15-819/18-879: Hybrid Systems Analysis & Theorem Proving

01: Safety-critical Hybrid Systems

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Outline

1 Applications
   - Air Traffic Control
   - Hybrid Systems / Cyber-Physical Systems
   - Train Control
   - Car Control
   - UAV
   - Chemical/Physical Process Control
   - Biomedical Applications
   - Advanced Chip Design

2 Hybrid Systems
   - Labeled Transition Systems
   - Finite Automata
   - Hybrid Automata
   - Hybrid Systems

3 Differential Equations
How can we build computerized controllers for physical systems that are guaranteed to meet their design goals?
Hybrid systems
Logic-based analysis
Symbolic / numerical techniques
Automatic theorem proving
Model checking
Verification
Balance theory, practice & applications
30% Homework, 15% Midterm, 55% Project
Project: Theory and/or implementation and/or application
Whitepaper (4p), proposal (10p), report
Course Outline

1. Safety-critical Hybrid Systems
2. Propositional Logic
3. First-order Logic
4. Numerical Analysis versus Symbolic Verification
5. Propositional Tableau Procedures
6. First-order Tableau Procedures
7. Dynamic Logic Programs and Dynamical Systems
8. Hybrid Dynamical Systems & Hybrid Programs
9. Aircraft, Train, and Car Control
10. Dynamic Verification Calculi
11. Decision Procedures
12. Theorem Proving Modulo
13. Differential Equations, Differential Variance and Invariance
14. Disturbances in Hybrid Systems Control
15. Proof Theory of Hybrid Systems
16. Fixedpoint Model Checking Engines
• Differential equations (Peano, Picard, Lipschitz)
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
Differential equations (Peano, Picard, Lipschitz)
Hybrid systems
Propositional logic
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
- Propositional logic
- First-order logic
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
- Propositional logic
- First-order logic
- Automated theorem proving
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
- Propositional logic
- First-order logic
- Automated theorem proving
- Model checking (discrete / hybrid)
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
- Propositional logic
- First-order logic
- Automated theorem proving
- Model checking (discrete / hybrid)
- Quantifier elimination
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
- Propositional logic
- First-order logic
- Automated theorem proving
- Model checking (discrete / hybrid)
- Quantifier elimination
- Algebraic geometry
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
- Propositional logic
- First-order logic
- Automated theorem proving
- Model checking (discrete / hybrid)
- Quantifier elimination
- Algebraic geometry
- Differential algebra
Questionnaire

- Differential equations (Peano, Picard, Lipschitz)
- Hybrid systems
- Propositional logic
- First-order logic
- Automated theorem proving
- Model checking (discrete / hybrid)
- Quantifier elimination
- Algebraic geometry
- Differential algebra
- Computer algebra
1 Applications
   - Air Traffic Control
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2 Hybrid Systems
   - Labeled Transition Systems
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   - Hybrid Systems

3 Differential Equations
Hybrid Systems

interacting discrete and continuous dynamics
Air Traffic Control

Hybrid Systems
interacting discrete and continuous dynamics

\[
\begin{align*}
x_1' &= v \cos \vartheta \\
y_1' &= u \cos \varsigma \\
x_2' &= v \sin \vartheta \\
y_2' &= u \sin \varsigma
\end{align*}
\]
\[
\begin{bmatrix}
    x_1' = v \cos \vartheta \\
    y_1' = u \cos \varsigma \\
    x_2' = v \sin \vartheta \\
    y_2' = u \sin \varsigma
\end{bmatrix}
\]
Air Traffic Control
Verification?
looks correct
Verification?

looks correct NO!
Air Traffic Control

Verification?
looks correct NO!

\[
\begin{bmatrix}
    x_1' \\
    x_2' \\
    \vartheta'
\end{bmatrix} =
\begin{bmatrix}
    -v + u \cos \vartheta + \omega x_2 \\
    u \sin \vartheta - \omega x_1 \\
    \varpi - \omega
\end{bmatrix}
\]
Air Traffic Control

\[
\begin{bmatrix}
    x_1' \\
    x_2' \\
    \vartheta'
\end{bmatrix} =
\begin{bmatrix}
    -v + u \cos \vartheta + \omega x_2 \\
    u \sin \vartheta - \omega x_1 \\
    \varpi - \omega
\end{bmatrix}
\]

Example ("Solving" differential equations)

\[
x_1(t) = \frac{1}{\omega \varpi} \left( x_1 \omega \varpi \cos t \omega - u \omega \cos t \omega \sin \vartheta + u \omega \cos t \omega \cos t \varpi \sin \vartheta - v \varpi \sin t \omega \\
+ x_2 \omega \varpi \sin t \omega - u \omega \cos \vartheta \cos t \varpi \sin t \omega - u \omega \sqrt{1 - \sin^2 \vartheta} \sin t \omega \\
+ u \omega \cos \vartheta \cos t \omega \sin t \varpi + u \omega \sin \vartheta \sin t \omega \sin t \varpi \right) \ldots
\]

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Air Traffic Control

Example ("Solving" differential equations)

\[
\begin{align*}
\forall t \geq 0 \quad & \frac{1}{\omega \varpi} \left( x_1 \omega \varpi \cos t \omega - u \omega \cos t \omega \sin \vartheta + u \omega \cos t \omega \cos t \varpi \sin \vartheta - v \varpi \sin t \omega 
\right.
\left. + x_2 \omega \varpi \sin t \omega - u \omega \cos \vartheta \cos t \varpi \sin t \omega - u \omega \sqrt{1 - \sin \vartheta^2} \sin t \omega 
\right.
\left. + u \omega \cos \vartheta \cos t \varpi \sin t \omega + u \omega \sin \vartheta \sin t \omega \sin t \varpi \right) \ldots
\end{align*}
\]
Human at ATC detected conflict
Human instructed Tupolev to descend
TCAS instructed Tupolev to climb and Boeing to descend
Boeing couldn’t notify human (busy)
Pilots on both aircraft descended
Mid-air collision (less than a minute after conflict detected)
Mid-air Collision at Überlingen, Germany 2002
Mathematical model for complex physical systems:

**Definition (Hybrid Systems)**
systems with interacting discrete and continuous dynamics

Technical characteristics:

**Definition (Cyber-Physical Systems)**
(Distributed network of) computerized control for physical system
ETCS objectives:

1. Collision free
2. Maximise throughput & velocity (320 km/h = 200 mph)
3. $2.1 \times 10^6$ passengers/day
Parametric Hybrid Systems

continuous evolution along differential equations + discrete change
European Train Control System

Parametric Hybrid Systems

continuous evolution along differential equations + discrete change

$$\forall \text{MA} \exists \text{SB} \quad \text{"train always safe"}$$
Parametric Hybrid Systems

continuous evolution along differential equations + discrete change
Parametric Hybrid Systems

continuous evolution along differential equations + discrete change

- Challenge: verification
- Which constraints for parameter $SB$?

$\forall MA \exists SB$ “train always safe”
Train engineer disobeyed stop signal at single track section
No warning issued to train dispatcher
First sight 4 seconds before impact
Freight train triggers emergency brakes 2 seconds before impact
Head-on Train Collision at Chatsworth, CA 2008
Car Control

- Adaptive cruise control keeps safe distance?
- Lane change assistant
- Safe control with wireless interactions in CAR2CAR and USCAR
- Virtual car platooning
Safe and stable UAV flight control
Mixing UAV swarms into pilot flight control areas
Refueling of UAV: mixed human operation and micro turbulences
Many other robotic applications
Computerized Chemical/Physical Process Control

Control objective:
Stabilize neutron multiplication factor

1. Containment Structure
2. Pressurizer, Steam Generator, Control Rods, Reactor Vessel
3. Generator, Turbine, Condenser

4. 235U
5. 238U
6. 235U

1 2 3
X X X
Computerized Chemical/Physical Process Control

\[ e^{(k-1) \frac{t}{\lambda}} \]

\[ k < 1 \]
Computerized Chemical/Physical Process Control

Control objective
Stabilize neutron multiplication factor
Biomedical Applications: Glucose/Insulin Regulation

Control objective
Maintain glucose in bounded range

Based on [Daly98]
Hybrid Effects in Chip Design

\[ L \]

\[ I_C \]

\[ C_1 \]

\[ C_2 \]
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2 Hybrid Systems
   - Labeled Transition Systems
   - Finite Automata
   - Hybrid Automata
   - Hybrid Systems

3 Differential Equations
Labeled Transition Systems

Definition (Labeled Transition System)

- Transition relation on $Q \times A \times Q$, denoted as $q \xrightarrow{a} q^+$, along with
  - (possibly infinite) set $A$ of transition actions,
  - (possibly infinite) set $Q$ of states.
Labeled Transition Systems

Definition (Labeled Transition System)

- Transition relation on $Q \times A \times Q$, denoted as $q \xrightarrow{a} q^+$, along with
- (possibly infinite) set $A$ of transition actions,
- (possibly infinite) set $Q$ of states.
Definition (Trace)

Finite/infinite series of states $q_0, q_1, q_2, \ldots \in Q$ such that $q_i \xrightarrow{a_i} q_{i+1}$ with some $a_i \in A$ for all $i$. 

![Diagram of Labeled Transition Systems]
Definition (Trace)

Finite/infinite series of states \( q_0, q_1, q_2, \ldots \in Q \) such that \( q_i \xrightarrow{a_i} q_{i+1} \) with some \( a_i \in A \) for all \( i \).
Labeled Transition Systems

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Labeled Transition Systems

**Definition (Trace)**

Finite/infinite series of states $q_0, q_1, q_2, \ldots \in Q$ such that $q_i \xrightarrow{a_i} q_{i+1}$ with some $a_i \in A$ for all $i$. 

![Diagram of a labeled transition system with states and transitions labeled with $a_i$.]
Definition (Model Checking Problem)

Given initial states $Q_0 \subseteq Q$ and bad states $B \subseteq Q$ for a transition system, check whether there is a trace from some $q_0 \in Q_0$ to some $q_b \in B$. 

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Labeled Transition Systems

Definition (Model Checking Problem)

Given initial states \( Q_0 \subseteq Q \) and bad states \( B \subseteq Q \) for a transition system, check whether there is a trace from some \( q_0 \in Q_0 \) to some \( q_b \in B \).
Labeled Transition Systems

Definition (Image Computation)

\[
\text{Post}_A(Y) := \{q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A\}
\]
Definition (Image Computation)

\[ \text{Post}_A(Y) := \{ q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A \} \]
Labeled Transition Systems

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\[ \text{Post}_A(Y) := \{ q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A \} \]
Labeled Transition Systems

**Definition (Image Computation)**

\[ \text{Post}_A(Y) := \{ q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A \} \]
Definition (Image Computation)

\[ Post_A(Y) := \{ q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A \} \]
Labeled Transition Systems

Definition (Image Computation)

$$\text{Post}_A(Y) := \{ q^+ \in Q : q \xrightarrow{a} q^+ \text{ for some } q \in Y, a \in A \}$$

$$\text{Post}_A^*(Y) := \mu Z. (Y \cup Z \cup \text{Post}_A(Z))$$
Finite Automata

Definition (Nondeterministic Finite Automata)

- Transition relation on $Q \times A \times Q$, denoted as $q \xrightarrow{a} q^+$, along with
- finite set $A$ of transition actions,
- finite set $Q$ of states, initial states $Q_0 \subseteq Q$. 
Finite Automaton for Collision Avoidance
Finite Automaton for Collision Avoidance

- Cruise
- Close
- Approach
- Left
- Right
- Better
- Straight
- Far
- Right
- Back
- Far
- Left
- Back

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Finite Automaton for Collision Avoidance

- cruise
- close
- appr
  - left
  - right
- straight
- far
- right
- far
- left
- back
- back

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Collision avoidance is a property of controlled movement!
Definition (Hybrid Automata)

- Finite directed graph: vertices $M$ (modes), edges $E$ (control switches)
- Continuous state space $\mathbb{R}^n$
- Flows $\varphi_v$, where $\varphi_v(t; x) \in \mathbb{R}^n$ is the state reached after staying in mode $v$ for time $t \geq 0$ when continuous evolution starts in state $x \in \mathbb{R}^n$
- Invariant conditions $inv_v \subseteq \mathbb{R}^n$ for $v \in M$
- Jump relations $jump_e \subseteq \mathbb{R}^n \times \mathbb{R}^n$ for edges $e \in E$
  - Usually comprising guard on current state and reset relations
Definition (Hybrid Automata)

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- Jump relations $jump_e \subseteq \mathbb{R}^n \times \mathbb{R}^n$ for edges $e \in E$

Is this a good definition?
Example (Mandelbrot Set)

For complex numbers $c \in \mathbb{C}$ define $f_0(c) = c$ and $f_{n+1}(c) = f_n(c)^2 + c$. Then the Mandelbrot set is

$$\{ c \in \mathbb{C} : f_n(c) \not\to \infty \text{ as } n \to \infty \}$$
Example (Mandelbrot Set)

For complex numbers $c \in \mathbb{C}$ define $f_0(c) = c$ and $f_{n+1}(c) = f_n(c)^2 + c$. Then the Mandelbrot set is

$$\{ c \in \mathbb{C} : f_n(c) \not\to \infty \text{ as } n \to \infty \}$$

Theorem (Lenore Blum, Cucker, Shub, Smale’90...98)

“The Mandelbrot set is undecidable over $\mathbb{R}$ / in Real Turing Machines”
Definition (Hybrid Automata)

- Finite directed graph: vertices $M$ (modes), edges $E$ (control switches)
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- Flows $\varphi_v$, where $\varphi_v(t; x) \in \mathbb{R}^n$ is the state reached after staying in mode $v$ for time $t \geq 0$ when continuous evolution starts in state $x \in \mathbb{R}^n$
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  Usually comprising guard on current state and reset relations

Is this a good definition?
Hybrid Automata

**Definition (Hybrid Automata)**

- Finite directed graph: vertices $M$ (*modes*), edges $E$ (*control switches*)
- Continuous state space $\mathbb{R}^n$
- *Flows* $\varphi_v$, where $\varphi_v(t; x) \in \mathbb{R}^n$ is the state reached after staying in mode $v$ for time $t \geq 0$ when continuous evolution starts in state $x \in \mathbb{R}^n$
- Invariant conditions $inv_v \subseteq \mathbb{R}^n$ for $v \in M$
- Jump relations $jump_e \subseteq \mathbb{R}^n \times \mathbb{R}^n$ for edges $e \in E$
  - Usually comprising guard on current state and reset relations

What if $inv_v$ is a Mandelbrot set?
Hybrid Automata

Definition (Hybrid Automata)

- Finite directed graph: vertices $M$ (modes), edges $E$ (control switches)
- Continuous state space $\mathbb{R}^n$
- Flows $\varphi_v$, where $\varphi_v(t; x) \in \mathbb{R}^n$ is the state reached after staying in mode $v$ for time $t \geq 0$ when continuous evolution starts in state $x \in \mathbb{R}^n$
- Invariant conditions $inv_v \subseteq \mathbb{R}^n$ for $v \in M$
- Jump relations $jump_e \subseteq \mathbb{R}^n \times \mathbb{R}^n$ for edges $e \in E$
  - Usually comprising guard on current state and reset relations

All relations decidable / definable in first-order real arithmetic . . .
Computationally relevant output needs computational input!
Hybrid Automaton for Collision Avoidance

\[ x' = v \sin \vartheta \]
\[ x' = v \cos \vartheta \]
\[ \| x - y \| \geq \alpha \]
\[ \begin{align*}
  x' &= v \sin \vartheta \\
  y' &= v \cos \vartheta \\
  \tau' &= 1 & \text{straight} \\
  \tau' &= -1 & \text{turn} \\
  \| x - y \| &< 10
\end{align*} \]
Hybrid Automaton for Collision Avoidance

\[
\begin{align*}
\dot{\vartheta} &:= \vartheta + \frac{\pi}{4} & \tau &\leq 0 \\
\left\| x - y \right\| &< 10 \\
\dot{\vartheta} &:= \vartheta + \frac{\pi}{4} \\
\vartheta &:= \vartheta - \frac{\pi}{4}
\end{align*}
\]

- **cruise**
  \[
  \begin{align*}
  x_1' &= v \sin \vartheta \\
  x_2' &= v \cos \vartheta \\
  \left\| x - y \right\| &\geq \alpha
  \end{align*}
  \]

- **left**
  \[
  \begin{align*}
  x_1' &= v \sin \vartheta \\
  x_2' &= v \cos \vartheta \\
  \tau' &= 1
  \end{align*}
  \]

- **right**
  \[
  \begin{align*}
  x_1' &= v \sin \vartheta \\
  x_2' &= v \cos \vartheta \\
  \tau' &= -1
  \end{align*}
  \]

- **straight**
  \[
  \begin{align*}
  x_1' &= v \sin \vartheta \\
  x_2' &= v \cos \vartheta
  \end{align*}
  \]
Hybrid Automaton for Collision Avoidance: Formal

\[ \varphi_v(t; x, y, \tau) = \begin{pmatrix} x_1 + tv \sin \vartheta \\ x_2 + tv \cos \vartheta \\ y_1 + tu \sin \varsigma \\ y_2 + tu \cos \varsigma \\ \tau + t \end{pmatrix} \]

- \( \varphi \): Transition function
- \( t \): Time
- \( x, y \): Position
- \( \tau \): Time delay
- \( v, u \): Speed
- \( \vartheta, \varsigma \): Orientation

\begin{align*}
\| x - y \| &< 10 \\
\| x - y \| &\geq \alpha \\
\vartheta &:= \vartheta + \frac{\pi}{4} \\
\vartheta &:= \vartheta - \frac{\pi}{4} \\
\tau &\leq 0 \\
\end{align*}

- **Cruise**
- **Left**
- **Right**
- **Straight**
Hybrid Automaton for Collision Avoidance: Formal

- \( \text{inv}_{\text{cruise}} \equiv \|x - y\| \geq \alpha \)
- $inv_{\text{cruise}} \equiv \|x - y\| \geq \alpha$
- $inv_{\text{right}} \equiv \tau \geq 0$

Diagram:

- Cruise: $\|x - y\| \geq \alpha$
  - $\tau \leq 0$: $\vartheta := \vartheta + \frac{\pi}{4}$
  - $\tau \geq 0$: $\vartheta := \vartheta - \frac{\pi}{4}$

- Left: $\|x - y\| < 10$
  - $\vartheta := \vartheta + \frac{\pi}{4}$

- Right: $\tau \geq 0$
  - $\vartheta := \vartheta - \frac{\pi}{4}$

- Straight: $\vartheta := \vartheta + \frac{\pi}{4}$
\( \text{inv}_{\text{cruise}} \equiv \|x - y\| \geq \alpha \)

\( \text{inv}_{\text{right}} \equiv \tau \geq 0 \)

\( \text{inv}_{\text{left}} \equiv \text{inv}_{\text{straight}} \equiv \text{true} \)
Hybrid Automaton for Collision Avoidance: Formal

- $inv_{\text{cruise}} \equiv \| x - y \| \geq \alpha$
- $inv_{\text{right}} \equiv \tau \geq 0$
- $inv_{\text{left}} \equiv inv_{\text{straight}} \equiv \text{true}$
- $jump_e(x, x^+) \equiv guard_e(x) \land reset_e(x, x^+)$
Hybrid Automaton for Collision Avoidance: Formal

- \( \text{inv}_{\text{cruise}} \equiv \|x - y\| \geq \alpha \)
- \( \text{inv}_{\text{right}} \equiv \tau \geq 0 \)
- \( \text{inv}_{\text{left}} \equiv \text{inv}_{\text{straight}} \equiv \text{true} \)
- \( \text{jump}_e(x, x^+) \equiv \text{guard}_e(x) \land \text{reset}_e(x, x^+) \)
- \( \text{guard}_{\text{cruise, left}} \equiv \|x - y\| < 10 \)
Hybrid Automaton for Collision Avoidance: Formal

- $inv_{cruise} \equiv \|x - y\| \geq \alpha$
- $inv_{right} \equiv \tau \geq 0$
- $inv_{left} \equiv inv_{straight} \equiv true$
- $jump_e(x, x^+) \equiv guard_e(x) \land reset_e(x, x^+)$
- $guard_{cruise, left} \equiv \|x - y\| < 10$
- $reset_{cruise, left} \equiv \vartheta^+ = \vartheta + \frac{\pi}{4}$
Hybrid Automaton for Collision Avoidance: Formal

- $inv_{cruise} \equiv \|x - y\| \geq \alpha$
- $inv_{right} \equiv \tau \geq 0$
- $inv_{left} \equiv inv_{straight} \equiv true$
- $jump_e(x, x^+) \equiv guard_e(x) \land reset_e(x, x^+)$
- $guard_{cruise,left} \equiv \|x - y\| < 10$
- $reset_{cruise,left} \equiv \vartheta^+ = \vartheta + \frac{\pi}{4}$
- $reset_{left,straight} \equiv \vartheta^+ = \vartheta - \frac{\pi}{4}$
Hybrid Automaton for Collision Avoidance: Formal

- $inv_{\text{cruise}} \equiv \|x - y\| \geq \alpha$
- $inv_{\text{right}} \equiv \tau \geq 0$
- $inv_{\text{left}} \equiv inv_{\text{straight}} \equiv \text{true}$
- $jump_e(x, x^+) \equiv guard_e(x) \land reset_e(x, x^+)$
- $guard_{\text{cruise, left}} \equiv \|x - y\| < 10$
- $reset_{\text{cruise, left}} \equiv \vartheta^+ = \vartheta + \frac{\pi}{4}$
- $reset_{\text{left, straight}} \equiv \vartheta^+ = \vartheta - \frac{\pi}{4}$
- $reset_{\text{straight, right}} \equiv \vartheta^+ = \vartheta - \frac{\pi}{4}$
Hybrid Automaton for Collision Avoidance: Formal

- \( \text{inv}_{\text{cruise}} \equiv \| x - y \| \geq \alpha \)
- \( \text{inv}_{\text{right}} \equiv \tau \geq 0 \)
- \( \text{inv}_{\text{left}} \equiv \text{inv}_{\text{straight}} \equiv \text{true} \)
- \( \text{jump}_e(x, x^+) \equiv \text{guard}_e(x) \land \text{reset}_e(x, x^+) \)
- \( \text{guard}_{\text{cruise}, \text{left}} \equiv \| x - y \| < 10 \)
- \( \text{reset}_{\text{cruise}, \text{left}} \equiv \vartheta^+ = \vartheta + \frac{\pi}{4} \)
- \( \text{reset}_{\text{left}, \text{straight}} \equiv \vartheta^+ = \vartheta - \frac{\pi}{4} \)
- \( \text{reset}_{\text{straight}, \text{right}} \equiv \vartheta^+ = \vartheta - \frac{\pi}{4} \)
- \( \text{guard}_{\text{right}, \text{cruise}} \equiv \tau \leq 0 \)
Hybrid Automaton for Collision Avoidance: Formal

- \( \text{inv}_{\text{cruise}} \equiv \|x - y\| \geq \alpha \)
- \( \text{inv}_{\text{right}} \equiv \tau \geq 0 \)
- \( \text{inv}_{\text{left}} \equiv \text{inv}_{\text{straight}} \equiv \text{true} \)
- \( \text{jump}_{\text{e}}(x, x^+) \equiv \text{guard}_{\text{e}}(x) \land \text{reset}_{\text{e}}(x, x^+) \)
- \( \text{guard}_{\text{cruise}, \text{left}} \equiv \|x - y\| < 10 \)
- \( \text{reset}_{\text{cruise}, \text{left}} \equiv \vartheta^+ = \vartheta + \frac{\pi}{4} \)
- \( \text{reset}_{\text{left}, \text{straight}} \equiv \vartheta^+ = \vartheta - \frac{\pi}{4} \)
- \( \text{reset}_{\text{straight}, \text{right}} \equiv \vartheta^+ = \vartheta - \frac{\pi}{4} \)
- \( \text{guard}_{\text{right}, \text{cruise}} \equiv \tau \leq 0 \)
- \( \text{reset}_{\text{right}, \text{cruise}} \equiv \vartheta^+ = \vartheta + \frac{\pi}{4} \)
**Example (Property)**

If the aircraft are far apart and have compatible speed, then—when following the protocol—they will never crash?

---

**Example (Property)**

If the aircraft enter collision avoidance, then—when following the protocol—will they ever leave again, i.e. follow their old route?
**Definition (Hybrid Automata → Hybrid System)**

- \( Q := (M \times \mathbb{R}^n) \cap \{(v, x) : x \in \text{inv}_v\} \)
- Discrete transition \((v, x) \xrightarrow{a} (v^+, x^+)\) iff there is an edge \( e \) from \( v \) to \( v^+ \) with input \( a \) such that \((x, x^+) \in \text{jump}_e\)
- Continuous transition \((v, x) \xrightarrow{r} (v, x^+)\) iff \( x^+ = \varphi_v(r; x) \) for \( r \geq 0 \) and \( \varphi_v(t; x) \in \text{inv}_v \) for all \( 0 \leq t \leq r \).
Hybrid Systems

Definition (Hybrid Automata \(\rightarrow\) Hybrid System)

- \(Q := (M \times \mathbb{R}^n) \cap \{(v, x) : x \in inv_v\}\)
- Discrete transition \((v, x) \xrightarrow{a} (v^+, x^+)\) iff there is an edge \(e\) from \(v\) to \(v^+\) with input \(a\) such that \((x, x^+) \in jump_e\)
- Continuous transition \((v, x) \xrightarrow{r} (v, x^+)\) iff \(x^+ = \varphi_v(r; x)\) for \(r \geq 0\) and \(\varphi_v(t; x) \in inv_v\) for all \(0 \leq t \leq r\).
Definition (Hybrid Automata → Hybrid System)

- \( Q := (M \times \mathbb{R}^n) \cap \{(v, x) : x \in inv_v\} \)
- Discrete transition \((v, x) \xrightarrow{a} (v^+, x^+)\) iff there is an edge \(e\) from \(v\) to \(v^+\) with input \(a\) such that \((x, x^+) \in \text{jump}_e\)
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André Platzer (CMU)
Outline

1 Applications
   - Air Traffic Control
   - Hybrid Systems / Cyber-Physical Systems
   - Train Control
   - Car Control
   - UAV
   - Chemical/Physical Process Control
   - Biomedical Applications
   - Advanced Chip Design

2 Hybrid Systems
   - Labeled Transition Systems
   - Finite Automata
   - Hybrid Automata
   - Hybrid Systems

3 Differential Equations
How to describe continuous change?

Relate continuously changing quantity and its rate of change (derivative)
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Relate continuously changing quantity and its rate of change (derivative)

\[
\begin{align*}
y'(t) &= f(t, y) \\
y(t_0) &= y_0
\end{align*}
\]
How to describe continuous change?

Relate continuously changing quantity and its rate of change (derivative)

\[ y'(t) = f(t, y) \]

\[ y(t_0) = y_0 \]

in which direction \( y \) evolves as time \( t \) progresses

where \( y \) starts at time \( t_0 \)
Intuition of Differential Equations

\[ x'(t) = \frac{1}{4}x \]
\[ x(t_0) = 1 \]
\[
\begin{align*}
\Delta = 1 & \quad \Delta = 2 \\
\frac{dx}{dt} = \frac{1}{4}x \\
x(t_0) = 1
\end{align*}
\]
Intuition of Differential Equations

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x'(t) &= \frac{1}{4} x \\
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\end{align*}
\]
Definition (Ordinary Differential Equation, ODE)

\( f : D \rightarrow \mathbb{R}^n \) on domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \). Then \( Y : I \rightarrow \mathbb{R}^n \) is solution of IVP

\[
\begin{bmatrix}
y'(t) = f(t, y) \\
y(t_0) = y_0
\end{bmatrix}
\]

on interval \( I \subseteq \mathbb{R} \), iff, for all \( t \in I \),

1. \( (t, Y(t)) \in D \)
2. \( Y'(t) \) exists and \( Y'(t) = f(t, Y(t)) \).
3. \( Y(t_0) = y_0 \)

Accordingly for higher-order differential equations, i.e., differential equations involving higher-order derivatives \( y^{(n)}(t) \).

If \( f \in C(D, \mathbb{R}^n) \), then \( Y \in C^1(I, \mathbb{R}^n) \).
What is a solution of the following IVP?

\[
\begin{bmatrix}
    y'(x) = -2xy \\
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Solution:

\[y(x) = e^{-x^2}\]
ODE Examples

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Solution:

\[y(x) = e^{-x^2}\]

Proof.

\[
y'(x) = \frac{de^{-x^2}}{dx} = e^{-x^2}(-2x) = -2xy(x)
\]

\[y(0) = e^{-0^2} = 1\]
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...non-analytic

...non-elementary
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Existence

Peano

Theorem (Existence theorem of Peano’1890)

\( f \in C(D, \mathbb{R}^n) \) on open, connected domain \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) with \( (x_0, y_0) \in D \). Then, IVP has a solution. Further, every solution can be continued arbitrarily close to the border of \( D \).

Example (Solvable)

\[
\begin{bmatrix}
y' = \sqrt{|y|} \\
y(0) = 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
y'(x) = 3x^2y - \frac{1}{y} \sin x \cos y \\
y(0) = 1
\end{bmatrix}
\]
Existence

Example (Solvable but not uniquely)

\[
\begin{bmatrix}
  y' &= \sqrt{|y|} \\
  y(0) &= 0
\end{bmatrix}
\]

\[y_1(x) = 0\]

\[y_2(x) = \frac{x^2}{4}\]
Example (Continuable but limited)

\[
\begin{bmatrix}
    y' = 1 + y^2 \\
y(0) = 0
\end{bmatrix}
\]
Definition (Lipschitz-continuous)

$f : D \rightarrow \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is Lipschitz-continuous for $y$ iff there is an $L \in \mathbb{R}$ such that for all $(x, y), (x, \bar{y}) \in D$:

$$\|f(x, y) - f(x, \bar{y})\| \leq L\|y - \bar{y}\|$$
Lipschitz-Continuity

Definition (Lipschitz-continuous)

A function $f : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is **Lipschitz-continuous** for $y$ iff there is an $L \in \mathbb{R}$ such that for all $(x, y), (x, \bar{y}) \in D$:

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Lipschitz-Continuity

Definition (Lipschitz-continuous)

$f : D \rightarrow \mathbb{R}^n$ with $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is Lipschitz-continuous for $y$ iff there is an $L \in \mathbb{R}$ such that for all $(x, y), (x, \bar{y}) \in D$:

$$\|f(x, y) - f(x, \bar{y})\| \leq L\|y - \bar{y}\|$$
Define a function \( f: D \to \mathbb{R}^n \) with \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) as Lipschitz-continuous for \( y \) if there exists an \( L \in \mathbb{R} \) such that for all \((x, y), (x, \bar{y}) \in D:\)

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\|f(x, y) - f(x, \bar{y})\| \leq L\|y - \bar{y}\|
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\( f : D \to \mathbb{R}^n \) with \( D \subseteq \mathbb{R} \times \mathbb{R}^n \) is \textit{Lipschitz-continuous} for \( y \) iff there is an \( L \in \mathbb{R} \) such that for all \( (x, y), (x, \bar{y}) \in D \):

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\| f(x, y) - f(x, \bar{y}) \| \leq L \| y - \bar{y} \|
\]

If \( \frac{\partial f(x, y)}{\partial y} \) exists and is bounded on \( D \) then \( f \) is Lipschitz-continuous. \( f \) is \textit{locally Lipschitz-continuous} iff for each \( (x, y) \in D \), there is a neighbourhood in which \( f \) is Lipschitz-continuous.
Existence and Uniqueness
Picard-Lindelöf / Cauchy-Lipschitz

Theorem (Uniqueness theorem of Picard-Lindelöf’1894)

In addition to Peano premisses, let $f$ be locally Lipschitz-continuous for $y$ (e.g. $f \in C^1(D, \mathbb{R}^n)$). Then, there is a unique solution of IVP.
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Proposition (Global uniqueness theorem of Picard-Lindelöf)

\( f \in C([0, a] \times \mathbb{R}^n, \mathbb{R}^n) \) Lipschitz-continuous for \( y \). Then, there is a unique solution of IVP on \([0, a]\).
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