
- G: undirected graph \((V,E)\).
- Matching: set of edges \(M \subseteq E\) s.t. every node has degree \(\leq 1\).
- Max cardinality matching in \(G\) or max weight matching.
- "Perfect" matching = every node has degree = 1.

Some examples

- jobs & machines
- boys & girls

Bipartite matching is a simpler case. Existence proofs, "Fast" algorithms, etc.

1. Via a reduction to maximum flows.

\[
\text{Max flow} = \text{max cardinal matching}.
\]

\[
\text{Max flow} \leq \text{Augmenting paths}.
\]

\[
\text{run in time } O(mf) = O(mn).
\]

Even, Tarjan, and Dinic showed that for unit capacity graphs, the
runtime is \(O(m \min\{m^{1/2}, n^{3/4}\})\).

2. Via direct algorithms (also known as "augmenting paths").

- Given a matching \(M \subseteq E\) (technically means \(M \subseteq E(G)\)).
- An augmenting path alternates between \(M\) and \(\overline{M}\) edges,
- an open vertex is not matched in \(M\).
- An augmenting path is an alternating path \(P\) between 2 open vertices. (must be odd length)

\[
\Rightarrow M \cup P \text{ is also an matching, of greater cardinality.}
\]

A symmetric difference

Fact [Berge] \(M\) is a maximum cardinality matching if and only if
no augmenting path with respect to \(M\).
Proof: if I any path ≠ clearly M not maximum.

Sp M maximum not max. let M* be maximum.
allowing
M ∩ M* = a path and cycles.

Since |M*| > |M| must have at least one odd length alternating path w/o M.
into both ends open in M ⇒ M-augment path.

⇒ just find M-augment patho until reach max matching.

How to do this?

Bipartite Case is easier. G = (V, R, E)

N.B. showing that M is a max matching is tricky using this characterization. Have to show a "co-NP type" certificate. Here's another way to show that M is Max matching.

Konig's theorem: G, let MN(G) = max matching

then MM(G) = VCC(G).

[it is easy to show that #VC, #matchings, M ≤ VC]

So we could also show a VC if cardinality |M| and hence prove optimality.

Very good. But how to find true matching?

Idea basically looks like FFon the red/oriograph.

Take all open vertices level 0. (marked).

Then at level i, (open), ≤ L

must be

level i-2 = all unmarked vertices connected to level i-1 vertices by non-M edges.

level i-1 = if we see an open vertex at level i-1, then found an odd length alt path b/w 2 open => augment.
(where can edge from level \( i \) (even) go?)

not to \( L \), because bipartite so to \( R \).

(to previous odd levels, or to unmarked
new vertices)

level \( i \geq 2 \): take matched edge out of
the level \( i \) vertices. Since we
did not find any open \( |L| \)-level nodes, must do this.

Do until all vertices matched or unreachable.

Note: if \( \Gamma \) an \( M \)-augmenting path, we will find it this way, in \( O(\text{m}) \) time.

\( \Rightarrow \) total time \( O(\text{mn}) \).

([Really just doing Ford-Fulkerson].)

Can we use König's theorem to prove optimality?

Sure. Let \( X \) = marked vertices. Then

Claim: \( C^* := (L - X) \cup (R \cap X) \) is a vertex cover of size \( M \).

\( \text{Pt}: \) if \( e \) not covered \( \Rightarrow \) \( e \) is marked

\( = (L, r) \) but \( r \) is not marked.

\( \Rightarrow \) \( e \) cannot be in matching else when \( r \) is marked,

\( \text{next } x \text{ would be marked.} \)

\( \text{would just have been marked.} \)

\( \text{when } r \text{ is marked, } x \text{ would be marked next.} \)

\( \Rightarrow \) no such \( e \) exists.

Next: \( |C^*| \leq |M| \)

- Every vertex in \( ROX \) has a matching edge incident to it. (else augment \( \Gamma \)!)
- Every vertex in \( L - X \) has . . . . . (else would be picked in level 0).
- There are no edges between \( L - X \) and \( ROX \), so distinct edges.

\( \Rightarrow \) \( |C^*| \leq |M| \).
Not true in non-bipartite graphs

For general graphs this theory is richer/looser.

Tutte-Berge: Suppose pick a set $U \subseteq V$, and delete it from $G$.

* set components $K_1, K_2, \ldots, K_k$

How big a matching can graph have?

$$|U| + \sum_{i=1}^{k} \left\lfloor \frac{|K_i|}{2} \right\rfloor = |U| + \frac{(|V|/2) - \#\text{odd comp.}}{2}$$

One edge per vertex in $U$

if $\text{odd}(G \setminus U) = \#\text{odd components in } G \setminus U$,

$$|M| \leq \min_{U \subseteq V} \left| U \right| + \frac{|V| - \text{odd}(G \setminus U)}{2}$$

Theorem: for a graph $G$,

$$[\text{Tutte-Berge}] \quad |MM(G)| = \min_{U \subseteq V} \left| U \right| + \frac{|V| - \text{odd}(G \setminus U)}{2}$$

Observation: If $G$ is bipartite, let $U = \text{vertex cover of } G$. ($G \setminus U$) has only isolated vertices (no edges), so $\text{odd}(G \setminus U) = |V| - |U|$.

$$\Rightarrow \text{RHS} \leq \left| U \right| = |V| = 1 + \text{VC}^*.$$  

This is clearly a bond stronger than $MM^* \leq \text{VC}^*$.

Take $U = \emptyset$  \Rightarrow  \text{RHS} = \frac{0 + 3 - 1}{2} = 1 = MM^*.$
How to prove this? Via an algorithm [Edmonds Blossom Alg].

Again: want to find M-augmenting path (if one exists).

Ead: find M-aug path, P, M ⊆ MASP, repeat.

Unfortunately: can only show:

Theorem 1: Suppose G contains an M-augmenting path P. Then

any algorithm that finds either

(a) an M-augmenting path or
(b) a blossom.

What's a blossom?

A blossom

(an alternating path

starting at open vertex)

and a blossom (an almost alternating cycle) of odd length)

If we find a blossom (with stem S and blossom B)

(i) toggle edges on stem (i.e., M ⊆ MASP)

as set flower into empty stem

(open!!)

(ii) Shrink B into new vertex V. Call graph G/B.

(iii) Find a (M/B)-augmenting path in G/B (recursively)

just drop the edges of M on B. Keep all others.

(iv) Extend P' to an M-augmenting path P, in G.

Hence: in either case find M-augmenting path. ⇒ make progress.
Q1: How does this also work?
Q2: Why is this OK? For this we need Thm 2.

**Theorem 2:**

If a M-ary path in \( G \) \( \iff \) a M/bay path in \( G/B \).

**Pf:**

- If M-ary path \( P \) does not hit \( B \) \( \Rightarrow \) Still in \( G/B \).
- Else: at least one end of \( P \) not on \( B \).
  - \( P \) has 2 ends open, \( B \) has one open node.
  - Say \( u \). Let \( v \) be first node on \( P/B \). Then
    - \( P[B] \leftarrow u \rightarrow v \) is M/bay path in \( G/B \).

[Recall: \( v_B \) is open in \( G/B \)].

**\( \Leftarrow \):**

Consider \( P \) in \( G/B \). Either misses \( v_B \) then in \( G \) also

Or, one end is \( v_B \). Then extend.

---

**Now to theorem 1.**

Also: Start with all open guys in level 0. (Mark them).

Do a simultaneous "BFS" from \( L0 \) as follows.

Given \( L(2i) \), do the following:

look for all unmatched edges from \( L2i \). Say \( u \in L2i \), \( v \) end.

(i) if \( v \) unmarked, \( v \in L(2i+1) \), mark. (Even odd)

(ii) if \( v \) at same level, aug or blossom!

Why? Look at paths open \( 
\overrightarrow{u} \) \( \in \) \( u \) and \( (uv) \) & M.
(iii) If \( v \) at previous odd level \( \text{OK} \). (e-o).

(iii) If \( v \) at previous even level \( L_{2j} \) \((j<i)\). Not possible! else \( u \) would be at level \( 2j+1 < 2i \).

\[ \Rightarrow \text{either success or all edges (even-odd)} \]

Given \( L_{2i+1} \), get \( L_{2i+2} \) as follows: \( u \in L_{2i+1} \) \((u,v) \in EM\). Match \( j \) edges, nyj!

(4) If \( v \) unmarked, \( v \in L_{2i+2} \), mark. \([0-e] \)

(iii) If \( v \in L_{2i+1} \), aug or blossom!!

(iii) \( v \) cannot be at previous levels.

\[ \Rightarrow \text{Continue until level} \equiv \text{empty} \]

No. if no success \( \Rightarrow \) all edges even-odd.

Now ARSOC: \( M \)

- Augment path \( P \) in \( G \), and we don't find aug or blossom.

- Want a contradiction:-

- Label each vertex with parity (even/odd).
  - Ends have parity \( E \) (open \( \Rightarrow \text{Lo} \Rightarrow \text{even} \)).
  - Endpoints of edges have opposite parity.
  - But path of odd length \( \Rightarrow \) even \# 4 vertices.

\[ \text{E \quad \bigcirc \quad E \quad \bigcirc \quad E \quad \bigcirc \quad E \quad \bigcirc \quad E \quad \bigcirc \quad E \quad \bigcirc \quad E \quad \bigcirc \quad E \quad \bigcirc \quad E \]
Finally: Tutte-Berge

\[ G \mapsto G' \] and here submends did not find any cross edge ("success")

If odd vertices in \( G' \) are bipartite graph

\[ \Rightarrow \text{take all odd vertices. Each one has a successor matched edge} \Rightarrow U = \text{odds} \]

And all unmarked nodes are matched (not open)

So \( (G' \setminus U) \) has 1 Even components all singletons = odd

all other components matched up = even

\[ \Rightarrow \frac{|V'| + |U| - \text{odd}(G' \setminus U)}{2} = \frac{2n' + |\text{odd}| - \text{Even}}{2} \]

\[ = \frac{n' + |\text{odd}| - (n' - |\text{odd}| + \text{rest})}{2} \]

\[ = \frac{2 \cdot \text{odd} + \text{rest}}{2} = \text{odd} + \text{rest} \]

\[ = M' \]
Now pull back to $G$:

- For each $u'$, we have
  \[ M' = \frac{n' + |u' - \text{odd}(u')|}{2} \]

- Want to extend back to $U$.
  \[ M = M' + \frac{B - 1}{2} \]

Note: $V_B$ was even $\Rightarrow$ not in $U'$ (and in fact in odd component) and open

$\Rightarrow$ replace $V_B$ by $B$ still means in odd component (size increases by $B - 1 = \text{even}$).

\[ |M| = M' + \frac{B - 1}{2} = \frac{n' + |u| - \text{odd}(u)}{2} + \frac{B - 1}{2} = \frac{n + |u| - \text{odd}(u)}{2} \]

Tutte's Perfect Matching Thm: $\begin{array}{c} G \text{ has a PM } \iff \ O(G \setminus U) \leq |U| + |U_{\bar{V}}| \end{array}$


decrease: find alternating path in time $O(mn)$

total: $O(mn^2)$.

Can do better: [Micali Vazirani] $O(m \sqrt{n})$

[Müsch Taubalowski] $O(n^3)$.