

Here we complete the proof of the following claim from Lecture 25:

Theorem 0.1. *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ satisfy $\max_S |\hat{f}(S)| = \epsilon$. Suppose we choose k strings $\mathbf{x}^1, \dots, \mathbf{x}^k$ independently and uniformly at random and test that $f(\mathbf{x}^i)f(\mathbf{x}^j)f(\mathbf{x}^i \circ \mathbf{x}^j) = 1$ for all $1 \leq i < j \leq k$. Then*

$$\Pr[\text{test passes}] \leq 2^{-\binom{k}{2}} + \epsilon.$$

Proof. We have

$$\begin{aligned} \Pr[\text{test passes}] &= \mathbf{E} \left[\prod_{\{i,j\} \in \binom{[k]}{2}} \left(\frac{1}{2} + \frac{1}{2} f(\mathbf{x}^i) f(\mathbf{x}^j) f(\mathbf{x}^i \circ \mathbf{x}^j) \right) \right] \\ &= 2^{-\binom{k}{2}} \sum_{\mathcal{E} \subseteq \binom{[k]}{2}} \mathbf{E} \left[\prod_{\{i,j\} \in \mathcal{E}} f(\mathbf{x}^i) f(\mathbf{x}^j) f(\mathbf{x}^i \circ \mathbf{x}^j) \right] \\ &\leq 2^{-\binom{k}{2}} + \text{avg}_{\emptyset \neq \mathcal{E} \subseteq \binom{[k]}{2}} \mathbf{E} \left[\prod_{\{i,j\} \in \mathcal{E}} f(\mathbf{x}^i) f(\mathbf{x}^j) f(\mathbf{x}^i \circ \mathbf{x}^j) \right]. \end{aligned}$$

Thus it suffices to show that

$$\mathbf{E} \left[\prod_{\{i,j\} \in \mathcal{E}} f(\mathbf{x}^i) f(\mathbf{x}^j) f(\mathbf{x}^i \circ \mathbf{x}^j) \right] \tag{1}$$

is at most ϵ , for any nonempty $\mathcal{E} \subseteq \binom{[k]}{2}$.

Assume without loss of generality that $\{1, 2\} \in \mathcal{E}$. It is always possible to find fixings $\mathbf{x}^3 = x^3, \dots, \mathbf{x}^k = x^k$ such that the expectation in (1) does not decrease. We claim that regardless of these fixings, we can upper-bound (1) by ϵ . To see this, consider the quantity inside the expectation once we fix x^3, \dots, x^k . We still have $f(\mathbf{x}^1)$, $f(\mathbf{x}^2)$, and $f(\mathbf{x}^1 \circ \mathbf{x}^2)$ inside the product. The remaining terms $f(\mathbf{x}^i)$ in the product become ± 1 constants, and the remaining terms $f(\mathbf{x}^i \circ \mathbf{x}^j)$ (where one of i, j is in $\{1, 2\}$ and the other is not) become certain other functions just of $f(\mathbf{x}^1)$ or $f(\mathbf{x}^2)$. If we collect together (by multiplying) all the functions of \mathbf{x}^1 , and do the same for \mathbf{x}^2 , we see that (1) becomes

$$\pm \mathbf{E}_{\mathbf{x}^1, \mathbf{x}^2} [g(\mathbf{x}^1)h(\mathbf{x}^2)f(\mathbf{x}^1 \circ \mathbf{x}^2)].$$

Now straightforward Fourier analysis gives that the above is

$$\begin{aligned} \pm \sum_{S \subseteq [n]} \hat{g}(S) \hat{h}(S) \hat{f}(S) &\leq \max_S |\hat{f}(S)| \sum_S |\hat{g}(S)| |\hat{h}(S)| \\ &\leq \epsilon \sqrt{\sum_S \hat{g}(S)^2} \sqrt{\sum_S \hat{h}(S)^2} \\ &= \epsilon, \end{aligned}$$

where we used Cauchy-Schwarz. □