What about non-bipartite graphs?

Note: \( \sum_j x_{ij} = 1 \quad \forall i \in V, \quad x_{ij} \geq 0 \) is not the same as convex hull of all perfect matchings.

\( K = \{ x \in \mathbb{R}^{|E|} \mid \sum_j x_{ij} = 1 \quad \forall i \in V, \quad x_{ij} \geq 0 \} \quad \text{Edmonds': the perfect matching polytope (i.e. convex hull of all PMs in } G \text{)}

is given by

\[
K = \{ x \in \mathbb{R}^{|E|} \mid \sum_{j} x_{ij} = 1 \quad \forall i \in V, \quad x_{ij} \geq 1 \quad \forall \text{odd sets } S \subseteq E \}
\]

Proof: Different ways to do it. Can use Blossom algorithm. Here indirect proof.

Let \( \mathcal{C}^{pm} = \text{convex hull of all } X_M : M = \text{perfect matching in } G \).

Since each \( X_M \in K_{PM} \), \( C^{PM} \subseteq K_{PM} \). So now suffice to show that \( K_{PM} \subseteq C^{PM} \). We induct on \( 1E_1 \).

Base case: \( 1E_1 = 1 \) then must have 2 vertices, and \( x_{uv} = 1 \). \( \Rightarrow \) trivial.

Inductive step: \( X \in K_{PM} \). and \( X \) has a vertex \( A \in K_{PM} \).

Want to show \( X \in C^{PM} \). Then if all vertices \( A \in K_{PM} \) in \( CH \Rightarrow \) all \( A \in K \in CH \).

If \( x_e = 0 \) then induct on \( G \setminus e \).

If \( e = 1 \) then induct on \( G \setminus u, v, u, v \), since all other edges in \( du, dv = 0 \).

\( x_e > 0 \) then if all vertices have degree 2 \( \Rightarrow x \) cannot be a vertex \( \in K_{PM} \).

\( \Rightarrow \) cycles

\( \Rightarrow \) if vertex of degree 2 (and all else \geq 2) \( \Rightarrow 1E_1 > 1V_1 \). \( \Rightarrow \) \( \geq n + 1 \) tight constraints.

\( \Rightarrow \) one non-trivial constraint tight \( \Rightarrow x^*(A) = 1 \).
\[ S = V \setminus S^* \]

\[ G/3, G/3 \] by contracting one side or other to ver. \[ x^*, \bar{x} \].

Since \( x(3U) = 1 \) both are in the KPM phytype of the respective graphs.

\[ Z^* = \sum_{\text{N in G/3}} x_N X_N \]

\[ X^* = \sum_{\text{N in G/3}} x_N X_N \]

Now match them up to get \[ z = \frac{1}{N} \sum x x_i \]

Here's a different proof that for bipartite graphs, the perfect matching phytype is

\[ K_P = \{ z \in \mathbb{R}^E \mid \sum_{x} x_{ij} = 1, \sum_{x} x_{ij} = 1, \exists \} \subseteq \mathbb{R}^m \]

**Pf:** Consider any vertex \( x \in K \). Want to show it is a perfect matching.

Since \( \forall M, x_M \in K \), this will prove that \( K = CH(CPMs) \).

\( X \in K \) is a vertex. So obtained by \( m \) tight constraints (linearly indep.)

There are \( 2n + m \) constraints.

Also \( \leq (2n-1) + m \) LC constraints (since \( \sum_{x} x_{ij} = 1 \) \( \leq \sum_{x} x_{ij} = 1 \))

\( \Rightarrow \) at least most \( 2n-1 \) of the interstig constraints are tight (and all others)

\( \Rightarrow \) at least \( m - (2n-1) \) of the tight constraints at \( x \) are \( x_{ij} = 0 \).

\( \Rightarrow \) at most \( 2n-1 \) edges have non-zero values.

But \( \exists \) 2n vertices, and each vertex has \( 2n-1 \) edge out of it

So this and there is a vertex with degree \( 1 \).

\( \Rightarrow \) 1 edge with value \( = 1 \).

\[ x_{12} = 1 \]

Now induct on the rest of the graph \( G/E, V \).