Smoothed Analysis

Instead of having out \( \max_{I \cap n} \text{runtime}(I) \)

we consider \( \max_{I \cap n} \sum_{I' \in \text{neighborhood}(I)} \text{runtime}(I') \)

\[ \text{Smooth out the runtime} \]

What is the definition of the neighborhood?

depends on the problem — but we want it to be

big enough to that we can smooth out the performance.

"If smooth complexity is small, means that the worst cases are isolated"

**Major results:** Spielman & Teng showed that a certain pricing rule for simplex has

polynomial smoothed complexity under a certain noise model.

Improvements by Deshpande & Spielman, and Veeravagu.

Results by Bans & Veeravagu for k-sparse and other problems.

Example: the 2OPT heuristic for TSP. (Local search).

\[ \text{Start with an arbitrary tour. While } J \text{ a pair} \]

\[ \text{Swap } (ac) \text{ with } (ad) \rightarrow (cd) \text{ gives a lower cost solution, take it.} \]

Knew that \( J \) exponentially long improving paths in some instances.

But does well in practice.

Simplified Smoothed Model: Fix a graph \( G = (V, E) \).

For each edge \( e \), give a density function \( f_e : E \rightarrow [0, 1] \).

\( \rightarrow \) the edge lengths are drawn independently from \( f_e \).

Now run the 2OPT heuristic. (On the graph, not metric completion).

Claim: longest path in the improvement graph is \( O(n \log n \cdot \Phi) \).

Directed edges are tours, 1-step from \( T_i \) to \( T_2 \) if fastest way to get

from \( T_i \) to \( T_2 \), which is improving.
N.B. this means that no matter where we start, after $O(n \log n)$ steps we get in a local optimum.

Pf: initial tour has $n = n$. Suppose the smallest improvement were $\Delta$, then would take $\geq \frac{\Delta}{E}$ steps.

Claim: $P_{e \in E} \Delta \leq n^c \epsilon p$

$\Rightarrow Pr[T > t] \leq Pr[\Delta \leq n^c] \leq n^c \frac{\epsilon p}{t} \epsilon$

$E[T] = \sum_{t=0}^{n^c} Pr[T > t] \leq n^c \epsilon p \epsilon \ln(\epsilon n) \leq O(n^c \log n \epsilon^p)$. 

Pf of claim: fix the 4 edges, $n$ ways. Now have fixed the largest 3 edges,

$P_e \Delta \leq n^c \epsilon p \epsilon$

$\Rightarrow n^c \epsilon p$.

This was a simplified model $\epsilon$-ipita.

in [ERV: Erglov, Boas, Vöck by] they construct for a smoothed model of choosing points in $R^d$ Euclidean space $R^d$ (for $d$ constant),

$E[\text{length} + 20pT \epsilon \text{improving more}] \leq O(n^{4+\frac{1}{3}} \log(n\epsilon^p) \epsilon^p)$

And can ignore if choose start point smartly, etc.

Note: they find a local optimum, which may be quite bad. (but not too bad)

[ERV]: the expected approximation ratio for a locally optimal tour chosen from a distribution $\delta$, where each point $v$ is picked from $f_v : [0,1]^d \rightarrow [0,p]$ is $O(\sqrt{p})$. [local opt not 20p!]
Knapack: sizes/weights \( w_i \), profits \( p_i \), size \( = 1 \) (say).

[Nemhauser/Ullmann] Build the pareto curve and pick the best one from it.

- Pareto curve, \( x \in \mathbb{R}^{0 \times 1} \): on the pareto curve if \( \forall x', \quad w.x \leq w.x' \Rightarrow p.x \leq p.x' \)

\[ \text{profit} \quad \text{empty} \quad x_2 \quad x_3 \]
\[ \text{weight} \]

- Let \( \mathcal{P}(i) = \) pareto curve \( \Rightarrow \) solutions for items \( [1..i] \).

**Fact:** \( \mathcal{P}(i+1) \subseteq \mathcal{P}(i) \cup \{ x + e_i | x \in \mathcal{P}(i) \} \)

\[ \Rightarrow \text{can compute } \mathcal{P}(n) \text{ in time } O \left( \sum_{i=1}^{n} \mathcal{P}(i) \right) \].

- **Smoothing model:** say the weights are chosen randomly, with \( w_i \in [0,1] \) \( \rightarrow \) \( [0,\phi] \).

\[ \phi = \text{"smoothing parameter". Can generalize this to } w_i : [0,\phi] \rightarrow [0,\phi] \text{ etc., but see paper. (suggestions: this sounds?)} \]

Claim: \( E[\text{size of pareto curve on n items}] = O(n^2 \phi) \)

**Proof:** Let's break the weight axis into pieces of width \( \frac{1}{k} \), and hence the size of \( \mathcal{P}(n) \) can be written as
\[
(\star) \quad 1 + \lim_{k \to \infty} \sum_{i=0}^{k} \mathbb{I}(\text{there exists } x \in \mathcal{P}(n) \text{ with } w.x \in \left( \frac{i}{k}, \frac{i+1}{k} \right])
\]

**Note:** because smoothing, no two solution can have same weight (w/1).

**Note 2:** all weights \( \in [0,1] \) so total size \( \in [0,n] \).

\[
(\star) = 1 + \lim_{k \to \infty} \sum_{i=0}^{\left\lfloor \frac{n}{k} \right\rfloor} \mathbb{I}(\exists x \in \mathcal{P}(n) \text{ with } w.x \in \left( \frac{i}{k}, \frac{i+1}{k} \right])
\]

For any threshold \( t \), define \( \Delta(t) = \begin{cases} \infty & \text{if no such } x \text{ exists} \\ 0 & \text{for } x \text{ least upper bound } \leq t \end{cases} \)

at \( t \):
\[
2^* = \arg\max \left\{ w.x \left| x \in \mathbb{R}_{0 \times 1}^{n} \land w.x \leq t \right\} \] is winner
\[
2 = \arg\min \left\{ w.x \left| x \in \mathbb{R}_{0 \times 1}^{n} \land p.x > p.x^* \right\} \] is lower.
Note: many ways to define \( \text{inner} \& \text{outer}, \) these definitions are carefully chosen to make proofs work.

\[
\Rightarrow (\ast) = 1 + \lim_{k \to \infty} \sum_{i=1}^{\frac{n}{k}} \mathbb{1}(\Delta(i/k) \in [0, \varepsilon/k])
\]

Claim: \( \forall t \) \( \mathbb{P}[\Delta(t) \in (0, \varepsilon)] \leq n^2 \phi \).

\[
\Rightarrow \mathbb{E}[\text{Penetocurve}] \leq \sum_{k=n^{\phi}}^{n^2 \phi} \sum_{i=1}^{\frac{n}{k}} \mathbb{P}[\Delta(i/k) \in (0, \varepsilon/k)] \leq \sum_{k=n^{\phi}}^{n^2 \phi} \frac{n}{k} \leq n^2 \phi.
\]

**Proof & Claim:**

Define

\[
x^* = \arg\max \{ p \cdot x \mid x \geq 0 \land p \cdot x \leq t \}
\]

\[
x_i = \arg\min \{ w \cdot x \mid x \geq 1 \land p \cdot x > p \cdot x \}
\]

\[
\Delta(t) = \begin{cases} 
  \sum_{i} w \cdot x_i - t & \text{if } x_i \text{ exists} \\
  \infty & \text{otherwise}
\end{cases}
\]

**Subclaim:** either \( \Delta(t) = \infty \) or \( \Delta(t) = \Delta^i(t) \) for some \( i \).

*Proof:* if there is some \( x^* \) and \( x_i^* \) at \( t \), then \( x_i \) is \( t \) \text{ not in } x^* \text{ (since } w_i \text{ are } 0) \text{.}

Now this will give \( x^* \) and \( x_i \). Details easy.

\[
\Rightarrow \sum_{\Delta(t) \in [0, \varepsilon]} \mathbb{P}
\]

**Subclaim:** \( \mathbb{P}[\Delta^i(t) \in (0, \varepsilon)] \leq \phi \).

*Proof:* fix \( w_i \) all except \( i \). Now \( x_i \) fixed. Also identify \( x_i \) at each \( t \) min money together. Chance that its weight falls into \( t \) within interval \( \leq \phi \).
Similarly can imagine profit is random, weight is adversarial.

- Now we did not really see that profit was generated by a modular (additive) function $\pi_i$, but just that there was a total ordering of solutions based on profit, and $x^*$ was defined as the one higher in this total ordering than $x$. We did use that weights were additive.

- So can extend to model with general weights, but random profits (note that we now need a rank of $n$ elements according to "weights", and we do need that profits are additive). This extends to any combinatorial optimization problem with solutions set $S \subseteq \{0, 1\}^n$, profit function $p(x) = \sum p_i x_i$ (with $p_i$'s being $\phi$-smooth).

- Roglin & Teng, Moitra & O'Donnell extend to multiobjective approx.

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Finally: simplex smoothed analysis.

- Shadow vertex pivot rule. [Garey, Johnson]
- Smoothly rounded: given $\tilde{A}, \tilde{b}, c, \sigma$ we want to solve

$$\max c^T x$$
$$s.t. (\tilde{A} + \sigma G)x \leq (\tilde{b} + g)$$

$G, g$ have independent Gaussian entries, mean 0, standard deviation $\sigma$. Max $\| (\tilde{A} + \sigma G) x \|_{\infty}$

[Like saying adversary can choose means for the random $\tilde{A}$, but not actual values.]

- ST showed: let $\tilde{c}, \tilde{x}$ be vectors (fixed), $a_1, \ldots, a_n \in \mathbb{R}$ Gaussian vectors (independent) with means having norms $\leq 1$, standard deviation $\sigma$.

$$\mathbb{R} \ni \tilde{x} = 1 \sum a_i x_i \leq 1 \forall i$$. Then $P$ projected onto plane $(\tilde{a}, \tilde{c})$ has at most $\text{poly}(n^d, \sigma)$ vertices.

Problem: $u$ is not indep of the $a_i$'s, since it depends on starting point. How to find starting point anyways?

[STOC] and [Vershynin] handle these. Also, Vershynin shows: poly $(d, \log n, \epsilon)$!