Here we complete the proof of the following claim from Lecture 25:

Theorem 0.1. *Let $f : \{-1,1\}^n \to \{-1,1\}$ satisfy $\max_S |\hat{f}(S)| = \epsilon$. Suppose we choose k strings $\boldsymbol{x}^1, \dots \boldsymbol{x}^k$ independently and uniformly at random and test that $f(\boldsymbol{x}^i)f(\boldsymbol{x}^j)f(\boldsymbol{x}^i \circ \boldsymbol{x}^j) = 1$ for all $1 \le i < j \le k$. Then*

$$\mathbf{Pr}[\textit{test passes}] \le 2^{-\binom{k}{2}} + \epsilon.$$

Proof. We have

$$\begin{aligned}
\mathbf{Pr}[\text{test passes}] &= \mathbf{E}\left[\prod_{\{i,j\}\in\binom{[k]}{2}} \left(\tfrac{1}{2} + \tfrac{1}{2} f(\boldsymbol{x}^i)f(\boldsymbol{x}^j)f(\boldsymbol{x}^i \circ \boldsymbol{x}^j)\right)\right] \\
&= 2^{-\binom{k}{2}} \sum_{\mathcal{E}\subseteq\binom{[k]}{2}} \mathbf{E}\left[\prod_{\{i,j\}\in\mathcal{E}} f(\boldsymbol{x}^i)f(\boldsymbol{x}^j)f(\boldsymbol{x}^i \circ \boldsymbol{x}^j)\right] \\
&\le 2^{-\binom{k}{2}} + \operatorname*{avg}_{\emptyset\neq\mathcal{E}\subseteq\binom{[k]}{2}} \mathbf{E}\left[\prod_{\{i,j\}\in\mathcal{E}} f(\boldsymbol{x}^i)f(\boldsymbol{x}^j)f(\boldsymbol{x}^i \circ \boldsymbol{x}^j)\right].
\end{aligned}$$

Thus it suffices to show that

$$\mathbf{E}\left[\prod_{\{i,j\}\in\mathcal{E}} f(\boldsymbol{x}^i)f(\boldsymbol{x}^j)f(\boldsymbol{x}^i \circ \boldsymbol{x}^j)\right] \tag{1}$$

is at most ϵ, for any nonempty $\mathcal{E} \subseteq \binom{[k]}{2}$.

Assume without loss of generality that $\{1,2\} \in \mathcal{E}$. It is always possible to find fixings $\boldsymbol{x}^3 = x^3, \dots, \boldsymbol{x}^k = x^k$ such that the expectation in (1) does not decrease. We claim that regardless of these fixings, we can upper-bound (1) by ϵ. To see this, consider the quantity inside the expectation once we fix $x^3, \dots, x^k$. We still have $f(\boldsymbol{x}^1)$, $f(\boldsymbol{x}^2)$, and $f(\boldsymbol{x}^1 \circ \boldsymbol{x}^2)$ inside the product. The remaining terms $f(x^i)$ in the product become ± 1 constants, and the remaining terms $f(\boldsymbol{x}^i \circ \boldsymbol{x}^j)$ (where one of i,j is in $\{1,2\}$ and the other is not) become certain other functions just of $f(\boldsymbol{x}^1)$ or $f(\boldsymbol{x}^2)$. If we collect together (by multiplying) all the functions of $\boldsymbol{x}^1$, and do the same for $\boldsymbol{x}^2$, we see that (1) becomes

$$\pm \mathop{\mathbf{E}}_{\boldsymbol{x}^1,\boldsymbol{x}^2}\left[g(\boldsymbol{x}^1)h(\boldsymbol{x}^2)f(\boldsymbol{x}^1 \circ \boldsymbol{x}^2)\right].$$

Now straightforward Fourier analysis gives that the above is

$$\begin{aligned}
\pm \sum_{S\subseteq[n]} \hat{g}(S)\hat{h}(S)\hat{f}(S) &\le \max_S |\hat{f}(S)| \sum_S |\hat{g}(S)||\hat{h}(S)| \\
&\le \epsilon \sqrt{\sum_S \hat{g}(S)^2}\sqrt{\sum_S \hat{h}(S)^2} \\
&= \epsilon,
\end{aligned}$$

where we used Cauchy-Schwarz. □