1 Introduction

More information can be found in [App98, Ch 18.1-18.3] and [Muc97].

Last lecture, we have seen strength reduction. In order to perform
strength reduction, however, we need to know which of the variables change
linearly in the loop. These are called induction variables.

2 Induction Variables

Once we have identified a loop (e.g., natural loop), one of the central ques-
tions about it is, which variables are induction variables of the loop. In
for-loops, there are syntactical indicators if the step is of the form $i++$. But
that alone does not make $i$ a proper induction variable, because there could
be further assignments to $i$. Furthermore, other variables could effectively
be induction variables, even if they are not written in the step part of a
for-loop. Finally, induction variables are also of interest for while or repeat-
until loops, where they are not identified syntactically. Consequently, we
need an analysis to identify induction variables.

Generally, we will consider variables to be induction variables if their
value is linear in the number of loop iterations. We call a variable $i$ a basic
induction variable if the only assignments to $i$ in the loop body are of the
form $i = i + c$ (or $i = i - c$) for a loop-invariant expression $c$. We call
variable $j$ a derived induction variable if it only assumes values of the form
$j = a_j \times i + b_j$ for a basic induction variable $i$ and loop-invariant expressions
$a_j, b_j$. 

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s = 0
i = 0
l1: if (i ≥ n) goto l2
j = 4*i  // derived j=4*i+0
k = j + a  // derived k=1*j+a=(1*4)*i+a
x = M(k)
s = s + x
i = i + 1  // basic
goto l1
l2:

Figure 1: Example computing sum of 32bit array contents

In fact, both basic and derived induction variables are linear functions of the basic induction variable. All of them have the form \( j = a_j * i + b_j \) (where \( i = 1 * i + 0 \) is a special case).

There are several ways to find a derived induction variable \( j \) in a loop. We show one way following [App98]. Variable \( j \) is a derived induction variable if it is only defined once in the loop body with a definition \( j = a_j * k + b_j \) for a (derived or basic) induction variable \( k \) and loop-invariant expressions \( a_j, b_j \). If this variable \( k \) is a derived induction variable then we also require that its (unique) definition in the loop body is the only definition of \( k \) reaching the definition of \( j \) and that the corresponding basic induction variable \( i \) for \( k \) is not redefined on any path between the respective definitions of \( j \) and \( k \).

\[
\begin{align*}
    l : j &= i ± c \\
    \neg inv(c) &\quad \rightarrow \quad \neg IV(j) & BIV_1 \\
        &\quad \quad \quad \rightarrow \quad \neg IV(i) & BIV_2
\end{align*}
\]

\[
\begin{align*}
    l : j &= a_j * i ± b_j \\
    \neg inv(a_j) \lor \neg inv(b_j) &\quad \rightarrow \quad \neg IV(j) & IV_1 \\
        &\quad \quad \quad \rightarrow \quad \neg IV(i) & IV_2
\end{align*}
\]

\[
\begin{align*}
    l : j &= \Phi(i_1, \ldots, i_n) \\
    \neg inv(i_k) &\quad \rightarrow \quad \neg IV_B(i_k) \\
        &\quad \quad \quad \rightarrow \quad \neg IV(j) & \Phi IV
\end{align*}
\]
For $IV_1$ and $IV_2$, the case where $a_j$ does not appear (corresponding to $a_j = 1$) actually includes $BIV_1$ and $BIV_2$ as a special case.

For computing induction variables, we proceed as follows. We first just assume that all variables were induction variables. Then we successively throw candidates out that do not match the conditions. For SSA programs, this is particularly easy.

\[
S = \text{set of all variables} \\
\text{repeat until fixedpoint:} \\
\quad \text{remove } j \text{ from } S \text{ if } j \text{ not computed as one of the forms} \\
\quad \text{basic:} \\
\quad \quad j = i \pm c \text{ for an } i \in S \text{ and a loop-invariant } c \\
\quad \text{derived:} \\
\quad \quad j = a_j \ast i \pm b_j \text{ for an } i \in S \text{ and loop-invariant } a_j, b_j \\
\quad \text{flow:} \\
\quad \quad j = \Phi(i_1, \ldots, i_n) \text{ and each } i_k \text{ loop-invariant or basic}\in S
\]

3 Strength Reduction for Induction Variables

If we have found a basic induction variable $i$ that is initialized to $i_0$ before the loop and a derived induction variable $j$, then we can replace $j$ by a new induction variable $j'$ as follows. Then we replace the loop

\[
i = i_0 \\
\text{while (e) } \{ \\
\quad \ldots \\
\quad j = a_j \ast i + b_j \\
\quad \ldots j \ldots \\
\quad i = i+c \\
\quad \ldots j \ldots \\
\}\]

according to the strength reduction optimization by

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### 18.4 Induction Variables

\[ i = i_0 \]
\[ j' = a_j * i_0 + b_j \]

while (e) {
  
  \[ j = j' \] // j updates by shadow j'
  
  \[ i = i+c \]
  
  \[ j' = j' + a_j * c \] // increment j' at every change of i
  
  \[ j \]
}

After every assignment to the basic induction variable \( i \), we increment the new variable \( j' \). And the (single) assignment \( j = a_j * i + b_j \) gets replaced by \( j = j' \). Note that \( a_j * c \) can either be computed by constant folding or is loop-invariant and can be moved outside. Finally, we can rely on copy propagation to optimize \( j \) away as much as possible. We can also use reassociation and constant folding to accumulate successive increments of \( j' \) within the loop body into one assignment if that is permitted.

Figure 2 on p 5 shows the result of strength reduction optimization of Fig. 1 on p 2. Note that the variable \( j' \) is quite useless, because its only purpose has become to assign to itself. This is what the neededness analysis from lecture 5 on dataflow analysis can figure out and eliminate \( j' \). The only use of \( j' \) is to define itself and it’s dead after the loop too.

### 4 Almost Useless Variables

If the induction variable \( i \) is still used in the loop body or loop test \( e \) then the assignments to \( i \) can either be kept, or, instead, uses of \( i \) can be recomputed from \( j \) and replaced by \((j - b_j) \div a_j\). The latter really only makes sense when this division can be simplified arithmetically. At least we know that \( j \) changes in multiples of \( a_j \).

In Figure 2 (bottom) there is an almost useless variable \( i \). The reasoning is by using that \( k \) is derived from \( j \) by \( k = a_k * j + b_k \), which is derived from \( i \) by \( j = a_j * i + b_j \). Consequently,

\[ i = (j - b_j) \div a_j = (((k - b_k) \div a_k) - b_j) \div a_j \]

Thus, \( i \geq n \) is equivalent to

\[ (((k - b_k) \div a_k) - b_j) \div a_j \geq n \]
s = 0
i = 0
j' = 0  // j' not needed => dead
k' = a
l1: if (i $\geq$ n) goto l2
j = j'
  // dead
k = k'
x = M(k)
s = s + x
i = i + 1
j' = j' + 4  // not needed
k' = k' + 4
goto l1
l2:

Neededness analysis removes useless j'. Copy propagation of k = k' gives

s = 0
i = 0
k' = a
l1: if (i $\geq$ n) goto l2
x = M(k')
s = s + x
i = i + 1  // almost useless
k' = k' + 4
goto l1
l2:

Figure 2: Example from Figure 2 after strength reduction for j and k.
Inserting the relations from Fig. 1, we get

\[((k - a) \div 1) - 0) \div 4 \geq n\]

i.e.

\[(k - a) \div 4 \geq n\]

This is equivalent to the following, because we know that \(k\) will only change in multiples of its linear factor 4

\[k - a \geq 4 \ast n\]

i.e.,

\[k \geq 4 \ast n + a\]

This \(4 \ast n + a\) is a loop-invariant expression that can be computed before the loop.

After optimizing the almost useless variable \(i\) away, we get Figure 3.

```c
s = 0
k' = a
e = 4*n + a
l1: if (k' \geq e) goto l2
x = M(k')
s = s + x
k' = k' + 4
goto l1
l2:
```

Figure 3: Strength reduced example from Figure 2 after eliminating the almost useless variable \(i\).

References
