15-819/18-879: Logical Analysis of Hybrid Systems
05: Differential Equations

André Platzer

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1 Differential Equations

- Intuition
- ODE & IVP
- Examples
- Peano Existence
- Picard-Lindelöf Uniqueness
Outline

1. Differential Equations
   - Intuition
   - ODE & IVP
   - Examples
   - Peano Existence
   - Picard-Lindelöf Uniqueness
How to describe continuous change?

Relate continuously changing quantity and its rate of change (derivative)
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Relate continuously changing quantity and its rate of change (derivative)

\[
\begin{align*}
y'(t) &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]
How to describe continuous change?

Relate continuously changing quantity and its rate of change (derivative)

\[
\begin{align*}
y'(t) &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]

in which direction \(y\) evolves as time \(t\) progresses

where \(y\) starts at time \(t_0\)
\[
\begin{aligned}
  x'(t) &= \frac{1}{4} x(t) \\
  x(t_0) &= 1
\end{aligned}
\]
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Intuition for Differential Equations

\[
x' \left( t \right) = \frac{1}{4} x \left( t \right)
\]

\[
x(\ t_0) = 1
\]

\[
\Delta = 1 \quad \Delta = 2 \quad \Delta = 4
\]

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Intuition for Differential Equations

\[ x'(t) = \frac{1}{4}x(t) \]
\[ x(t_0) = 1 \]

\[ \Delta = \frac{1}{2} \]
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\[
x'(t) = \frac{1}{4}x(t) \\
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\]
\[
\begin{align*}
\Delta = \frac{1}{2} & \\
\Delta = 1 & \\
\Delta = 2 & \\
\Delta = 4 & \\
\end{align*}
\]
\[
\begin{align*}
x(t + \Delta) & := x(t) + \frac{1}{4}x(t)\Delta \\
x(t_0) & := 1
\end{align*}
\]
Definition (Ordinary Differential Equation, ODE)

\[ f : D \to \mathbb{R}^n \text{ on domain } D \subseteq \mathbb{R} \times \mathbb{R}^n \text{ (i.e., open connected). Then } \]

\[ Y : I \to \mathbb{R}^n \text{ is solution of IVP} \]

\[ \begin{bmatrix} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{bmatrix} \]

on interval \( I \subseteq \mathbb{R} \), iff, for all \( t \in I \),
Definition (Ordinary Differential Equation, ODE)

\( f : D \rightarrow \mathbb{R}^n \) on domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) (i.e., open connected). Then \( Y : I \rightarrow \mathbb{R}^n \) is solution of IVP

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on interval \( I \subseteq \mathbb{R} \), iff, for all \( t \in I \),

1. \( (t, Y(t)) \in D \)
### Definition (Ordinary Differential Equation, ODE)

Let $f : D \rightarrow \mathbb{R}^n$ be a function defined on the domain $D \subseteq \mathbb{R} \times \mathbb{R}^n$ (i.e., open connected). Then $Y : I \rightarrow \mathbb{R}^n$ is a solution of the initial-value problem (IVP)

\[
\begin{align*}
    y'(t) &= f(t, y) \\
    y(t_0) &= y_0
\end{align*}
\]

on interval $I \subseteq \mathbb{R}$, if and only if, for all $t \in I$,

1. $(t, Y(t)) \in D$
2. $Y'(t)$ exists and $Y'(t) = f(t, Y(t))$.

Accordingly, for higher-order differential equations, i.e., differential equations involving higher-order derivatives $y^{(n)}(t)$. If $f \in C(D, \mathbb{R}^n)$, then $Y \in C^1(I, \mathbb{R}^n)$. 
Definition (Ordinary Differential Equation, ODE)

\[ f : D \to \mathbb{R}^n \] on domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) (i.e., open connected). Then \( Y : I \to \mathbb{R}^n \) is solution of IVP

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Accordingly for higher-order differential equations, i.e., differential equations involving higher-order derivatives \( y^{(n)}(t) \).

If \( f \in C(D, \mathbb{R}^n) \), then \( Y \in C^1(I, \mathbb{R}^n) \).
What is a solution of the following IVP?

\[
\begin{bmatrix}
    y'(t) = y^2 \\
    y(0) = 1
\end{bmatrix}
\]

Solution:

\[y(t) = 1 - t\]

Proof.

\[
y'(t) = \frac{d}{dt}(1 - t)^2 = 0
\]

\[
-2(1 - t)(-1) = y(t)
\]

\[
y(0) = 1
\]

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Solution:

\[y(t) = \frac{1}{1-t}\]

Proof.

\[y'(t) = \frac{d}{dt} \frac{1}{1-t} = \frac{0 - \frac{d(1-t)}{dt}}{(1-t)^2} = \frac{1}{(1-t)^2} = y(t)^2\]

\[y(0) = \frac{1}{1-0} = 1\]
What is a solution of the following IVP?

\[
\begin{bmatrix}
  y'(t) = -2ty \\
  y(0) = 1
\end{bmatrix}
\]

Solution:

\[y(t) = e^{-t^2}\]

Proof.

\[y'(t) = de^{-t^2}/dt = e^{-t^2}(-2t) = -2ty\]

\[y(0) = e^{0^2} = 1\]
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\begin{cases}
y'(t) = -2ty \\ y(0) = 1
\end{cases}
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Solution:

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Proof.

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**Note:** The solution for $x'(t) = x^2 + x^4$ is non-analytic.
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<td>$x' = x, x(0) = x_0$</td>
<td>$x(t) = x_0e^t$</td>
</tr>
<tr>
<td>$x' = x^2, x(0) = x_0$</td>
<td>$x(t) = \frac{x_0}{1-tx_0}$</td>
</tr>
<tr>
<td>$x' = \frac{1}{x}, x(0) = 1$</td>
<td>$x(t) = \sqrt{1 + 2t} \ldots$</td>
</tr>
<tr>
<td>$y'(x) = -2xy, y(0) = 1$</td>
<td>$y(x) = e^{-x^2}$</td>
</tr>
<tr>
<td>$x'(t) = tx, x(0) = x_0$</td>
<td>$x(t) = x_0e^{\frac{t^2}{2}}$</td>
</tr>
<tr>
<td>$x' = \sqrt{x}, x(0) = x_0$</td>
<td>$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$</td>
</tr>
<tr>
<td>$x' = y, y' = -x, x(0) = 0, y(0) = 1$</td>
<td>$x(t) = \sin t, y(t) = \cos t$</td>
</tr>
<tr>
<td>$x' = 1 + x^2, x(0) = 0$</td>
<td>$x(t) = \tan t$</td>
</tr>
<tr>
<td>$x'(t) = \frac{2}{t^3}x(t)$</td>
<td>$x(t) = e^{-\frac{1}{t^2}}$ non-analytic</td>
</tr>
<tr>
<td>$x' = x^2 + x^4$</td>
<td>non-elementary</td>
</tr>
<tr>
<td>$x'(t) = e^{t^2}$</td>
<td>}</td>
</tr>
</tbody>
</table>
Existence: Peano

Theorem (Existence theorem of Peano’1890)

\[ f \in C(D, \mathbb{R}^n) \text{ on open, connected domain } D \subseteq \mathbb{R} \times \mathbb{R}^n \text{ with } (x_0, y_0) \in D. \]

Then, IVP has a solution. Further, every solution can be continued arbitrarily close to the border of D.

Example (Solvable)

\[
\begin{bmatrix}
  y' = \sqrt{|y|} \\
  y(0) = 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  y'(x) = 3x^2y - \frac{1}{y} \sin x \cos y \\
  y(0) = 1
\end{bmatrix}
\]
Example (Solvable but not uniquely)

\[
\begin{bmatrix}
y' = \sqrt{|y|} \\
y(0) = 0
\end{bmatrix}
\]

\[y_2(x) = \frac{x^2}{4}\]

\[y_1(x) = 0\]
Example (Solvable but not uniquely)

\[ y' = \sqrt{|y|} \]
\[ y(0) = 0 \]

\[ y_2(x) = \frac{x^2}{4} \]

\[ y_1(x) = 0 \]

Example (Solvable but not uniquely)

\[ y' = \sqrt[3]{y} \]
\[ y(0) = 0 \]

\( y(t) = \left(\frac{2}{3} t\right)^{\frac{3}{2}} \) or \( y(t) = 0 \)
Example (Continuable but limited)

\[
\begin{bmatrix}
y' &= 1 + y^2 \\
y(0) &= 0
\end{bmatrix}
\]

\[\Rightarrow y(t) = \tan(t)\]
Example (Continuable but limited)

\[
\begin{bmatrix}
  y' = 1 + y^2 \\
  y(0) = 0
\end{bmatrix} \implies y(t) = \tan t
\]
Lipschitz-Continuity

Definition (Lipschitz-continuous)

\( f : D \rightarrow \mathbb{R}^n \) with \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) is Lipschitz-continuous for \( y \) iff there is an \( L \in \mathbb{R} \) such that for all \( (x, y), (x, \bar{y}) \in D \):

\[
\| f(x, y) - f(x, \bar{y}) \| \leq L \| y - \bar{y} \|
\]
Definition (Lipschitz-continuous)

A function \( f : D \rightarrow \mathbb{R}^n \) with \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) is \textit{Lipschitz-continuous} for \( y \) iff there is an \( L \in \mathbb{R} \) such that for all \((x, y), (x, \bar{y}) \) in \( D \):

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\[
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\]
**Definition (Lipschitz-continuous)**

A function $f : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is **Lipschitz-continuous** for $y$ if for all $(x, y), (x, \bar{y}) \in D$:

$$\|f(x, y) - f(x, \bar{y})\| \leq L \|y - \bar{y}\|$$

where $L \in \mathbb{R}$ is a constant.
Definition (Lipschitz-continuous)

$f : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is Lipschitz-continuous for $y$ iff there is an $L \in \mathbb{R}$ such that for all $(x, y), (x, \bar{y}) \in D$:

$$\|f(x, y) - f(x, \bar{y})\| \leq L \|y - \bar{y}\|$$
Definition (Lipschitz-continuous)

$f : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is **Lipschitz-continuous** for \( y \) iff there is an $L \in \mathbb{R}$ such that for all $(x, y), (x, \bar{y}) \in D$:

$$
\|f(x, y) - f(x, \bar{y})\| \leq L\|y - \bar{y}\|
$$

If $\frac{\partial f(x,y)}{\partial y}$ exists and is bounded on $D$ then $f$ is Lipschitz-continuous. $f$ is **locally Lipschitz-continuous** iff for each $(x, y) \in D$, there is a neighborhood in which $f$ is Lipschitz-continuous.
Lipschitz-Continuity

**Definition (Lipschitz-continuous)**

Let $f : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be a function. $f$ is **Lipschitz-continuous** for $y$ iff there exists an $L \in \mathbb{R}$ such that for all $(x, y), (x, \bar{y}) \in D$:

$$
\|f(x, y) - f(x, \bar{y})\| \leq L \|y - \bar{y}\|
$$

If $\frac{\partial f(x,y)}{\partial y}$ exists and is bounded on $D$ then $f$ is Lipschitz-continuous. $f$ is **locally Lipschitz-continuous** iff for each $(x, y) \in D$, there is a neighborhood in which $f$ is Lipschitz-continuous.

If $f \in C^1(D, \mathbb{R}^n)$ then locally Lipschitz-continuous, as $f'$ locally bounded.
Theorem (Uniqueness theorem of Picard-Lindelöf’1894)

In addition to Peano premisses, let $f$ be locally Lipschitz-continuous for $y$ (e.g. $f \in C^1(D, \mathbb{R}^n)$). Then, there is a unique solution of IVP.
Existence & Uniqueness: Picard-Lindelöf / Cauchy-Lipschitz

**Theorem (Uniqueness theorem of Picard-Lindelöf’1894)**

In addition to Peano premisses, let $f$ be locally Lipschitz-continuous for $y$ (e.g. $f \in C^1(D, \mathbb{R}^n)$). Then, there is a unique solution of IVP.

**Proposition (Global uniqueness theorem of Picard-Lindelöf)**

$f \in C([0, a] \times \mathbb{R}^n, \mathbb{R}^n)$ Lipschitz-continuous for $y$. Then, there is a unique solution of IVP on $[0, a]$.
Example (Unique solution but not global)

\[
\begin{bmatrix}
y' &= -y^2 \\
y(0) &= -1
\end{bmatrix}
\]
P. Hartman.  
*Ordinary Differential Equations.*  

A. Platzer.  
*Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics.*  

W. T. Reid.  
*Ordinary Differential Equations.*  

W. Walter.  
*Ordinary Differential Equations.*  