15-819/18-879: Hybrid Systems Analysis & Theorem Proving
11: Differential-algebraic Dynamic Logic

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Outline

1 Motivation

2 Differential-algebraic Programs
   - Design Motives
   - Syntax
     - Discrete Jump Constraints
     - Differential-algebraic Constraints
     - Differential-algebraic Programs
   - Semantics
     - Differential-algebraic Constraints
     - Discrete Jump Constraints
     - Differential-algebraic Programs

3 Air Traffic Control

4 Differential-algebraic Dynamic Logic DAL
   - Syntax
   - Semantics
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   - Syntax
   - Semantics
Hybrid Systems

continuous evolution along differential equations + discrete change
Hybrid Systems

continuous evolution along differential equations + discrete change
Hybrid Systems

continuous evolution along differential equations + discrete change

\[
\begin{bmatrix}
    x_1' \\
    x_2' \\
    \vartheta'
\end{bmatrix} =
\begin{bmatrix}
    -v_1 + v_2 \cos \vartheta + \omega x_2 \\
    v_2 \sin \vartheta - \omega x_1 \\
    \varpi - \omega
\end{bmatrix}
\]
Example ("Solving" differential equations)

\[
x_1(t) = \frac{1}{\omega \varpi} \left( x_1 \omega \varpi \cos t \omega - v_2 \omega \cos t \omega \sin \vartheta + v_2 \omega \cos t \varpi \sin \vartheta - v_1 \varpi \sin t \omega \\
+ x_2 \omega \varpi \sin t \omega - v_2 \omega \cos \vartheta \cos t \varpi \sin t \omega - v_2 \omega \sqrt{1 - \sin^2 \vartheta} \sin t \omega \\
+ v_2 \omega \cos \vartheta \cos t \varpi \sin t \omega + v_2 \omega \sin \vartheta \sin t \omega \sin t \varpi \right) \ldots
\]
Example ("Solving" differential equations)

$$\forall t \geq 0 \frac{1}{\omega \varpi} \left( x_1 \omega \varpi \cos t \varpi - v_2 \omega \cos t \varpi \sin \theta + v_2 \omega \cos t \varpi \cos t \omega \sin \theta - v_1 \varpi \sin t \omega ight. $$

$$+ x_2 \omega \varpi \sin t \omega - v_2 \omega \cos \theta \cos t \omega \sin t \omega - v_2 \omega \sqrt{1 - \sin \theta^2} \sin t \omega$$

$$+ v_2 \omega \cos \theta \cos t \omega \sin t \omega + v_2 \omega \sin \theta \sin t \omega \sin t \omega \sin t \varpi \right) \ldots$$
Verification of Hybrid Systems & Air Traffic Control

Symbolic Verification
- constant/nilpotent dynamics
- otherwise “no” solutions
- sound

Numerical Verification
- challenging dynamics
- approximation errors
- unsound, ... see [PC07]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{\vartheta}
\end{bmatrix} =
\begin{bmatrix}
-v_1 + v_2 \cos \vartheta + \omega x_2 \\
v_2 \sin \vartheta - \omega x_1 \\
\varpi - \omega
\end{bmatrix}
\]
Verification of Hybrid Systems & Air Traffic Control

How To Get What We Really Need?
✓ challenging dynamics, e.g., curved flight
✓ automatic verification
✓ sound
Idea: Exploit Vector Field of Differential Equations

“Definition” (Differential Invariant)

“Property that remains true in the direction of the dynamics”
Idea: Exploit Vector Field of Differential Equations

“Definition” (Differential Invariant)

“Property that remains true in the direction of the dynamics”
### Verification Approaches for Hybrid Systems

**Table:**

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<thead>
<tr>
<th>problem</th>
<th>technique</th>
<th>Op</th>
<th>Par</th>
<th>T</th>
<th>Cl</th>
<th>Aut</th>
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<td>TL-MC</td>
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<td>×</td>
<td>✓</td>
<td>×</td>
<td>✓</td>
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<tr>
<td>$\models (Ax(\text{TRM}) \rightarrow z &lt; m)$</td>
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<td>✓</td>
<td>×</td>
<td>✓</td>
<td>×</td>
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</tbody>
</table>

**Diagrams:**

- Diagram 1: A hybrid system with entry and exit points.
- Diagram 2: A hybrid system with a circular trajectory $C$.

**André Platzer (CMU)**
Verification Approaches for Hybrid Systems

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differential-algebraic dynamic logic

$DAL = DL + DAP$
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differential-algebraic dynamic logic

\[
\text{DAL} = \text{FOL}_\mathbb{R} + \text{ML}
\]
differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL}_\mathbb{R} + \text{DL} \]
differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL}_\mathbb{R} + \text{DL} + \text{DAP} \]

\[
\begin{align*}
[ d_1' &\leq -\omega d_2 \land d_2' \leq \omega d_1 \lor d_1' \leq 4 ] \|d\| \geq 1
\end{align*}
\]
DAL Motives: Differential-algebraic Program Model

differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL}_\mathbb{R} + \text{DL} + \text{DAP} \]

\[
[ \begin{array}{c}
  d_1 := -d_2; \\
  d_1' \leq -\omega d_2 \land d_2' \leq \omega d_1 \lor d_1' \leq 4
\end{array} ] \|d\| \geq 1
\]
DAL Motives: Differential-algebraic Program Model

Differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL}_\mathbb{R} + \text{DL} + \text{DAP} \]

\[
\begin{align*}
    d_1 &:= -d_2; \\
    d_1' &\leq -\omega d_2 \land d_2' \leq \omega d_1 \lor d_1' \leq 4 \quad \|d\| \geq 1
\end{align*}
\]

differential-algebraic program

= first-order completion of hybrid programs
Example (Discrete Jump / DJ constraints)

- \( d_1 := -d_2 \)
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  

  simultaneous effect, rotate left by $\frac{\pi}{2}$
Example (Discrete Jump / DJ constraints)

- \( d_1 := -d_2 \)
- \( d_1 := -d_2 \land d_2 := d_1 \)  \( \text{simultaneous effect, rotate left by } \frac{\pi}{2} \)
- \( d_1 := -d_2 \land d_1 := 0 \)
Example (Discrete Jump / DJ constraints)

- \( d_1 := -d_2 \)
- \( d_1 := -d_2 \land d_2 := d_1 \) \( \) simultaneous effect, rotate left by \( \frac{\pi}{2} \)
- \( d_1 := -d_2 \land d_1 := 0 \) \( \) incompatible jump
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  \hspace{1em} \text{simultaneous effect, rotate left by $\frac{\pi}{2}$}
- $d_1 := -d_2 \land d_1 := 0$  \hspace{1em} \text{incompatible jump}
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  simultaneous effect, rotate left by $\frac{\pi}{2}$
- $d_1 := -d_2 \land d_1 := 0$  incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- $\neg (d_1 := 5)$
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  
  simultaneous effect, rotate left by $\frac{\pi}{2}$
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  incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- $\neg (d_1 := 5)$  
  what is really assigned to $d_1$ now?
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  simultaneous effect, rotate left by $\pi/2$
- $d_1 := -d_2 \land d_1 := 0$  incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- $\neg(d_1 := 5)$  what is really assigned to $d_1$ now?
- $\exists a (\omega := a^2 \land a < 5)$
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$ simultaneous effect, rotate left by $\frac{\pi}{2}$
- $d_1 := -d_2 \land d_1 := 0$ incompatible jump
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- $\neg(d_1 := 5)$ what is really assigned to $d_1$ now?
- $\exists a (\omega := a^2 \land a < 5)$
- $\forall a \omega := a^2$
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  \hspace{1em} \text{simultaneous effect, rotate left by $\frac{\pi}{2}$}
- $d_1 := -d_2 \land d_1 := 0$  \hspace{1em} \text{incompatible jump}
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- $\neg(d_1 := 5)$  \hspace{1em} \text{what is really assigned to $d_1$ now?}
- $\exists a (\omega := a^2 \land a < 5)$
- $\forall a \omega := a^2$  \hspace{1em} \omega can hardly assume all those values at once
Example (Discrete Jump / DJ constraints)

- \( d_1 := -d_2 \)
- \( d_1 := -d_2 \land d_2 := d_1 \) simultaneous effect, rotate left by \( \frac{\pi}{2} \)
- \( d_1 := -d_2 \land d_1 := 0 \) incompatible jump
- \((d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)\)
- \( \neg (d_1 := 5) \) what is really assigned to \( d_1 \) now?
- \( \exists a (\omega := a^2 \land a < 5) \)
- \( \forall a \omega := a^2 \) \( \omega \) can hardly assume all those values at once
- \( \exists a a := d_1 \)
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$ \quad \text{simultaneous effect, rotate left by } \frac{\pi}{2}
- $d_1 := -d_2 \land d_1 := 0$ \quad \text{incompatible jump}
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- $\neg(d_1 := 5)$ \quad \text{what is really assigned to } d_1 \text{ now?}
- $\exists a \ (\omega := a^2 \land a < 5)$
- $\forall a \ \omega := a^2$ \quad \omega \text{ can hardly assume all those values at once}
- $\exists a \ a := d_1$ \quad \text{is just } \textit{true} \text{ as no visible effects or constraints}
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  \hspace{1em} \text{simultaneous effect, rotate left by $\frac{\pi}{2}$}
- $\times \ d_1 := -d_2 \land d_1 := 0$  \hspace{1em} \text{incompatible jump}
- $(d_1 > 0 \rightarrow d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \rightarrow d_1 := d_2 \land d_2 := -d_1)$
- $\times \ \neg(d_1 := 5)$  \hspace{1em} what is really assigned to $d_1$ now?
- $\exists a \ (\omega := a^2 \land a < 5)$
- $\times \ \forall a \ \omega := a^2$  \hspace{1em} $\omega$ can hardly assume all those values at once
- $\exists a \ a := d_1$  \hspace{1em} is just $true$ as no visible effects or constraints
- $\bullet \ d_1 > 0 \rightarrow \exists a \ (a < 5 \land d_1 := a^2 + 1)$
Example (Discrete Jump / DJ constraints)

- $d_1 := -d_2$
- $d_1 := -d_2 \land d_2 := d_1$  simultaneous effect, rotate left by $\frac{\pi}{2}$
- $d_1 := -d_2 \land d_1 := 0$  incompatible jump
- $(d_1 > 0 \implies d_1 := -d_2 \land d_2 := d_1) \land (d_1 \leq 0 \implies d_1 := d_2 \land d_2 := -d_1)$
- $\neg (d_1 := 5)$  what is really assigned to $d_1$ now?
- $\exists a \ (\omega := a^2 \land a < 5)$
- $\forall a \omega := a^2$  $\omega$ can hardly assume all those values at once
- $\exists a \ a := d_1$  is just true as no visible effects or constraints

$? \ d_1 > 0 \implies \exists a \ (a < 5 \land d_1 := a^2 + 1)$  what happens if $d_1 \leq 0$
Example (Differential-algebraic / DA constraints)

- $x_1' = d_1 \land x_2' = d_2$
Example (Differential-algebraic / DA constraints)

- $x'_1 = d_1 \land x'_2 = d_2$
- $x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1$

$\exists \omega \left( d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \right)$

$\forall \omega \left( d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \right)$

$\omega$ cannot have all those slopes at once

$\exists \omega \left( d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \right) \lor \left( d'_1 \leq d'_2 \leq 2 d_1 \right)$

$d_1 > 0 \rightarrow x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1$

What is the slope of $d_1$ now?
Example (Differential-algebraic / DA constraints)

- \( x'_1 = d_1 \land x'_2 = d_2 \)
- \( x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \)
- \( d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0 \)
Example (Differential-algebraic / DA constraints)

- $x'_1 = d_1 \land x'_2 = d_2$
- $x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1$
- $d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0$
- $d'_1 = -d_2 \land d'_1 = 1$
Example (Differential-algebraic / DA constraints)

- $x'_1 = d_1 \land x'_2 = d_2$
- $x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1$
- $d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0$
- $d'_1 = -d_2 \land d'_1 = 1$ incompatible slope
Example (Differential-algebraic / DA constraints)

- $x'_1 = d_1 \land x'_2 = d_2$
- $x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1$
- $d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0$

× $d'_1 = -d_2 \land d'_1 = 1$ incompatible slope

- $(d_1 > 0 \rightarrow d'_1 = -d_2 \land d'_2 = d_1) \land (d_1 \leq 0 \rightarrow d'_1 = d_2 \land d'_2 = -d_1)$
Example (Differential-algebraic / DA constraints)

- \( x'_1 = d_1 \land x'_2 = d_2 \)
- \( x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \)
- \( d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0 \)
- \( d'_1 = -d_2 \land d'_1 = 1 \) incompatible slope

- \( (d_1 > 0 \rightarrow d'_1 = -d_2 \land d'_2 = d_1) \land (d_1 \leq 0 \rightarrow d'_1 = d_2 \land d'_2 = -d_1) \)
- \( \neg (d'_1 = 5) \)
Example (Differential-algebraic / DA constraints)

- \( x'_1 = d_1 \wedge x'_2 = d_2 \)
- \( x'_1 = d_1 \wedge x'_2 = d_2 \wedge d'_1 = -\omega d_2 \wedge d'_2 = \omega d_1 \)
- \( d'_1 = -\omega d_2 \wedge d'_2 = \omega d_1 \wedge d_1 \geq 0 \)
- \( d'_1 = -d_2 \wedge d'_1 = 1 \) incompatible slope
- \( (d_1 > 0 \rightarrow d'_1 = -d_2 \wedge d'_2 = d_1) \wedge (d_1 \leq 0 \rightarrow d'_1 = d_2 \wedge d'_2 = -d_1) \)
- \( \neg (d'_1 = 5) \) what is the slope of \( d_1 \) now?
Example (Differential-algebraic / DA constraints)

- \( x_1' = d_1 \land x_2' = d_2 \)
- \( x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \land d_2' = \omega d_1 \)
- \( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land d_1 \geq 0 \)
- \( d_1' = -d_2 \land d_1' = 1 \) incompatible slope
- \( (d_1 > 0 \rightarrow d_1' = -d_2 \land d_2' = d_1) \land (d_1 \leq 0 \rightarrow d_1' = d_2 \land d_2' = -d_1) \)
- \( \neg(d_1' = 5) \) what is the slope of \( d_1 \) now?
- \( \exists \omega (d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1) \)
Example (Differential-algebraic / DA constraints)

- \( x_1' = d_1 \land x_2' = d_2 \)
- \( x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \land d_2' = \omega d_1 \)
- \( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land d_1 \geq 0 \)
- \( d_1' = -d_2 \land d_1' = 1 \) incompatible slope
- \((d_1 > 0 \rightarrow d_1' = -d_2 \land d_2' = d_1) \land (d_1 \leq 0 \rightarrow d_1' = d_2 \land d_2' = -d_1)\)
- \(\neg(d_1' = 5)\) what is the slope of \(d_1\) now?
- \(\exists \omega \ (d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1)\)
- \(\forall \omega \ (d_1' = -\omega d_2 \land d_2' = \omega d_1)\)
Example (Differential-algebraic / DA constraints)

- $x'_1 = d_1 \land x'_2 = d_2$
- $x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1$
- $d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0$
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- $(d_1 > 0 \rightarrow d'_1 = -d_2 \land d'_2 = d_1) \land (d_1 \leq 0 \rightarrow d'_1 = d_2 \land d'_2 = -d_1)$
- $\neg(d'_1 = 5)$ what is the slope of $d_1$ now?
- $\exists \omega (d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land -1 \leq \omega \leq 1)$
- $\forall \omega (d'_1 = -\omega d_2 \land d'_2 = \omega d_1)$ \hspace{1em} $\omega$ cannot have all those slopes at once
Example (Differential-algebraic / DA constraints)

- \( x'_1 = d_1 \land x'_2 = d_2 \)
- \( x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \)
- \( d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0 \)
- \( \times \quad d'_1 = -d_2 \land d'_1 = 1 \) incompatible slope
- \((d_1 > 0 \rightarrow d'_1 = -d_2 \land d'_2 = d_1) \land (d_1 \leq 0 \rightarrow d'_1 = d_2 \land d'_2 = -d_1)\)
- \( \times \quad \neg(d'_1 = 5) \) what is the slope of \( d_1 \) now?
- \( \exists \omega \ (d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land -1 \leq \omega \leq 1) \)
- \( \times \quad \forall \omega \ (d'_1 = -\omega d_2 \land d'_2 = \omega d_1) \quad \omega \) cannot have all those slopes at once
- \( \exists a \ a' = d_1 \)
Example (Differential-algebraic / DA constraints)

- $x'_1 = d_1 \land x'_2 = d_2$
- $x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1$
- $d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0$

$\times$ $d'_1 = -d_2 \land d'_1 = 1$ incompatible slope

- $(d_1 > 0 \rightarrow d'_1 = -d_2 \land d'_2 = d_1) \land (d_1 \leq 0 \rightarrow d'_1 = d_2 \land d'_2 = -d_1)$
- $\neg (d'_1 = 5)$ what is the slope of $d_1$ now?
- $\exists \omega \ (d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land -1 \leq \omega \leq 1)$
- $\forall \omega \ (d'_1 = -\omega d_2 \land d'_2 = \omega d_1)$ $\omega$ cannot have all those slopes at once
- $\exists a a' = d_1$ is just true as no visible effects or constraints
Example (Differential-algebraic / DA constraints)

- \( x'_1 = d_1 \land x'_2 = d_2 \)
- \( x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \)
- \( d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land d_1 \geq 0 \)
- \( d'_1 = -d_2 \land d'_1 = 1 \) incompatible slope
- \( (d_1 > 0 \rightarrow d'_1 = -d_2 \land d'_2 = d_1) \land (d_1 \leq 0 \rightarrow d'_1 = d_2 \land d'_2 = -d_1) \)
- \( \neg (d'_1 = 5) \) what is the slope of \( d_1 \) now?
- \( \exists \omega (d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land -1 \leq \omega \leq 1) \)
- \( \forall \omega (d'_1 = -\omega d_2 \land d'_2 = \omega d_1) \) \( \omega \) cannot have all those slopes at once
- \( \exists a \ a' = d_1 \) is just true as no visible effects or constraints
- \( \exists \omega (d'_1 = -\omega d_2 \land d'_2 = \omega d_1 \land -1 \leq \omega \leq 1) \lor (d'_1 \leq d'_2 \leq 2d_1) \)
**Example (Differential-algebraic / DA constraints)**

- \( x_1' = d_1 \land x_2' = d_2 \)
- \( x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \land d_2' = \omega d_1 \)
- \( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land d_1 \geq 0 \)
- \( d_1' = -d_2 \land d_1' = 1 \) incompatible slope
- \((d_1 > 0 \rightarrow d_1' = -d_2 \land d_2' = d_1) \land (d_1 \leq 0 \rightarrow d_1' = d_2 \land d_2' = -d_1)\)
- \( \neg(d_1' = 5) \) what is the slope of \( d_1 \) now?
- \( \exists \omega (d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1) \)
- \( \forall \omega (d_1' = -\omega d_2 \land d_2' = \omega d_1) \) \( \omega \) cannot have all those slopes at once
- \( \exists a a' = d_1 \) is just true as no visible effects or constraints
- \( \exists \omega (d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1) \lor (d_1' \leq d_2' \leq 2d_1) \)
- \( d_1 > 0 \rightarrow x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \)
General Continuous Change

Example (Differential-algebraic / DA constraints)

- \( x_1' = d_1 \land x_2' = d_2 \)
- \( x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \land d_2' = \omega d_1 \)
- \( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land d_1 \geq 0 \)
- \( d_1' = -d_2 \land d_1' = 1 \) incompatible slope
- \( (d_1 > 0 \rightarrow d_1' = -d_2 \land d_2' = d_1) \land (d_1 \leq 0 \rightarrow d_1' = d_2 \land d_2' = -d_1) \)
- \( \neg (d_1' = 5) \) what is the slope of \( d_1 \) now?
- \( \exists \omega \left( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1 \right) \)
- \( \forall \omega \left( d_1' = -\omega d_2 \land d_2' = \omega d_1 \right) \) \( \omega \) cannot have all those slopes at once
- \( \exists a \ a' = d_1 \) is just true as no visible effects or constraints
- \( \exists \omega \left( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1 \right) \lor (d_1' \leq d_2' \leq 2d_1) \)
- \( d_1 > 0 \rightarrow x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \) what happens if \( d_1 \leq 0 \)
Reasonable Change

We only allow change $x := \theta$ or $x'$ in affirmative subformulas:

**Definition (Affirmative subformula)**

Formula $G$ is *affirmative subformula* of first-order formula $F$, iff:

1. $G$ is a positive subformula of $F$, i.e., occurs with an even number of negations, and
2. no variable $y$ that occurs in $G$ is in the scope of a universal quantifier $\forall y$ of a positive subformula of $F$ (or $\exists y$ of negative subformula of $F$)
Discrete Jump Constraints

Definition (DJ constraint)

- DJ constraint: FOL\(_R\) formula \(\mathcal{J}\) over \(\Sigma\) with additional atomic formulas of the form \(x := \theta\) where \(x \in \Sigma, \theta \in \text{Trm}(\Sigma)\).
- The latter are called assignments and are only allowed in affirmative subformulas of DJ-constraints that are not in the scope of a quantifier for \(x\) of \(\mathcal{J}\).
- DJ-constraint without assignments is called *jump-free*.
- Variable \(x\) is (possibly) *changed* in \(\mathcal{J}\) iff an assignment of the form \(x := \theta\) occurs in \(\mathcal{J}\).
\[(x_1 := \theta_1 \land \ldots \land x_n := \theta_n \land x_1 > 0) \lor (x_1 := \vartheta_1 \land \ldots \land x_n := \vartheta_n \land x_1 < 0)\]

- if \(x_1 > 0\), then simultaneously changes \(x_i\) to the respective \(\theta_i\), and
Discrete Jump Constraints

\[(x_1 := \theta_1 \land \ldots \land x_n := \theta_n \land x_1 > 0) \lor (x_1 := \vartheta_1 \land \ldots \land x_n := \vartheta_n \land x_1 < 0)\]

- if \(x_1 > 0\), then simultaneously changes \(x_i\) to the respective \(\theta_i\), and
- if \(x_1 < 0\), changes the \(x_i\) to \(\vartheta_i\), instead.
Discrete Jump Constraints

\[(x_1 := \theta_1 \land \ldots \land x_n := \theta_n \land x_1 > 0) \lor (x_1 := \vartheta_1 \land \ldots \land x_n := \vartheta_n \land x_1 < 0)\]

- if \(x_1 > 0\), then simultaneously changes \(x_i\) to the respective \(\theta_i\), and
- if \(x_1 < 0\), changes the \(x_i\) to \(\vartheta_i\), instead.
- if \(x_1 = 0\), then no disjunct applies as evaluates to \textit{false} so that no jump is possible at all, which will deadlock the system.
Discrete Jump Constraints

\[(x_1 := \theta_1 \land \ldots \land x_n := \theta_n \land x_1 > 0) \lor (x_1 := \vartheta_1 \land \ldots \land x_n := \vartheta_n \land x_1 < 0)\]

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- Overlapping cases as in \((x := x - 1 \land x \geq 0) \lor x := 0\) allow any disjunct to take effect by a nondeterministic choice.
Discrete Jump Constraints

\[(x_1 := \theta_1 \land \ldots \land x_n := \theta_n \land x_1 > 0) \lor (x_1 := \vartheta_1 \land \ldots \land x_n := \vartheta_n \land x_1 < 0)\]

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- Overlapping cases as in \((x := x - 1 \land x \geq 0) \lor x := 0\) allow any disjunct to take effect by a nondeterministic choice.
- Quantifiers express unbounded discrete nondeterministic choices.

\[\exists u_1 \exists u_2 \ (e_1 := u_1 \land e_2 := u_2 \land \exists \lambda > 0 \exists \mu > 0 \ (\lambda d_1 = \mu u_1 \land \lambda d_2 = \mu u_2))\]
Definition (DA constraints)

- DA-constraint: FOL$_\mathbb{R}$ formula $\mathcal{D}$ over $\Sigma \cup \Sigma'$, in which symbols of $\Sigma'$ only occur in affirmative subformulas that are not in the scope of a quantifier of $\mathcal{D}$ for that symbol.
- $\Sigma'$ is the set of all differential symbols $x^{(n)}$ with $n \in \mathbb{N}$ for state variables $x \in \Sigma$. Write $x'$ for $x^{(1)}$ ...
- DA-constraint without differential symbols is called non-differential.
- Variable $x$ is (possibly) changed in $\mathcal{D}$ iff $x^{(n)}$ occurs in $\mathcal{D}$ for an $n \geq 1$.
- $\text{ord}_x \mathcal{D}$ is the highest order $n \in \mathbb{N}$ of a differential symbol $x^{(n)}$ occurring in $\mathcal{D}$, otherwise not defined.
Differential-algebraic Constraints

\[(x' = \theta \land x > 0) \lor (x' = -x^2 \land x < 0)\]

- ongoing continuous evolution respecting differential and non-differential constraints during the whole evolution;
Differential-algebraic Constraints

\[(x' = \theta \land x > 0) \lor (x' = -x^2 \land x < 0)\]

- ongoing continuous evolution respecting differential and non-differential constraints during the whole evolution;
- evolves along \(x' = \theta\) while \(x > 0\), evolves along \(x' = -x^2\) when \(x < 0\)
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- never allowed to enter the region where neither case applies \((x = 0)\).
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- never allowed to enter the region where neither case applies (\(x = 0\)).
- Overlapping cases allow any disjunct to take effect by a nondeterministic choice:

\[ \exists \omega \left( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1 \right) \lor \left( d_1' = d_2' = 0 \right) \]
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\[
\exists \omega \left( d_1' = -\omega d_2 \land d_2' = \omega d_1 \land -1 \leq \omega \leq 1 \right) \lor (d_1' = d_2' = 0)\]

- Quantifiers express continuous nondeterministic choices:

\[
\exists u \left( d_1' = -(\omega + u)d_2 \land d_2' = (\omega + u)d_1 \land -0.1 \leq u \leq 0.1 \right)\]

expresses that the system follows a continuous evolution in which, at each time, the differential equations are respected for some choice of \(u\) in \(-0.1 \leq u \leq 0.1\), possibly different at each point in time.
Simplify: Fully Homogeneous Constraints

Free nondeterministic change of $y$ is expressible:

- $\exists a \ y := a$, or
- $\exists a \ y' = a$

Convention (Fully Homogeneous)

Expect changes of all changed variables are specified explicitly in all cases of the constraints to improve readability: A DA-constraint or DJ-constraint $C$ is homogeneous iff, in each disjunct of a disjunctive normal form of $C$, every changed variable of $C$ is changed exactly once.
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\[
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\]

\[
\times \ (d_1 > 0 \land d_1 := a^2 + 1) \lor d_2 \leq 0
\]

\[
d_1 > 0 \rightarrow x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2
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- $d_1 > 0 \rightarrow \exists a \ (a < 5 \land d_1 := a^2 + 1)$  what happens if $d_1 \leq 0$
- $(d_1 > 0 \land d_1 := a^2 + 1) \lor d_2 \leq 0$
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$\times \ (d_1 > 0 \land d_1 := a^2 + 1) \lor d_2 \leq 0$

$\times \ d_1 > 0 \rightarrow x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2$  what happens if $d_1 \leq 0$

$\bullet \ (d_1 > 0 \land x'_1 = d_1 \land x'_2 = d_2 \land d'_1 = -\omega d_2) \lor (d_1 \leq 0 \land x'_1 = 1)$
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$$\begin{align*}
\times \ d_1 > 0 & \rightarrow \exists a \ (a < 5 \land d_1 := a^2 + 1) \quad \text{what happens if } d_1 \leq 0 \\
\times \ (d_1 > 0 \land d_1 := a^2 + 1) \lor d_2 \leq 0 \\
\times \ d_1 > 0 & \rightarrow x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2 \quad \text{what happens if } d_1 \leq 0 \\
\times \ (d_1 > 0 \land x_1' = d_1 \land x_2' = d_2 \land d_1' = -\omega d_2) \lor (d_1 \leq 0 \land x_1' = 1)
\end{align*}$$
### Definition (Differential-algebraic program $\alpha$)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<td>$D$</td>
<td>(continuous DA-constraint)</td>
</tr>
<tr>
<td>$J$</td>
<td>(discrete DJ-constraint)</td>
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<tr>
<td>$\alpha; \beta$</td>
<td>(seq. composition)</td>
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<tr>
<td>$\alpha \cup \beta$</td>
<td>(nondet. choice)</td>
</tr>
<tr>
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$\{ \}$ Kleene algebra
Definition (Differential-algebraic program $\alpha$)

- $D$ (continuous DA-constraint)
- $J$ (discrete DJ-constraint)
- $\alpha; \beta$ (seq. composition)
- $\alpha \cup \beta$ (nondet. choice)
- $\alpha^*$ (nondet. repetition)

Kleene algebra

**TRM** $\equiv (ctrl; fly)^*$

- $ctrl \equiv (\exists a \omega := a) \uplus (\exists a \omega := a)$
- $fly \equiv x' = d' \land d_1' = -\omega d_2 \land d_1' = \omega d_1$

$\land \ldots$
Differential-algebraic Programs: Syntax

Definition (Differential-algebraic program $\alpha$)

- $D$ (continuous DA-constraint)
- $J$ (discrete DJ-constraint)
- $\alpha; \beta$ (seq. composition)
- $\alpha \cup \beta$ (nondet. choice)
- $\alpha^*$ (nondet. repetition)

Kleene algebra

$TRM \equiv (ctrl; fly)^*$

$ctrl \equiv (\exists \omega : a \mid \parallel x - y \parallel \geq 5; \exists a \omega := a)$

$\cup (\exists \omega : a \mid \parallel x - y \parallel < 5; \exists a \omega := a)$

$fly \equiv x' = d' \land d'_1 = -\omega d_2 \land d'_1 = \omega d_1$

$\land y' = e' \land e'_1 = -\omega e_2 \land e'_1 = \omega e_1$

$\land ...$
### Definition (Differential-algebraic program $\alpha$)

- $D$ (continuous DA-constraint)
- $J$ (discrete DJ-constraint)
- $\alpha; \beta$ (seq. composition)
- $\alpha \cup \beta$ (nondet. choice)
- $\alpha^*$ (nondet. repetition)

### TRM

$$TRM \equiv (ctrl; fly)^*$$

**ctrl**

$$\equiv (\?\|x - y\| \geq 5; \exists a \omega := a) \cup (\?\|x - y\| < 5; d := \omega(x - c)')$$

**fly**

$$\equiv x' = d' \land d'_1 = -\omega d_2 \land d'_1 = \omega d_1 \land y' = e' \land e'_1 = -\omega e_2 \land e'_1 = \omega e_1 \land ...$$
<table>
<thead>
<tr>
<th>DA-program class</th>
<th>System class</th>
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<tr>
<td>conjunctive DA-constraints</td>
<td>continuous dynamical systems</td>
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<td>general DA-programs</td>
<td>hybrid dynamical systems + first-order dynamics</td>
</tr>
</tbody>
</table>
Definition (Differential-algebraic programs $\alpha$: transition semantics)

$$\nu \models \mathcal{I}_{x:=}^{w(x)} = \nu = w \text{ when not changed}$$
Definition (Differential-algebraic programs $\alpha$: transition semantics)

$$v \xrightarrow{\mathcal{D}} w$$

$$\varphi(t)$$

$$(x, t)$$

$$(v, w)$$
Definition (Differential-algebraic programs $\alpha$: transition semantics)

$\nu \xrightarrow{D} \chi \wedge \nu \phi(t)$

$\chi$

$\nu \xrightarrow{D} \chi \wedge \nu \phi(t)$

$\chi$

$t$
Differential-algebraic Programs: Transition Semantics

Definition (Differential-algebraic programs $\alpha$: transition semantics)

\[ \varphi(t) \]

\[ \chi \]

\[ D \]

\[ \land \chi \]

\[ v \xrightarrow{D} w \]

\[ x \]

\[ t \]
Definition (Differential-algebraic programs $\alpha$: transition semantics)

$\alpha; \beta$

$\alpha$ $\beta$

$v$ $s$ $w$
Definition (Differential-algebraic programs $\alpha$: transition semantics)

\[
\begin{align*}
\alpha; \beta \\
\text{v} \quad \text{s} \quad \text{w}
\end{align*}
\]
Definition (Differential-algebraic programs $\alpha$: transition semantics)

\[ \alpha; \beta \]

\[ \begin{align*}
\nu & \xrightarrow{\alpha} s \\
& \xrightarrow{\beta} w
\end{align*} \]
Definition (Differential-algebraic programs $\alpha$: transition semantics)

$\alpha^*$

$V \xrightarrow{\alpha} S_1 \xrightarrow{\alpha} S_2 \ldots \xrightarrow{\alpha} S_n \xrightarrow{\alpha} W$
Definition (Differential-algebraic programs $\alpha$: transition semantics)
Definition (Differential-algebraic programs $\alpha$: transition semantics)

$\alpha \cup \beta$
Definition (Differential-algebraic programs $\alpha$: transition semantics)
In which state can we interpret $d_1' = -\omega d_2 \land d_2' = \omega d_1$?
In which state can we interpret $d'_1 = -\omega d_2 \land d'_2 = \omega d_1$?

Not in a single state, because derivatives not defined
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Along flow, $d'_1$ makes sense and DA-constraint can be interpreted locally.
In which state can we interpret $d'_1 = -\omega d_2 \land d'_2 = \omega d_1$?

- Not in a single state, because derivatives not defined.
- Along flow, $d'_1$ makes sense and DA-constraint can be interpreted locally.

**Definition (Differential state flow $\varphi : [0, r] \rightarrow \text{States}$)**

$\varphi$ componentwise continuous on $[0, r]$: $\varphi(\zeta)(x)$ continuous in $\zeta$ for $x \in \Sigma$. 

*Differentially augmented state* $\bar{\varphi}(\zeta)$ of $\varphi$ at $\zeta \in [0, r]$ agrees with $\varphi(\zeta)$ except that it assigns values to some $x^{(n)} \in \Sigma'$:

$$
\bar{\varphi}(\zeta)(x^{(n)}) = \begin{cases} 
\frac{d^n \varphi(t)(x)}{dt^n}(\zeta) & \text{if } n\text{-times continuously differentiable in } t \text{ at } \zeta \\
\text{undefined} & \text{otherwise}
\end{cases}
$$

$\varphi$ is *state flow of the order of* $D$, iff value of each differential symbol occurring in DA-constraint $D$ is defined on $[0, r]$. 
Definition (Interpretation of differential-algebraic constraints)

Let $\varphi$ state flow of the order of DA-constraint $\mathcal{D}$ and duration $r \geq 0$. $\varphi \models \mathcal{D}$ iff, for all $\zeta \in [0, r]$,

1. $\bar{\varphi}(\zeta) \models_{\mathbb{R}} \mathcal{D}$ using standard semantics $\models_{\mathbb{R}}$ of first-order real arithmetic, and

2. $\llbracket z \rrbracket \bar{\varphi}(\zeta) = \llbracket z \rrbracket \bar{\varphi}(0)$ for all variables $z$ that are not changed by $\mathcal{D}$. 
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Let \( \varphi \) state flow of the order of DA-constraint \( \mathcal{D} \) and duration \( r \geq 0 \).

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2. \( \left[ z \right] \bar{\varphi}(\zeta) = \left[ z \right] \bar{\varphi}(0) \) for all variables \( z \) that are not changed by \( \mathcal{D} \).

- Only variables whose differential symbols occur in \( \mathcal{D} \) need continuously differentiable values.
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- Only variables whose differential symbols occur in \( \mathcal{D} \) need continuously differentiable values.
- Quantified variables can change arbitrarily, even discontinuously.
Definition (Interpretation of differential-algebraic constraints)

Let $\varphi$ state flow of the order of DA-constraint $\mathcal{D}$ and duration $r \geq 0$. $\varphi \models \mathcal{D}$ iff, for all $\zeta \in [0, r]$,

1. $\bar{\varphi}(\zeta) \models_{\mathbb{R}} \mathcal{D}$ using standard semantics $\models_{\mathbb{R}}$ of first-order real arithmetic, and

2. $\llbracket z \rrbracket \bar{\varphi}(\zeta) = \llbracket z \rrbracket \bar{\varphi}(0)$ for all variables $z$ that are not changed by $\mathcal{D}$.

- Only variables whose differential symbols occur in $\mathcal{D}$ need continuously differentiable values.
- Quantified variables can change arbitrarily, even discontinuously.
- In $\exists u \, x' = u^2$, the value of $u^2$ (not $u$) varies continuously, because $x'$ does.
Definition (Interpretation of differential-algebraic constraints)

Let $\varphi$ state flow of the order of DA-constraint $D$ and duration $r \geq 0$. $\varphi \models D$ iff, for all $\zeta \in [0, r]$,

1. $\bar{\varphi}(\zeta) \models_{\mathbb{R}} D$ using standard semantics $\models_{\mathbb{R}}$ of first-order real arithmetic, and

2. $\llbracket z \rrbracket \bar{\varphi}(\zeta) = \llbracket z \rrbracket \bar{\varphi}(0)$ for all variables $z$ that are not changed by $D$.

- Only variables whose differential symbols occur in $D$ need continuously differentiable values.
- Quantified variables can change arbitrarily, even discontinuously.
- In $\exists u \ x' = u^2$, the value of $u^2$ (not $u$) varies continuously, because $x'$ does.
- For $r = 0$, atomic formulas with differential symbols take no effect ($true$ as positive).
Definition (Interpretation of discrete jump constraints)

$(v, w) \models J$ defined as follows:

$(v, w) \models x := \theta$ :\iff $\lbrack x \rbrack_w = \lbrack \theta \rbrack_v$

$(v, w) \models \theta_1 \geq \theta_2$ :\iff $\lbrack \theta_1 \rbrack_v \geq \lbrack \theta_2 \rbrack_v$

$(v, w) \models \phi \land \psi$ :\iff $(v, w) \models \phi$ and $(v, w) \models \psi$

$(v, w) \models \neg \phi$ :\iff $(v, w) \models \phi$ does not hold

$(v, w) \models \forall x \phi$ :\iff $(v_x, w) \models \phi$ for all $v_x$ that agree with $v$ except for the value of $x$

$(v, w) \models \exists x \phi$ :\iff $(v_x, w) \models \phi$ for some $v_x$ that agrees with $v$ except for the value of $x$

where $\lbrack z \rbrack_w = \lbrack z \rbrack_v$ for all variables $z$ that are not changed in $J$. 
Definition (Differential-algebraic programs $\alpha$)

$$
\rho(D) = \left\{ (\varphi(0), \varphi(r)) : \varphi \models D \text{ of order of } D, \text{ duration } r \geq 0 \right\}
$$

$$
\rho(J) = \left\{ (v, w) : (v, w) \models J \right\}
$$

$$
\rho(\alpha \cup \beta) = \rho(\alpha) \cup \rho(\beta)
$$

$$
\rho(\alpha; \beta) = \rho(\alpha) \circ \rho(\beta)
$$

$$
\rho(\alpha^*) = \left\{ (v, w) : \text{ there is } v \xrightarrow{\rho(\alpha)} v_1 \xrightarrow{\rho(\alpha)} v_2 \ldots \xrightarrow{\rho(\alpha)} w \right\}
$$
1 Motivation

2 Differential-algebraic Programs
   - Design Motives
   - Syntax
     - Discrete Jump Constraints
     - Differential-algebraic Constraints
     - Differential-algebraic Programs
   - Semantics
     - Differential-algebraic Constraints
     - Discrete Jump Constraints
     - Differential-algebraic Programs

3 Air Traffic Control

4 Differential-algebraic Dynamic Logic DAL
   - Syntax
   - Semantics
\[\begin{align*}
  x_1' &= v \cos \vartheta \\
  x_2' &= v \sin \vartheta \\
  y_1' &= u \cos \varsigma \\
  y_2' &= u \sin \varsigma
\end{align*}\]
\[ \begin{bmatrix} x_1' = v \cos \vartheta \\ x_2' = v \sin \vartheta \\ y_1' = u \cos \varsigma \\ y_2' = u \sin \varsigma \end{bmatrix} \]
\[ \begin{align*}
    x_1' &= v \cos \vartheta \\
    x_2' &= v \sin \vartheta \\
    y_1' &= u \cos \varsigma \\
    y_2' &= u \sin \varsigma
\end{align*} \]
Air Traffic Control

\[
\begin{align*}
x_1' &= v \cos \vartheta = d_1 & y_1' &= u \cos \varsigma \\
x_2' &= v \sin \vartheta = d_2 & y_2' &= u \sin \varsigma
\end{align*}
\]
Differential Axiomatization of Flight Dynamics

\[
\begin{bmatrix}
  x_1' = v \cos \vartheta = d_1 & y_1' = u \cos \varsigma = e_1 \\
  x_2' = v \sin \vartheta = d_2 & y_2' = u \sin \varsigma = e_2 \\
  d_1' = & e_1' = \\
  d_2' = & e_2' = \\
\end{bmatrix}
\]

\[
d_1' = \\
d_2' =
\]
\[
\begin{align*}
\dot{x}_1 &= v \cos \vartheta = d_1 \\
\dot{x}_2 &= v \sin \vartheta = d_2 \\
\dot{d}_1 &= \\
\dot{d}_2 &= \\
\dot{y}_1 &= u \cos \varsigma = e_1 \\
\dot{y}_2 &= u \sin \varsigma = e_2 \\
\end{align*}
\]

\[
\begin{align*}
\dot{d}_1' &= (v \cos \vartheta)' \\
\dot{d}_2' &= (v \sin \vartheta)'
\end{align*}
\]
Differential Axiomatization of Flight Dynamics

\[
\begin{align*}
    x_1' &= v \cos \vartheta = d_1 & y_1' &= u \cos \zeta = e_1 \\
    x_2' &= v \sin \vartheta = d_2 & y_2' &= u \sin \zeta = e_2 \\
    d_1' &= \quad & e_1' &= \quad \\
    d_2' &= \quad & e_2' &= \\
    d_1' &= (v \cos \vartheta)' = v' \cos \vartheta + v(- \sin \vartheta) \vartheta' \\
    d_2' &= (v \sin \vartheta)' = v' \sin \vartheta + v(\cos \vartheta) \vartheta'
\end{align*}
\]
Differential Axiomatization of Flight Dynamics

\[
\begin{align*}
\dot{x}_1 &= v \cos \vartheta = d_1 \\
\dot{x}_2 &= v \sin \vartheta = d_2 \\
\dot{d}_1 &= \\
\dot{d}_2 &= \\
\dot{y}_1 &= u \cos \zeta = e_1 \\
\dot{y}_2 &= u \sin \zeta = e_2 \\
\dot{e}_1 &= \\
\dot{e}_2 &= \\
\end{align*}
\]

\[
\begin{align*}
d_1' &= (v \cos \vartheta)' = v' \cos \vartheta + v(- \sin \vartheta) \vartheta' = -(v \sin \vartheta) \omega \\
d_2' &= (v \sin \vartheta)' = v' \sin \vartheta + v(\cos \vartheta) \vartheta' = (v \cos \vartheta) \omega 
\end{align*}
\]
\[
\begin{bmatrix}
  x'_1 &= v \cos \vartheta = d_1 \\
  x'_2 &= v \sin \vartheta = d_2 \\
  y'_1 &= u \cos \zeta = e_1 \\
  y'_2 &= u \sin \zeta = e_2
\end{bmatrix}
\]

\[d'_1 = (v \cos \vartheta)' = v' \cos \vartheta + v(-\sin \vartheta)\vartheta' = -(v \sin \vartheta)\omega = -\omega d_2\]

\[d'_2 = (v \sin \vartheta)' = v' \sin \vartheta + v(\cos \vartheta)\vartheta' = (v \cos \vartheta)\omega = \omega d_1\]
\[
\begin{bmatrix}
    x_1' &= v \cos \vartheta = d_1 \\
    x_2' &= v \sin \vartheta = d_2 \\
    d_1' &= -\omega d_2 \\
    d_2' &= \omega d_1
\end{bmatrix} = \begin{bmatrix}
    y_1' &= u \cos \varsigma = e_1 \\
    y_2' &= u \sin \varsigma = e_2 \\
    e_1' &= -\varrho e_2 \\
    e_2' &= \varrho e_1
\end{bmatrix}
\]

\[
\begin{align*}
    d_1' &= (v \cos \vartheta)' = v' \cos \vartheta + v(- \sin \vartheta) \vartheta' = -(v \sin \vartheta) \omega = -\omega d_2 \\
    d_2' &= (v \sin \vartheta)' = v' \sin \vartheta + v(\cos \vartheta) \vartheta' = (v \cos \vartheta) \omega = \omega d_1
\end{align*}
\]
Differential Axiomatization of Flight Dynamics

\[
\begin{align*}
    x_1' &= v \cos \vartheta = d_1 & y_1' &= u \cos \varsigma = e_1 \\
    x_2' &= v \sin \vartheta = d_2 & y_2' &= u \sin \varsigma = e_2 \\
    d_1' &= -\omega d_2 & e_1' &= -\varrho e_2 \\
    d_2' &= \omega d_1 & e_2' &= \varrho e_1 \\
\end{align*}
\]

\[
\begin{align*}
    d_1' &= (v \cos \vartheta)' = v' \cos \vartheta + v(-\sin \vartheta)\vartheta' = -(v \sin \vartheta)\omega = -\omega d_2 \\
    d_2' &= (v \sin \vartheta)' = v' \sin \vartheta + v(\cos \vartheta)\vartheta' = (v \cos \vartheta)\omega = \omega d_1 \\
\end{align*}
\]

\[
v = \|d\| = \sqrt{d_1^2 + d_2^2}
\]
\[
\begin{bmatrix}
    x_1' = d_1 \\
    y_1' = e_1 \\
    x_2' = d_2 \\
    y_2' = e_2 \\
    d_1' = -\omega d_2 \\
    e_1' = -\varrho e_2 \\
    d_2' = \omega d_1 \\
    e_2' = \varrho e_1
\end{bmatrix}
\]

\[
d_1' = (v \cos \vartheta)' = v' \cos \vartheta + v(-\sin \vartheta)\vartheta' = -(v \sin \vartheta)\omega = -\omega d_2
\]

\[
d_2' = (v \sin \vartheta)' = v' \sin \vartheta + v(\cos \vartheta)\vartheta' = (v \cos \vartheta)\omega = \omega d_1
\]

\[v = \|d\| = \sqrt{d_1^2 + d_2^2}\]
Air Traffic Control Roundabout Maneuvers
Air Traffic Control Roundabout Maneuvers
Air Traffic Control Roundabout Maneuvers

\[ \begin{bmatrix}
    x_1' \\
    x_2' \\
    \vartheta'
\end{bmatrix} =
\begin{bmatrix}
    -v_1 + v_2 \cos \vartheta + \omega x_2 \\
    v_2 \sin \vartheta - \omega x_1 \\
    \varpi - \omega
\end{bmatrix} \]
Falsification versus Verification

no more counterexamples but how to verify?
Air Traffic Control Roundabout Maneuvers

### Example ("Solving" differential equations)

\[
x_1(t) = \frac{1}{\omega \varpi} \left( x_1 \omega \varpi \cos t \omega - v_2 \omega \cos t \omega \sin \vartheta + v_2 \omega \cos t \omega \cos t \varpi \sin \vartheta - v_1 \varpi \sin t \omega \\
+ x_2 \omega \varpi \sin t \omega - v_2 \omega \cos \vartheta \cos t \varpi \sin t \omega - v_2 \omega \sqrt{1 - \sin \vartheta^2} \sin t \omega \\
+ v_2 \omega \cos \vartheta \cos t \omega \sin t \varpi + v_2 \omega \sin \vartheta \sin t \omega \sin t \varpi \right) \ldots
\]
Air Traffic Control Roundabout Maneuvers

Example (“Solving” differential equations)

\[ \forall t \geq 0 \quad \frac{1}{\omega \omega} \left( x_1 \omega \omega \cos t \omega - v_2 \omega \cos t \omega \sin \vartheta + v_2 \omega \cos t \omega \cos t \varpi \sin \vartheta - v_1 \varpi \sin t \omega \\ + x_2 \omega \varpi \sin t \omega - v_2 \omega \cos \vartheta \cos t \varpi \sin t \omega - v_2 \omega \sqrt{1 - \sin \vartheta^2} \sin t \omega \\ + v_2 \omega \cos \vartheta \cos t \omega \sin t \varpi + v_2 \omega \sin \vartheta \sin t \omega \sin t \varpi \right) \ldots \]
Air Traffic Control Roundabout Maneuvers

\[ \phi \equiv \|x - y\|^2 \geq p^2 \equiv (x_1 - y_1)^2 + (x_2 - y_2)^2 \geq p^2 \]

\[ trm \equiv \text{free}; \text{ entry}; \ F(\omega) \land G(\omega) \]

\[ free \equiv \exists \omega \ F(\omega) \land \exists \varpi \ G(\varpi) \land \phi \]

\[ entry \equiv \text{will be derived later} \]
Proposition (Hybrid automata embedding)

There is an effective mapping $\iota$ such that the following diagram commutes:

$$
\begin{array}{c}
HA \\ \downarrow^* \\
Q^2 \\
\end{array} \quad \xrightarrow{\iota} \quad 
\begin{array}{c}
DAP(\Sigma) \\
States^2 \\
\end{array}
$$
1 Motivation

2 Differential-algebraic Programs
   • Design Motives
   • Syntax
     • Discrete Jump Constraints
     • Differential-algebraic Constraints
     • Differential-algebraic Programs
   • Semantics
     • Differential-algebraic Constraints
     • Discrete Jump Constraints
     • Differential-algebraic Programs

3 Air Traffic Control

4 Differential-algebraic Dynamic Logic DAL
   • Syntax
   • Semantics
differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL}_\mathbb{R} + \text{DL} \]
differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL}_\mathbb{R} + \text{DL} + \text{DAP} \]

\[ [d_1' \leq -\omega d_2 \land d_2' \leq \omega d_1 \lor d_1' \leq 4] \|d\| \geq 1 \]
DAL Motives: Differential-algebraic Program Model

differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL}_{\mathbb{R}} + \text{DL} + \text{DAP} \]

\[
\begin{align*}
[&d_1 := -d_2; \quad d_1' \leq -\omega d_2 \land d_2' \leq \omega d_1 \lor d_1' \leq 4] \, \|d\| \geq 1 \\
\|d\| \geq 1 & \quad \|d\| \geq 1 \\
\|d\| \geq 1 &
\end{align*}
\]
DAL Motives: Differential-algebraic Program Model

differential-algebraic dynamic logic

\[ \text{DAL} = \text{FOL} + \text{DL} + \text{DAP} \]

differential-algebraic program

\[
\begin{aligned}
& d_1 := -d_2; \\
& d_1' \leq -\omega d_2 \land d_2' \leq \omega d_1 \lor d_1' \leq 4 \\
\end{aligned}
\]

= first-order completion of

hybrid programs
### Definition (DAL Signature $\Sigma$)

Countable set of predicate or function symbols along with natural numbers as arities containing $0, 1, +, \cdot, /, =, \leq, >, \geq, <$ for reals
Definition (DAL Signature $\Sigma$)

Countable set of predicate or function symbols along with natural numbers as arities containing $0, 1, +, \cdot, /, =, \leq, >, \geq, <$ for reals

Definition (DAL Term $t$)

$t ::= \begin{align*}
  x & \quad \text{for variable } x \in V \\
  f(t_1, \ldots, t_n) & \quad \text{for function } f/n \in \Sigma \text{ of arity } n \geq 0
\end{align*}$
Definition (DAL Signature $\Sigma$)

Countable set of predicate or function symbols along with natural numbers as arities containing $0, 1, +, \cdot, /, =, \leq, \geq, >, <$ for reals.

Definition (DAL Formula $\phi, \psi$)

$\phi ::= \begin{align*}
[\alpha] \phi & \quad \text{“all } \alpha \text{ reachables”} \\
\langle \alpha \rangle \phi & \quad \text{“some } \alpha \text{ reachable”} \\
p(t_1, \ldots, t_n) & \quad \text{for predicate } p/n \in \Sigma \text{ of arity } n \geq 0 \\
\neg \phi & \quad \text{“not”} \\
(\phi \land \psi) & \quad \text{“and”} \\
(\phi \lor \psi) & \quad \text{“or”} \\
(\phi \to \psi) & \quad \text{“implies”}
\end{align*}$
Definition (Formulas $\phi$)

$[\alpha] \phi \rightarrow \phi$

$\vee \rightarrow \phi, \phi, \phi$

$\alpha$-span $\rightarrow \phi, \phi, \phi$

$\beta$-span $\rightarrow \phi, \phi$
Definition (Formulas $\phi$)

\[ v \langle \alpha \rangle \phi \langle \beta \rangle \phi^{\beta-\text{span}} \text{ and } \phi^{\alpha-\text{span}} \]
Definition (Formulas $\phi$)

$[\alpha]\phi$

$\alpha$-span
Definition (Formulas $\phi$)

$[\alpha]\phi$

$\langle\beta\rangle\phi$

$\alpha$-span

$\beta$-span
Definition (Formulas $\phi$)

- $\langle \beta \rangle \phi$
- $[\alpha] \phi$
- $\alpha$-span
- $\beta$-span
- $\langle \beta \rangle \phi$ span
Definition (Formulas $\phi$)

compositional semantics!
### Definition (Formulas $\phi$)

<table>
<thead>
<tr>
<th>Formula</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v \models \theta_1 \geq \theta_2$</td>
<td>$[\theta_1]_v \geq [\theta_2]_v$</td>
</tr>
<tr>
<td>$v \models \phi \land \psi$</td>
<td>$v \models \phi$ and $v \models \psi$</td>
</tr>
<tr>
<td>$v \models \neg \phi$</td>
<td>$v \models \phi$ does not hold</td>
</tr>
<tr>
<td>$v \models [\alpha] \phi$</td>
<td>$w \models \phi$ for all $w$ with $(v, w) \in \rho(\alpha)$</td>
</tr>
<tr>
<td>$v \models \langle \alpha \rangle \phi$</td>
<td>$w \models \phi$ for some $w$ with $(v, w) \in \rho(\alpha)$</td>
</tr>
</tbody>
</table>
Well-definedness and Divisions

**Assumption (Well-definedness)**

We assume all divisions $p/q$ in any formula are constraint $\phi$ are taken to mean $\phi \land q \neq 0$. 
Example (Zeno)

\[ (a' = -1 \land d \leq a; \ d := d/2)^* \]
Example (Zeno)

- $(a' = -1 \land d \leq a; \ d := d/2)^*$
- $(x \geq 0 \rightarrow x'' = -1) \land (x < 0 \rightarrow x'' = 1) \land y' = 1$
Time Anomalies and Zeno

**Example (Zeno)**

- \(a' = -1 \land d \leq a; \ d := d/2\)^{*}
- \((x \geq 0 \rightarrow x'' = -1) \land (x < 0 \rightarrow x'' = 1) \land y' = 1\)

**Assumption (Non-Zeno)**

State flow \(\varphi\) for DA-constraint \(\mathcal{D}\) is *non-Zeno*, if there only is a finite number of points in time where some variable needs to obey another differential constraint of \(\mathcal{D}\) than before the respective point in time:

Let \(\mathcal{D}_1 \lor \cdots \lor \mathcal{D}_n\) be a disjunctive normal form of \(\mathcal{D}\), then flow \(\varphi : [0, r] \rightarrow \text{States}\) is non-Zeno iff there are an \(m \in \mathbb{N}\) and \(0 = \zeta_0 < \zeta_1 < \cdots < \zeta_m = r\) and indices \(i_1, \ldots, i_m \in \{1, \ldots, n\}\) such that \(\varphi\) respects \(\mathcal{D}_{i_k}\) on the interval \([\zeta_{k-1}, \zeta_k]\), i.e., \(\varphi|_{[\zeta_{k-1}, \zeta_k]} = \mathcal{D}_{i_k}\) for all \(k \in \{1, \ldots, m\}\).
A. Platzer.
Differential-algebraic dynamic logic for differential-algebraic programs.
To appear.