15-819/18-879: Logical Analysis of Hybrid Systems

06: Propositional and First-Order Logic

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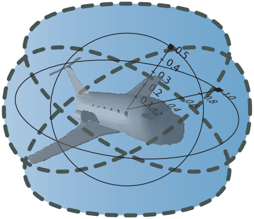
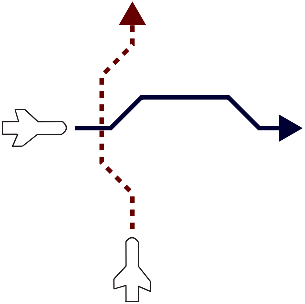


Outline

1. Propositional Logic
   - Motivation
   - Syntax
   - Semantics
   - Validity
2. First-order Logic
   - Motivation
   - Syntax
   - Semantics

Hybrid Automaton for Collision Avoidance



Example (Property)

If the aircraft are far apart and have compatible speed, then—when following the protocol—they will never crash?

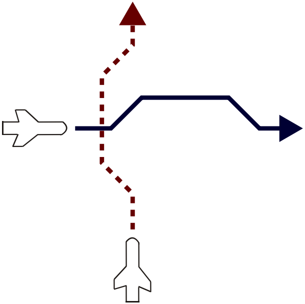
Example (Property)

If the aircraft enter collision avoidance, then—when following the protocol—will they ever leave again, i.e. follow their old route?

Example (System behavior)

If the aircraft are coming too close and I am flying northbound, then we will initiate left evasive actions.

Hybrid Automaton for Collision Avoidance



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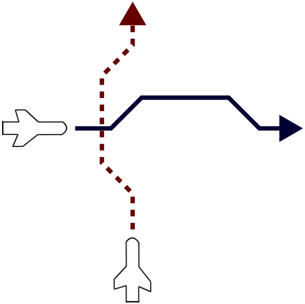

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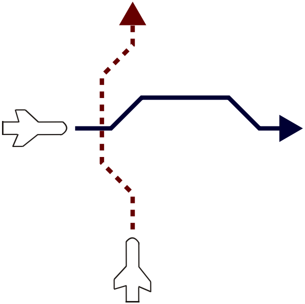

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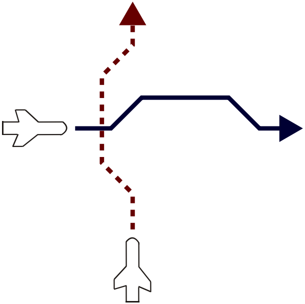

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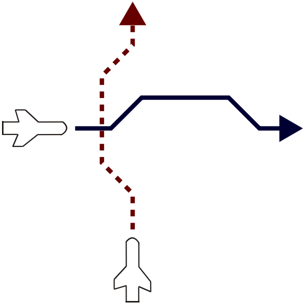
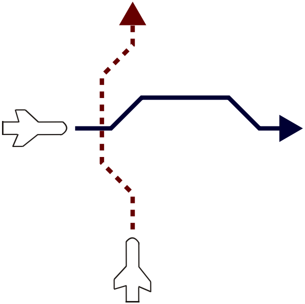


Example (Informal&incomplete reasoning)

If I turn left then so will he, thus we cannot come closer. Further, he will not turn back unless we have sufficient distance. Since we do not change course early later on, we will keep the distance and can never crash.

Hybrid Automaton for Collision Avoidance



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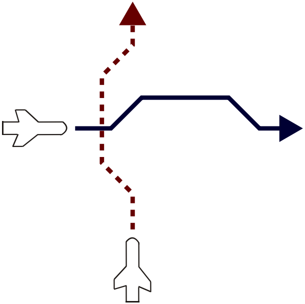

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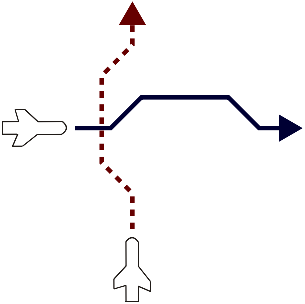

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In hybrid systems, there is significant logical structure in the properties, the system, the reasoning, ...

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Propositional Logic: Syntax

Definition (PL_0 Vocabulary V)

A set V of propositional variables / letters

Propositional Logic: Syntax

Definition (PL_0 Vocabulary V)

A set V of propositional variables / letters

Example

$$V = \{\mathrm{nb}_x, \mathrm{nb}_y, \mathrm{compat}, p_1, p_2, p_3, \dots\}$$

Propositional Logic: Syntax

Definition (PL_0 Vocabulary V)

A set V of propositional variables / letters

Definition (PL_0 Formula F, G)

$F ::=$

p	for any $p \in V$
$\neg F$	"not"
$(F \wedge G)$	"and"
$(F \vee G)$	"or"
$(F \rightarrow G)$	"implies"
$(F \leftrightarrow G)$	"equivalent/bi-implies"

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Example (Northbound flight of aircraft x and y is compatible)

$$(\mathrm{nb}_x \wedge \mathrm{nb}_y) \rightarrow \mathrm{compat}$$

Quiz

Is this a propositional formula?

1. $F \vee (G \wedge (H \leftrightarrow \neg F) \rightarrow J)$
2. $A \rightarrow (B \wedge (\neg F \leftrightarrow$
3. $x > y \leftrightarrow x - y > 0$
4. $A \rightarrow B \vee (\neg B \wedge C)$

Is this a propositional formula?

- ✓ $F \lor (G \land (H \leftrightarrow \neg F) \to J)$
2. $A \to (B \land (\neg F \leftrightarrow$
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Is this a propositional formula?

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Propositional Logic: Semantics

Definition (PL_0 Truth-assignment I)

Assignment $I : V \to \{true, false\}$ of truth-values to propositional variables

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Assignment $I : V \to \{true, false\}$ of truth-values to propositional variables

Example

v	nb_x	nb_y	compat
$I(v)$	*false*	*true*	*true*

Propositional Logic: Semantics

Definition (PL_0 Truth-assignment I)

Assignment $I : V \to \{true, false\}$ of truth-values to propositional variables

Definition (PL_0 Interpretation $[\![\cdot]\!]_I$)

$[\![\cdot]\!]_I$ is inductively defined as

$$
\begin{aligned}
[\![p]\!]_I &= I(p) \text{ for any } p \in V \\
[\![\neg F]\!]_I &= true \text{ iff } [\![F]\!]_I = false \\
[\![F \land G]\!]_I &= true \text{ iff } [\![F]\!]_I = true \text{ and } [\![G]\!]_I = true \\
[\![F \lor G]\!]_I &= true \text{ iff } [\![F]\!]_I = true \text{ or } [\![G]\!]_I = true \\
[\![F \to G]\!]_I &= true \text{ iff } [\![F]\!]_I = false \text{ or } [\![G]\!]_I = true \\
[\![F \leftrightarrow G]\!]_I &= true \text{ iff, either, } [\![F]\!]_I = true \text{ and } [\![G]\!]_I = true \\
&\qquad \text{or, instead, } [\![F]\!]_I = false \text{ and } [\![G]\!]_I = false
\end{aligned}
$$

Propositional Logic: Semantics

Example (Truth tables)

$I(\mathrm{nb}_x)$	$I(\mathrm{nb}_y)$	$I(\mathrm{compat})$	$[\![\mathrm{nb}_x \wedge nb_y]\!]_I$

Propositional Logic: Semantics

Example (Truth tables)

$I(\mathrm{nb}_x)$	$I(\mathrm{nb}_y)$	$I(\mathrm{compat})$	$[\![\mathrm{nb}_x \land nb_y]\!]_I$	$[\![\mathrm{nb}_x \land nb_y \to \mathrm{compat}]\!]_I$
true	*true*	*true*	*true*	

Propositional Logic: Semantics

Example (Truth tables)

$I(\text{nb}_x)$	$I(\text{nb}_y)$	$I(\text{compat})$	$[\![\text{nb}_x \land nb_y]\!]_I$	$[\![\text{nb}_x \land nb_y \to \text{compat}]\!]_I$
true	*true*	*true*	*true*	*true*
true	*true*	*false*	*true*	

Propositional Logic: Semantics

Example (Truth tables)

$I(\mathrm{nb}_x)$	$I(\mathrm{nb}_y)$	$I(\mathrm{compat})$	$\llbracket \mathrm{nb}_x \land nb_y \rrbracket_I$	$\llbracket \mathrm{nb}_x \land nb_y \rightarrow \mathrm{compat} \rrbracket_I$
true	*true*	*true*	*true*	*true*
true	*true*	*false*	*true*	*false*
true	*false*	*true*	*false*	

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true	*true*	*true*	*true*	*true*
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true	*false*	*true*	*false*	*true*
true	*false*	*false*	*false*	*true*
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Validity

Definition (Validity / Tautology)

Formula F is *valid* iff $[\![F]\!]_I = true$ for all assignments I, otherwise *invalid*

Example

Is this formula valid: $\text{nb}_x \land nb_y \rightarrow \text{separate}$?

Validity

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Example

Is this formula valid: $A \rightarrow (B \vee C \leftrightarrow (B \rightarrow \neg C \vee A))$?

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Is this formula valid: F ?

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How do you check validity of a formula?

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Validity and satisfiability for propositional logic are decidable (by exhaustive enumeration) in exponential time.

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Example

Is this formula valid: F ?

Problem

How do you check validity of a formula?

Proposition

Validity and satisfiability for propositional logic are decidable (by exhaustive enumeration) in exponential time. Better: SAT solving!

Quiz

1. Example for a valid formula?
2. Example for an invalid formula?
3. Example for an unsatisfiable formula?

Quiz

Is this a valid/invalid/satisfiable/unsatisfiable propositional formula?

1. $A \rightarrow (B \rightarrow A)$
2. $\neg(A \land B) \leftrightarrow \neg A \lor \neg B$
3. $A \rightarrow B \lor (\neg B \land C)$
4. $(A \rightarrow (\neg A \rightarrow B)) \rightarrow B$
5. $\big((A \leftrightarrow B) \leftrightarrow C\big) \leftrightarrow \big(A \leftrightarrow (B \leftrightarrow C)\big)$
6. $x < 0 \rightarrow x^2 > 0$
7. $\big(A \rightarrow (B \rightarrow C)\big) \leftrightarrow \big((A \rightarrow B) \rightarrow (A \rightarrow C)\big)$

Quiz

Is this a valid/invalid/satisfiable/unsatisfiable propositional formula?

1. ✓ $A \rightarrow (B \rightarrow A)$
2. $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
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Is this a valid/invalid/satisfiable/unsatisfiable propositional formula?

✓ $A \rightarrow (B \rightarrow A)$

✓ $\neg(A \land B) \leftrightarrow \neg A \lor \neg B$

× $A \rightarrow B \lor (\neg B \land C)$

4. $(A \rightarrow (\neg A \rightarrow B)) \rightarrow B$
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× $A \rightarrow B \lor (\neg B \land C)$

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Quiz

Is this a valid/invalid/satisfiable/unsatisfiable propositional formula?

✓ $A \to (B \to A)$

✓ $\neg(A \land B) \leftrightarrow \neg A \lor \neg B$

× $A \to B \lor (\neg B \land C)$

× $(A \to (\neg A \to B)) \to B$

✓ $((A \leftrightarrow B) \leftrightarrow C) \leftrightarrow (A \leftrightarrow (B \leftrightarrow C))$

6. $x < 0 \to x^2 > 0$

7. $(A \to (B \to C)) \leftrightarrow ((A \to B) \to (A \to C))$

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Is this a valid/invalid/satisfiable/unsatisfiable propositional formula?

- ✓ $A \rightarrow (B \rightarrow A)$
- ✓ $\neg(A \land B) \leftrightarrow \neg A \lor \neg B$
- × $A \rightarrow B \lor (\neg B \land C)$
- × $(A \rightarrow (\neg A \rightarrow B)) \rightarrow B$
- ✓ $((A \leftrightarrow B) \leftrightarrow C) \leftrightarrow (A \leftrightarrow (B \leftrightarrow C))$
- × $x < 0 \rightarrow x^2 > 0$
- 7 $(A \rightarrow (B \rightarrow C)) \leftrightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

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Is this a valid/invalid/satisfiable/unsatisfiable propositional formula?

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First-order Logic: Syntax

Definition (FOL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities

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Countable set of predicate or function symbols along with natural numbers as arities

Example (Aircraft vocabulary)

$$\Sigma = \{\mathrm{cruise}/1, \mathrm{separate}/2, \mathrm{faster}/2, \mathrm{ground}/1\} \cup \{\dots\}$$

First-order Logic: Syntax

Definition (FOL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities

Definition (FOL Term t)

$t ::=$

x	for variable $x \in V$
$f(t_1, \ldots, t_n)$	for function $f/n \in \Sigma$ of arity $n \geq 0$

First-order Logic: Syntax

Definition (FOL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities

Definition (FOL Formula F, G)

$F ::=$

$p(t_1, \ldots, t_n)$	for predicate $p/n \in \Sigma$ of arity $n \geq 0$
$\neg F$	"not"
$(F \land G)$	"and"
$(F \lor G)$	"or"
$(F \rightarrow G)$	"implies"
$(F \leftrightarrow G)$	"equivalent/bi-implies"
$\forall x\, F$	"universal quantifier/forall" for $x \in V$
$\exists x\, F$	"existential quantifier/exists" for $x \in V$

First-order Logic: Syntax

Definition (FOL Signature Σ)

Countable set of predicate or function symbols along with natural numbers as arities

Example (Aircraft vocabulary)

$$\Sigma = \{\text{cruise}/1, \text{separate}/2, \text{faster}/2, \text{ground}/1\} \cup \{\dots\}$$

Example (Cruising aircraft cannot crash)

$$(\forall x\, \text{cruise}(x)) \rightarrow \neg\exists x\, \exists y\, (\neg\text{separate}(x, y))$$

Quiz

Is this a first-order formula?

1. $F \lor (G \land (H \leftrightarrow \neg F) \to J)$
2. $\forall x\, (p(x) \to \exists y\, (p(y) \land \exists x\, \neg r(x, y)))$
3. $\forall x\, \exists y\, p(x \to f(y))$
4. $\forall x\, \forall y\, (x > y \leftrightarrow x - y > 0)$
5. $\forall x\, (\mathrm{human}(x) \to \mathrm{mortal}(x)) \land \mathrm{human}(\mathit{Sokrates}) \to \mathrm{mortal}(\mathit{Sokrates})$
6. $\forall x\, \forall y\, (e(x, y) \leftrightarrow \forall p\, (r(x) \land p(x) \leftrightarrow p(y)))$
7. $\forall x\, e(f(f(x)), x) \to \exists y\, e(f(f(f(y))), f(y))$

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Is this a first-order formula?

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× $\forall x\, \exists y\, p(x \rightarrow f(y))$

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- × $\forall x\, \exists y\, p(x \to f(y))$
- × $\forall x\, \forall y\, (x > y \leftrightarrow x - y > 0)$
- 5 $\forall x\, (\mathrm{human}(x) \to \mathrm{mortal}(x)) \land \mathrm{human}(Sokrates) \to \mathrm{mortal}(Sokrates)$
- 6 $\forall x\, \forall y\, (e(x,y) \leftrightarrow \forall p\, (r(x) \land p(x) \leftrightarrow p(y)))$
- 7 $\forall x\, e(f(f(x)), x) \to \exists y\, e(f(f(f(y))), f(y))$

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Is this a first-order formula?

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- ✓ $\forall x\, (p(x) \rightarrow \exists y\, (p(y) \land \exists x\, \neg r(x, y)))$
- × $\forall x\, \exists y\, p(x \rightarrow f(y))$
- × $\forall x\, \forall y\, (x > y \leftrightarrow x - y > 0)$
- ✓ $\forall x\, (\mathrm{human}(x) \rightarrow \mathrm{mortal}(x)) \land \mathrm{human}(Sokrates) \rightarrow \mathrm{mortal}(Sokrates)$

6. $\forall x\, \forall y\, (e(x, y) \leftrightarrow \forall p\, (r(x) \land p(x) \leftrightarrow p(y)))$
7. $\forall x\, e(f(f(x)), x) \rightarrow \exists y\, e(f(f(f(y))), f(y))$

Quiz

Is this a first-order formula?

- ✓ $F \lor (G \land (H \leftrightarrow \neg F) \rightarrow J)$
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Example (Cruising aircraft cannot crash)

$$(\forall x\, \mathrm{cruise}(x)) \rightarrow \neg\exists x\, \exists y\, (\neg \mathrm{separate}(x, y))$$

Is this formula true over the following signature?

Example (Aircraft vocabulary)

$$\Sigma = \{\mathrm{cruise}/1, \mathrm{separate}/2\} \cup \emptyset$$

Truth depends on the interpretation of symbols

First-order Logic: Semantics

Definition (FOL Interpretation I)

1. D non-empty set (*domain/universe*)
2. I assigns relations and functions on D to all symbols in Σ
   - function $I(f) : D^n \to D$ for each function symbol f of arity n
   - relation $I(p) \subseteq D^n$ for each predicate symbol p of arity n
   - element $I(c) \in D$ for each constant symbol (function of arity 0)
   - truth-value $I(p) \in \{true, false\}$ for each predicate symbol of arity 0

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   - truth-value $I(p) \in \{true, false\}$ for each predicate symbol of arity 0

Definition (FOL Variable Assignment β)

Assignment $\beta : V \to D$ of an element in domain D to each variable in V

First-order Logic: Semantics

Definition (FOL Valuation of terms $[\![\cdot]\!]_{I,\beta}$)

$[\![\cdot]\!]_{I,\beta}$ is inductively defined as

$$[\![x]\!]_{I,\beta} = \beta(x) \text{ for variable } x \in V$$
$$[\![f(t_1, \ldots, t_n)]\!]_{I,\beta} = I(f)([\![t_1]\!]_{I,\beta}, \ldots, [\![t_n]\!]_{I,\beta})$$

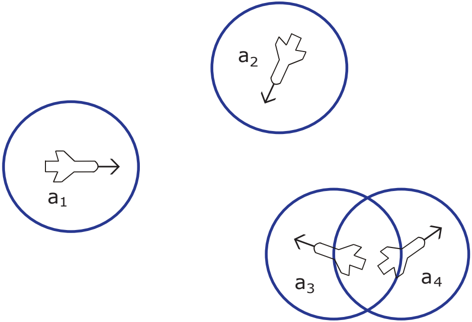
First-order Logic: Semantics

Definition (FOL Valuation of formulas $[\![\cdot]\!]_{I,\beta}$)

$[\![\cdot]\!]_{I,\beta}$ is inductively defined as

$$
\begin{aligned}
[\![p(t_1,\dots,t_n)]\!]_{I,\beta} &= \mathit{true} \text{ iff } ([\![t_1]\!]_{I,\beta},\dots,[\![t_n]\!]_{I,\beta}) \in I(p) \text{ for pred. } p \in \Sigma \\
[\![\neg F]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{false} \\
[\![F \land G]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{true} \text{ and } [\![G]\!]_{I,\beta} = \mathit{true} \\
[\![F \lor G]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{true} \text{ or } [\![G]\!]_{I,\beta} = \mathit{true} \\
[\![F \to G]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta} = \mathit{false} \text{ or } [\![G]\!]_{I,\beta} = \mathit{true} \\
[\![\forall x\, F]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta'} = \mathit{true} \text{ for all } \beta' \text{ that are like } \beta \\
&\qquad \text{except for the value of } x \\
[\![\exists x\, F]\!]_{I,\beta} &= \mathit{true} \text{ iff } [\![F]\!]_{I,\beta'} = \mathit{true} \text{ for some } \beta' \text{ that is like } \beta \\
&\qquad \text{except for the value of } x
\end{aligned}
$$

One Example for an Interpretation



Example (Cruising aircraft cannot crash)

$$(\forall x\, \text{cruise}(x)) \rightarrow \neg\exists x\, \exists y\, (\neg\text{separate}(x, y))$$

$$I_4(\text{cruise}) = \{a_1, a_3, a_4\}$$
$$I_4(\text{separate}) = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4)\}$$

Quiz

Is this a valid first-order formula?

1. $F \vee (G \wedge (H \leftrightarrow \neg F) \rightarrow J)$
2. $\forall x\, (p(x) \rightarrow \exists y\, (p(y) \wedge \exists x\, \neg r(x, y)))$
3. $\forall x\, (\text{human}(x) \rightarrow \text{mortal}(x)) \wedge \text{human}(Sokrates) \rightarrow \text{mortal}(Sokrates)$
4. $\forall x\, \forall y\, (e(x, y) \leftrightarrow \forall p\, (p(x) \leftrightarrow p(y)))$
5. $\forall x\, e(f(f(x)), x) \rightarrow \exists y\, e(f(f(f(y))), f(y))$
6. $\forall x\, (x > 0 \rightarrow \exists y\, x > y^2)$

Quiz

Is this a valid first-order formula?

× $F \lor (G \land (H \leftrightarrow \neg F) \to J)$

2. $\forall x\, (p(x) \to \exists y\, (p(y) \land \exists x\, \neg r(x, y)))$
3. $\forall x\, (\mathrm{human}(x) \to \mathrm{mortal}(x)) \land \mathrm{human}(Sokrates) \to \mathrm{mortal}(Sokrates)$
4. $\forall x\, \forall y\, (e(x, y) \leftrightarrow \forall p\, (p(x) \leftrightarrow p(y)))$
5. $\forall x\, e(f(f(x)), x) \to \exists y\, e(f(f(f(y))), f(y))$
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6. $\forall x\, (x > 0 \rightarrow \exists y\, x > y^2)$

Quiz

Is this a valid first-order formula?

- × $F \lor (G \land (H \leftrightarrow \neg F) \to J)$
- × $\forall x\, (p(x) \to \exists y\, (p(y) \land \exists x\, \neg r(x, y)))$
- ✓ $\forall x\, (\text{human}(x) \to \text{mortal}(x)) \land \text{human}(\mathit{Sokrates}) \to \text{mortal}(\mathit{Sokrates})$
- × $\forall x\, \forall y\, (e(x, y) \leftrightarrow \forall p\, (p(x) \leftrightarrow p(y)))$
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- × $\forall x\, (x > 0 \rightarrow \exists y\, x > y^2)$
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