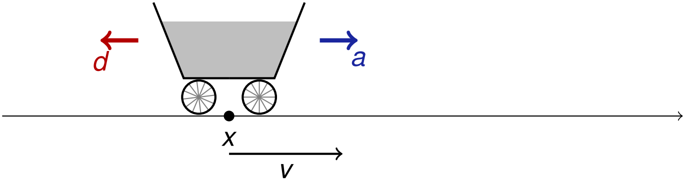

$$\big((a := 1 \cup a := -1); (d := 1 \cap d := -1); \{x' = v, v' = a + d\}\big)^*$$

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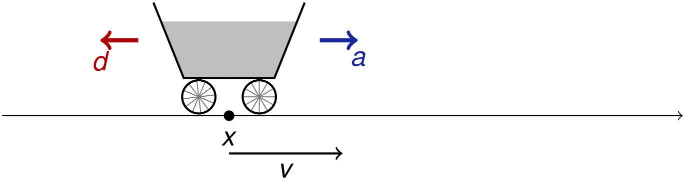
Example: Push-around Cart



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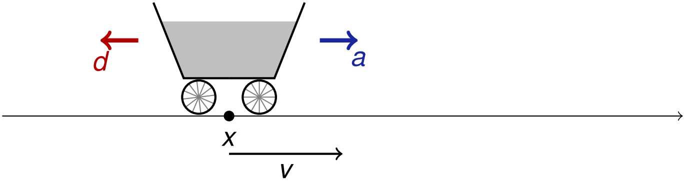
$$\text{HP}\ \big((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$$



$$\big((a:=1 \cup a:=-1);(d:=1 \cap d:=-1);\{x'=v,v'=a+d\}\big)^*$$

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HP $\big((d:=1 \cup d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\}\big)^*$

Hybrid systems can't say that a is Angel's choice and d is Demon's.
Only that there are choices.

Definition (Hybrid game α)

$$\alpha,\beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Differential Game Logic: Syntax

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$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P, Q \ ::= \ e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game

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All Reals | Some Reals

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

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Differential Game Logic: Syntax

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All Reals | Some Reals | Angel Wins

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

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Definition (dGL Formula P)

$$P, Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle \alpha \rangle P \mid [\alpha] P$$

All Reals | Some Reals | Angel Wins | Demon Wins

$$\langle (x := x+1; (x' = 1)^{d} \cup x := x-1)^{*} \rangle\, (0 \le x < 1)$$

$$\langle (x := x+1; (x' = 1)^{d} \cup (x := x-1 \cap x := x-2))^{*} \rangle (0 \le x < 1)$$

Simple Examples

$$\vDash \langle (x := x+1; (x' = 1)^d \cup x := x-1)^* \rangle (0 \le x < 1)$$

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Simple Examples

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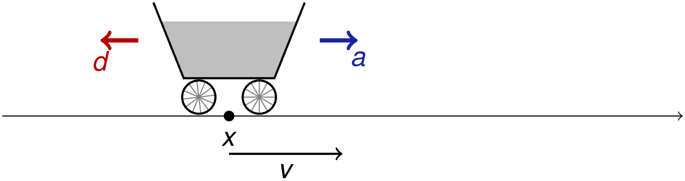
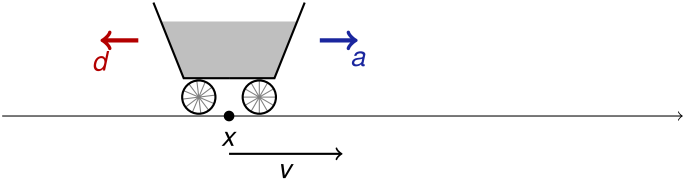
Example: Push-around Cart



$$v \geq 1 \rightarrow$$
$$\left[\left((d := 1 \cup d := -1)^{d}; (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^{*}\right] v \geq 0$$



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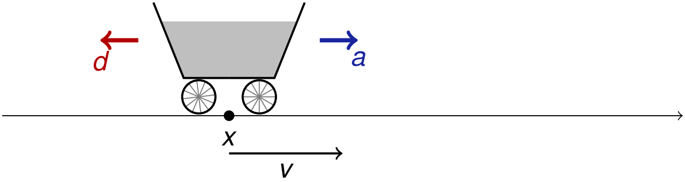
Example: Push-around Cart



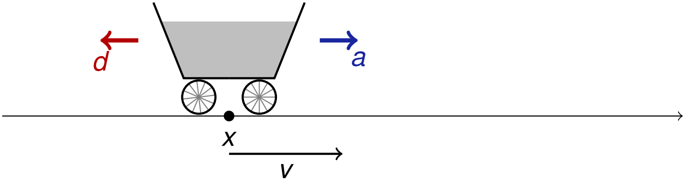
$\vDash v \geq 1 \rightarrow$ *d* before *a* can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, v \geq 0$$

$x \geq 0 \wedge v \geq 0 \rightarrow$

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, x \geq 0$$

Example: Push-around Cart

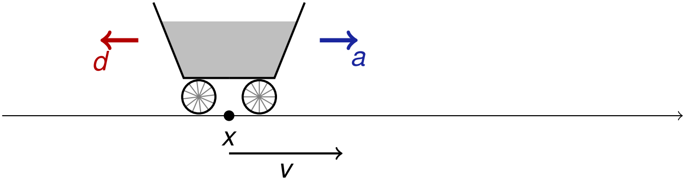


$\vDash v \geq 1 \rightarrow$ $\quad$ *d* before *a* can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

$\vDash x \geq 0 \land v \geq 0 \rightarrow$ $\quad$ *d* before *a* can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] x \geq 0$$

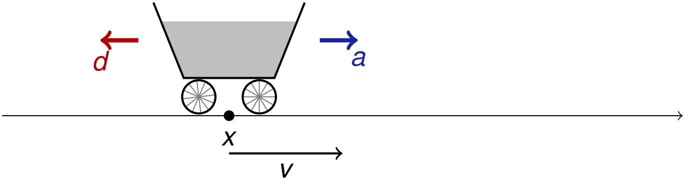


$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*] v \geq 0$$

$x \geq 0 \rightarrow$

$$\langle ((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^* \rangle x \geq 0$$



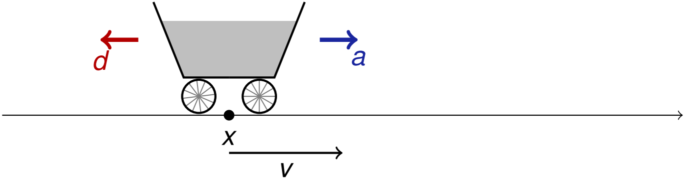
$\vDash v \geq 1 \rightarrow$ $\quad$ d before a can compensate

$$[((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*]\, v \geq 0$$

$\nvDash x \geq 0 \rightarrow$ $\quad$ boring by skip

$$\langle((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\})^*\rangle\, x \geq 0$$

Example: Push-around Cart



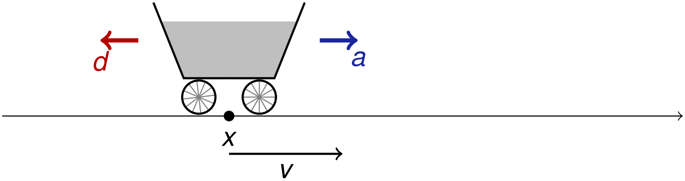
$\vDash v \geq 1 \rightarrow$

d before a can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

$$\left\langle\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$$

Example: Push-around Cart



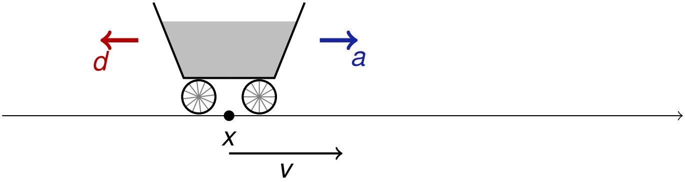
$\vDash v \geq 1 \rightarrow$ $\qquad$ d before a can compensate

$$\left[\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right] v \geq 0$$

$\not\vDash$ $\qquad$ counterstrategy $d := -1$

$$\left\langle\left((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\}\right)^*\right\rangle x \geq 0$$

Example: Push-around Cart



$\vDash v \geq 1 \rightarrow$ *d* before *a* can compensate

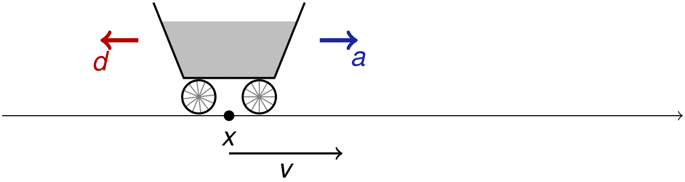
$$[((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*]\, v \geq 0$$

$\nvDash$ counterstrategy $d := -1$

$$\langle((d := 1 \cap d := -1); (a := 1 \cup a := -1); \{x' = v, v' = a + d\})^*\rangle\, x \geq 0$$

$$\langle((d := 1 \cap d := -1); (a := 2 \cup a := -2); \{x' = v, v' = a + d\})^*\rangle\, x \geq 0$$

Example: Push-around Cart



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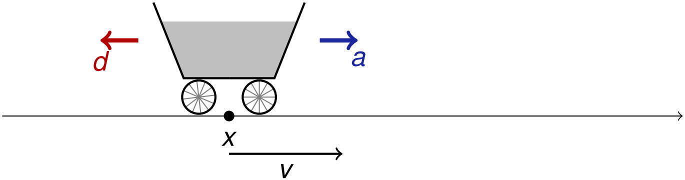

$\vDash v \geq 1 \rightarrow$ d before a can compensate

$$[((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v,v'=a+d\})^*]\, v \geq 0$$

$\not\vDash$ counterstrategy $d:=-1$

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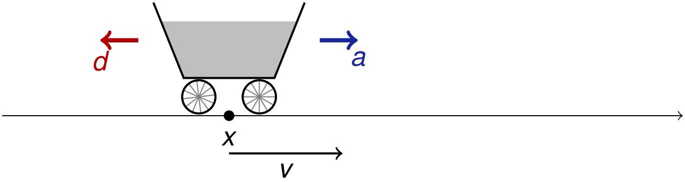
$$\vDash \langle((d:=1 \cap d:=-1);\,(a:=2 \cup a:=-2);\,\{x'=v,v'=a+d\})^*\rangle\, x \geq 0$$

$$\langle((d:=2 \cap d:=-2);\,(a:=2 \cup a:=-2);\; t:=0;\,\{x'=v,v'=a+d,t'=1\,\&\,t \leq 1\})^*\rangle\, x^2 \geq 100$$

Example: Push-around Cart



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$\not\vDash$ $\quad$ counterstrategy $d:=-1$

$$\left\langle\left((d:=1 \cap d:=-1);(a:=1 \cup a:=-1);\{x'=v, v'=a+d\}\right)^*\right\rangle x \geq 0$$

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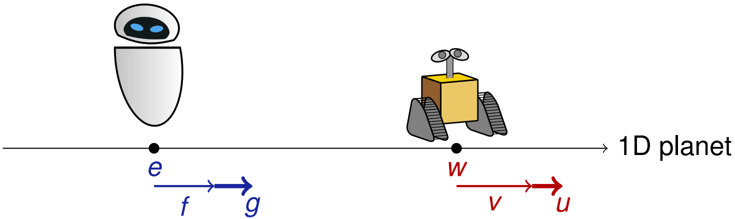
$$\vDash \left\langle\left((d:=2 \cap d:=-2);\ (a:=2 \cup a:=-2);\ t:=0;\ \{x'=v, v'=a+d, t'=1 \,\&\, t \leq 1\}\right)^*\right\rangle x^2 \geq 100$$

$a:=d$ then $a:=2\,\mathrm{sign}\,v$

Example: WALL·E and EVE Robot Dance

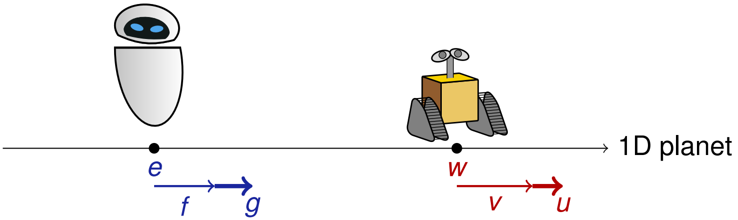


$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \rightarrow \\
&\langle ((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^{d} \\
&)^{\times} \rangle\, (w-e)^2 \le 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u

Example: WALL·E and EVE Robot Dance and the World

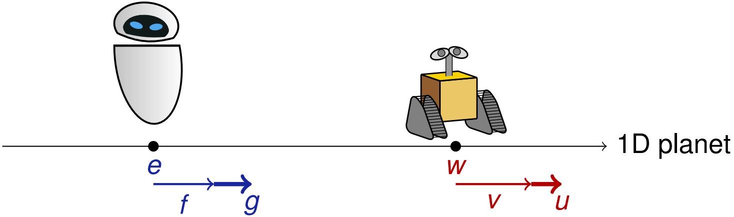


$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \rightarrow \\
&\Big\langle \Big((u := 1 \cap u := -1); \\
&\quad (g := 1 \cup g := -1); \\
&\quad t := 0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\}^{d} \\
&\Big)^{\times} \Big\rangle\, (w-e)^2 \le 1
\end{aligned}
$$

EVE at e plays Angel's part controlling g

WALL·E at w plays Demon's part controlling u and world time

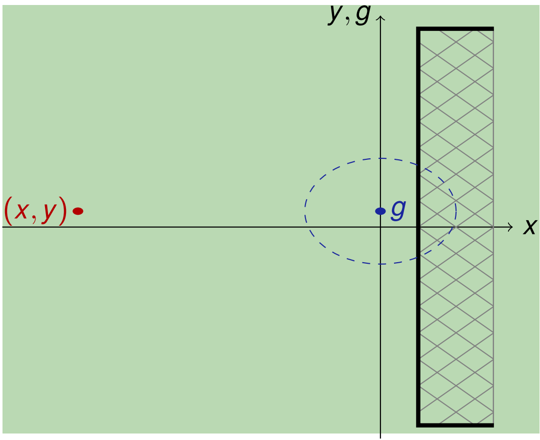
Example: EVE and WALL·E Robot Dance



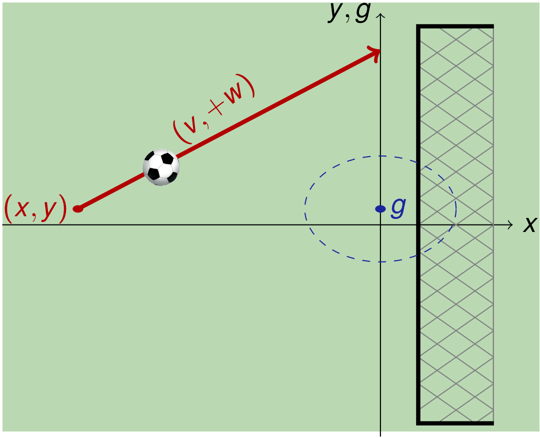
$$
\begin{aligned}
&(w-e)^2 \le 1 \land v = f \rightarrow \\
&[((u:=1 \cap u:=-1); \\
&\quad (g:=1 \cup g:=-1); \\
&\quad t:=0; \\
&\quad \{w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1\} \\
&)^{\times}]\ (w-e)^2 > 1
\end{aligned}
$$

WALL·E at w plays Demon's part controlling u and world time

EVE at e plays Angel's part controlling g



$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\Big\langle (w := +w \cap w := -w);$$
$$\Big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\} \Big)^* \Big\rangle\, x^2 + (y - g)^2 \leq 1$$



$$x < 0 \land v > 0 \land y = g \rightarrow$$
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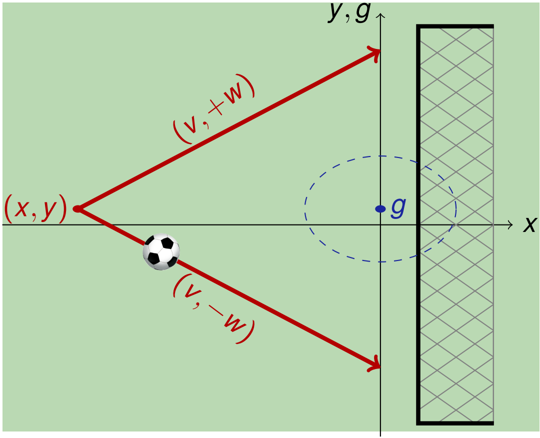
Example: Goalie in Robot Soccer



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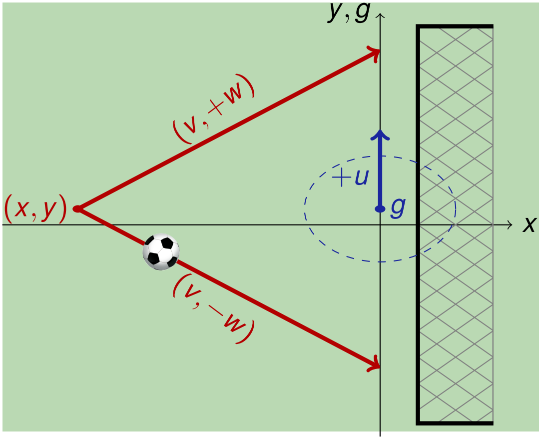

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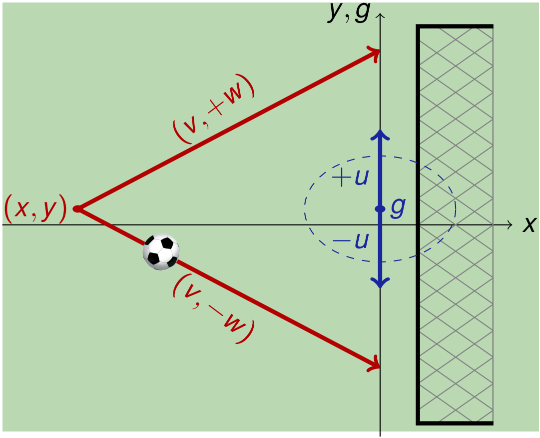
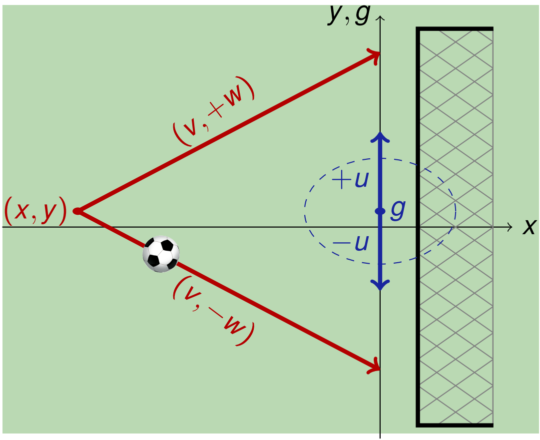


$$x < 0 \land v > 0 \land y = g \rightarrow \langle (w := +w \cap w := -w); ((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\})^* \rangle\, x^2 + (y-g)^2 \le 1$$

Example: Goalie in Robot Soccer



Goalie's Secret

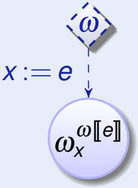
$$\left(\frac{x}{v}\right)^2 (u-w)^2 \le 1 \land$$

$$x < 0 \land v > 0 \land y = g \rightarrow$$
$$\langle (w := +w \cap w := -w);$$
$$\big((u := +u \cup u := -u); \{x' = v, y' = w, g' = u\}\big)^* \rangle\, x^2 + (y-g)^2 \le 1$$

Outline

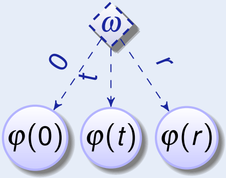
Definition (Hybrid game α: operational semantics)

$x := e$

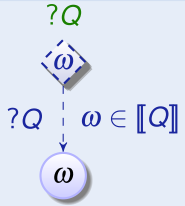


Definition (Hybrid game α: operational semantics)

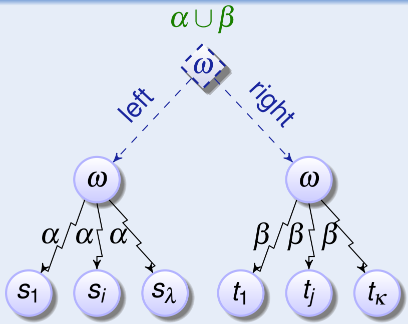
$$x' = f(x) \,\&\, Q$$



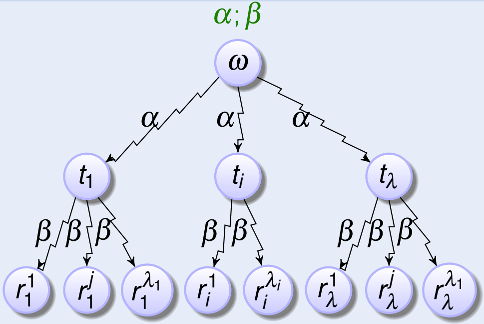
Definition (Hybrid game α: operational semantics)



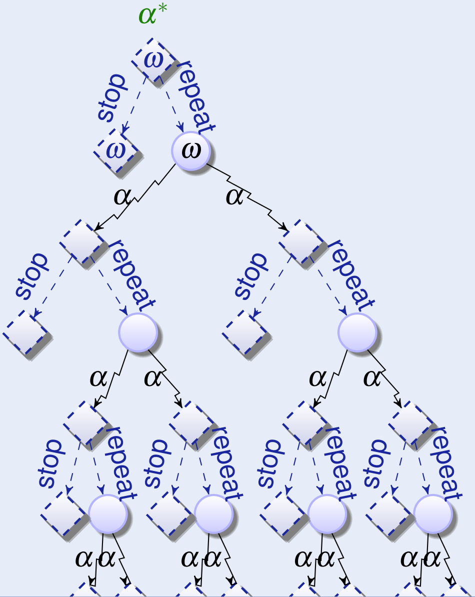
Definition (Hybrid game α: operational semantics)



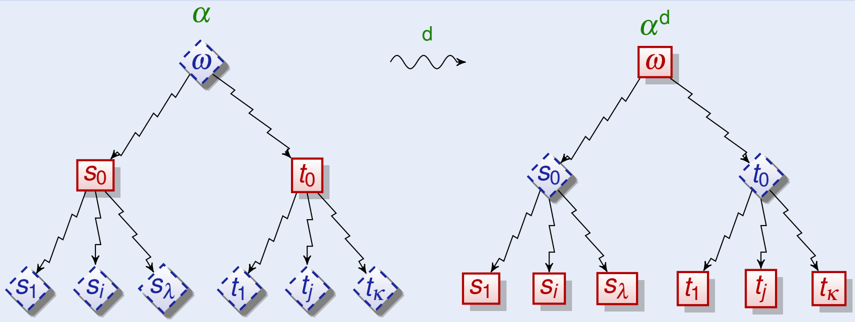
Definition (Hybrid game α: operational semantics)



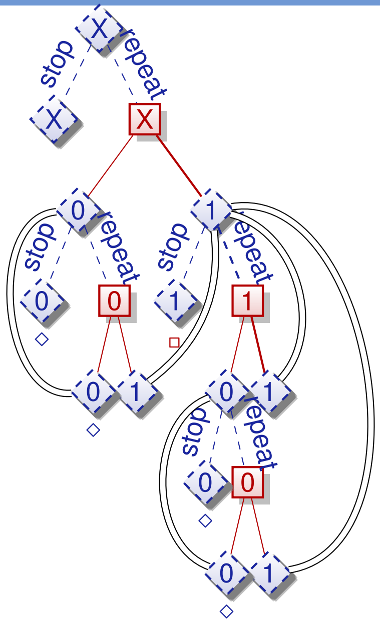
Definition (Hybrid game α: operational semantics)



Definition (Hybrid game α: operational semantics)

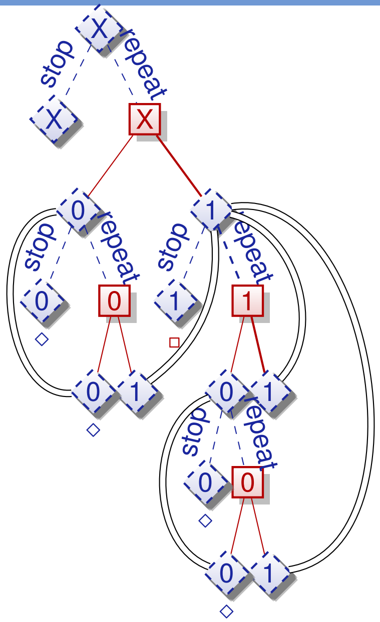


$$\langle(x:=0 \cap x:=1)^*\rangle x=0$$



$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$

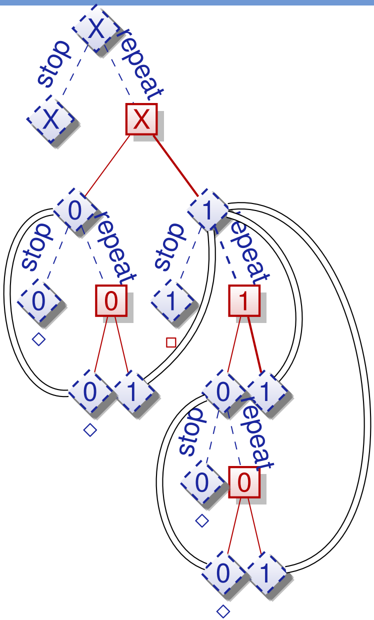


$\langle (x' = 1^{d}; x := 0)^{*} \rangle x = 0$

$\langle (x := 0; x' = 1^{d})^{*} \rangle x = 0$

$\langle (x := 0 \cap x := 1)^{*} \rangle x = 0$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$



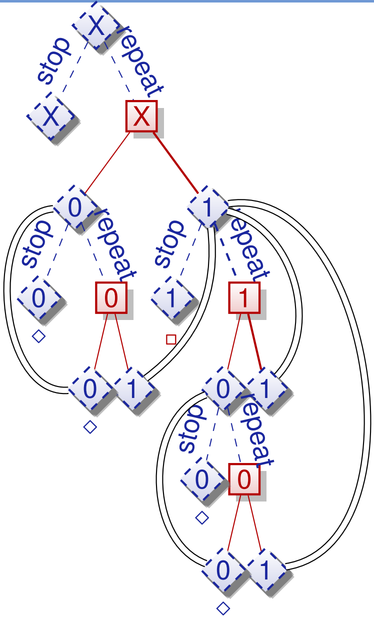
$\overset{<\infty}{\rightsquigarrow}$ true

$\langle (x' = 1^{\mathrm{d}}; x := 0)^* \rangle x = 0$

$\langle (x := 0; x' = 1^{\mathrm{d}})^* \rangle x = 0$

$\langle (x := 0 \cap x := 1)^* \rangle x = 0$

$\overset{\mathrm{wfd}}{\rightsquigarrow}$ false unless $x = 0$



$\overset{<\infty}{\rightsquigarrow}$ true

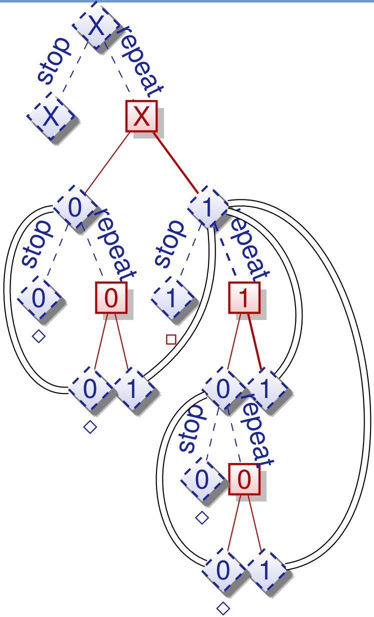
$$\langle (x' = 1^{\mathrm{d}}; x := 0)^* \rangle x = 0$$

$$\langle (x := 0; x' = 1^{\mathrm{d}})^* \rangle x = 0$$

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0$$

$\overset{\text{wfd}}{\rightsquigarrow}$ false unless $x = 0$

Well-defined games
can't be postponed forever!



Outline

Differential Game Logic: Syntax

Discrete Assign | Test Game | Differential Equation | Choice Game | Seq. Game | Repeat Game | Dual Game

Definition (Hybrid game α)

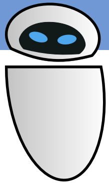
$$\alpha,\beta ::= x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha;\beta \mid \alpha^* \mid \alpha^d$$

Definition (dGL Formula P)

$$P,Q ::= e \geq \tilde{e} \mid \neg P \mid P \wedge Q \mid \forall x\, P \mid \exists x\, P \mid \langle\alpha\rangle P \mid [\alpha]P$$

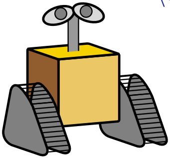
All Reals | Some Reals | Angel Wins | Demon Wins

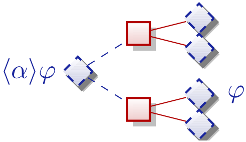
Summary



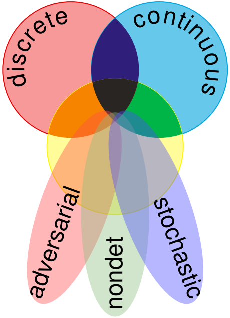
differential game logic

$\text{dGL} = \text{GL} + \text{HG} = \text{dL} + {}^{\text{d}}$





- Differential game logic
- Logic for hybrid games
- Compositional PL + logic
- Discrete + continuous + adversarial
- Operational semantics (informally)



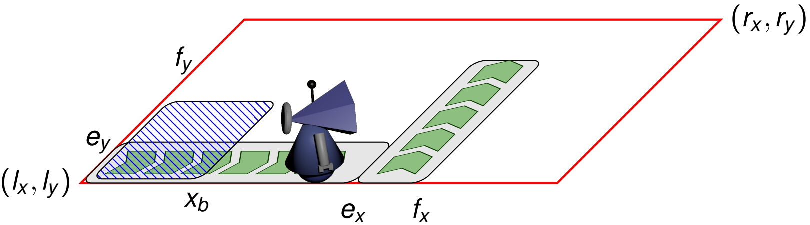
Next chapter

1. Formal semantics

Outline

7. Example: Robot Factory

Example: Robot Factory Decentralized Automation



Model

- (x, y) robot coordinates
- (v_x, v_y) velocities
- conveyor belts may instantaneously increase robot's velocity by (c_x, c_y)

Primary objectives of the robot

- Leave within time ε
- Never leave outer

Challenges

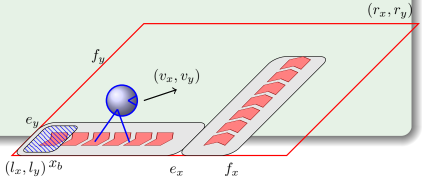
- Distributed, physical environment
- Possibly conflicting secondary objectives

Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \land y < e_y \land \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \leq x \land y \leq f_y \land \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$$

$$\Big)^*$$



Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

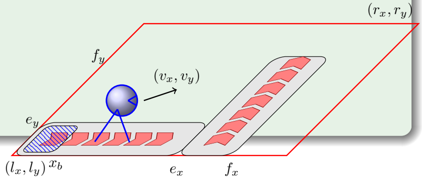
$$\cup (?(e_x \leq x \wedge y \leq f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ \big);$$

$$(\ a_x := *;\ ?(-A \leq a_x \leq A);$$

$$a_y := *;\ ?(-A \leq a_y \leq A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0\ )^d;$$

$$\Big)^*$$



Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\ \big);$$

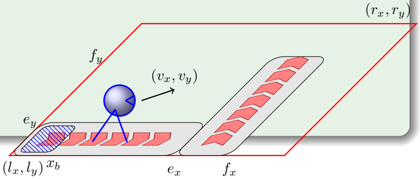
$$(\, a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0\ )^{d};$$

$$(\, x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \le \varepsilon\ );$$

$$\Big)^*$$



Example (Robot-Demon vs. Angel-Factory Environment)

$$\Big(\big(?true \cup (?(x < e_x \wedge y < e_y \wedge \mathrm{eff}_1 = 1);\ v_x := v_x + c_x;\ \mathrm{eff}_1 := 0) \qquad \text{// belt}$$

$$\cup (?(e_x \le x \wedge y \le f_y \wedge \mathrm{eff}_2 = 1);\ v_y := v_y + c_y;\ \mathrm{eff}_2 := 0)\big);$$

$$(a_x := *;\ ?(-A \le a_x \le A);$$

$$a_y := *;\ ?(-A \le a_y \le A); \qquad \text{// “independent” robot acceleration}$$

$$t_s := 0\,)^d;$$

$$\big((x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1\ \&\ t_s \le \varepsilon\,);$$

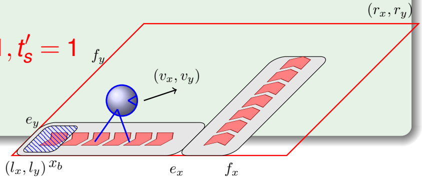
$$\cap (?(a_x v_x \le 0 \wedge a_y v_y \le 0)^d; \qquad \text{// brake}$$

$$\text{if } v_x = 0 \text{ then } a_x := 0 \text{ fi}; \qquad \text{// per direction: no time lock}$$

$$\text{if } v_y = 0 \text{ then } a_y := 0 \text{ fi};$$

$$(x' = v_x, y' = v_y, v_x' = a_x, v_y' = a_y, t' = 1, t_s' = 1$$

$$\&\ t_s \le \varepsilon \wedge a_x v_x \le 0 \wedge a_y v_y \le 0))\big)\Big)^*$$



Robot Factory Automation (*RF*)

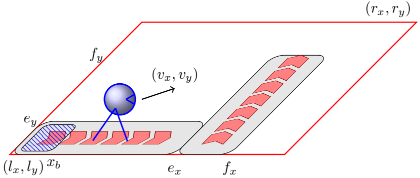
Proposition (Robot stays in □)

$$\models (x = y = 0 \land v_x = v_y = 0 \land \text{Controllability Assumptions})$$
$$\rightarrow [RF](x \in [l_x, r_x] \land y \in [l_y, r_y])$$

Proposition (Stays in □ and leaves ▨ on time)

$RF|_x$: RF projected to the x-axis

$$\models (x = 0 \land v_x = 0 \land \text{Controllability Assumptions})$$
$$\rightarrow [RF|_x](x \in [l_x, r_x] \land (t \geq \varepsilon \rightarrow x \geq x_b))$$



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