Cubical variations: HITs, $\pi_4(\mathbb{S}^3)$ and yacctt

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Cubes, cubes, cubes...

I will talk about various things related to cubes that I have been working on lately:

1. Higher inductive types in cubical sets and cubicaltt (*jww Thierry Coquand and Simon Huber*)
2. Computing the "Brunerie number": $n \in \mathbb{Z}$ such that $\pi_4(S^3) \simeq \mathbb{Z}/n\mathbb{Z}$ (*jww Guillaume Brunerie*)
3. `yacctt`: *yet another cartesian cubical type theory* (*jww Carlo Angiuli*)

HoTT/UF 2018

Benedikt Ahrens, Simon Huber and I are organizing:

4th Workshop on Homotopy Type Theory and Univalent Foundations

July 7-8, Oxford (Part of FLoC)

`https://hott-uf.github.io/2018/`

Invited speakers: **Martín Escardó, Paige North** and **Andrew Pitts**

Abstract submission deadline: **April 15** (1-2 pages)

HoTT/UF 2017-2018 MSCS special issue

We are also organizing:

Special Issue on Homotopy Type Theory and Univalent Foundations

Mathematical Structures in Computer Science

(Cambridge University Press)

`https://hott-uf.github.io/special-issue-17-18/`

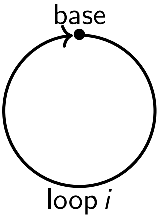
Submission is open to **everyone** and will be reviewed **on a rolling basis**.
Accepted papers will be published on the MSCS website via "FirstView"

Submission deadline: **December 31, 2018**

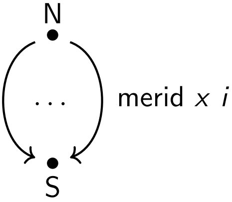
Higher inductive types (HITs)

Datatypes generated by regular "point" constructors and (higher) path constructors:

$\mathbb{S}^1$:



$\Sigma\mathbb{S}^1$:



These are usually added axiomatically to HoTT and justified semantically in "sufficiently nice" Quillen model categories (Lumsdaine-Shulman)

Our goal: give a constructive justification to HITs and design a type theory in which they compute

Cubical type theory: cubicaltt

Cubical Type Theory: a constructive interpretation of the univalence axiom (Cyril Cohen, Thierry Coquand, Simon Huber, AM) presents a dependent type theory in which univalence is a theorem

Theorem (CCHM)

cubicaltt has a model in cubical sets with a lot of structure

Theorem (Simon Huber)

cubicaltt enjoys canonicity for the type of natural numbers

Haskell implementation: `https://github.com/mortberg/cubicaltt`

Higher inductive types in cubicaltt

In a recent paper **On Higher Inductive Types in Cubical Type Theory** (*Thierry Coquand, Simon Huber, AM*) we extend cubicaltt and its semantics with a large class of HITs, exemplified by:

- The circle and spheres ($\mathbb{S}^n$),
- suspensions (ΣA),
- two versions of the torus ($\mathbb{T}$, $\mathbb{T}_\mathsf{F}$),
- propositional truncation ($\|A\|$), and
- pushouts

These illustrate many of the difficulties that one encounters when adding HITs to cubical type theory $\Rightarrow$ sketch a schema

Higher inductive types in cubicaltt

We do this in such a way that:

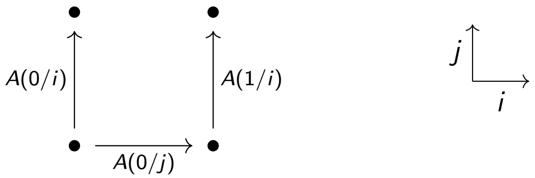
1. The higher inductive types are closed under universe levels: $\Sigma : U_n \to U_n$
2. Commute strictly with substitution: $(\Sigma A)\sigma = \Sigma(A\sigma)$
3. Satisfy judgmental/strict computation rules for **all** constructors

Justified by semantics in cubical sets $\Rightarrow$ consistency of HoTT with HITs and universes

cubicaltt: 101

Contexts:	$\Gamma ::= \bullet \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} \mid \Gamma, \varphi$
Formal interval, $\mathbb{I}$:	$r, s ::= 0 \mid 1 \mid i \mid 1 - r \mid r \wedge s \mid r \vee s$
Face formulas, $\mathbb{F}$:	$\varphi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$

$i\ j : \mathbb{I}, (i = 0) \vee (i = 1) \vee (j = 0) \vdash A$ corresponds to:



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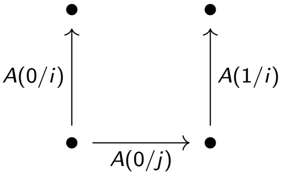

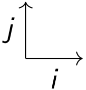

For every type A that we consider we must explain how to **transport** in it:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash a : A(0/i)}{\Gamma \vdash \mathsf{transport}^i\ A\ a : A(1/i)}$$

HITs in cubicaltt: the circle, $\mathbb{S}^1$

Formation:

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathbb{S}^1}$$

Introduction:

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathsf{base} : \mathbb{S}^1} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{loop}\, r : \mathbb{S}^1}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathsf{loop}\, 0 = \mathsf{base} : \mathbb{S}^1} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{loop}\, 1 = \mathsf{base} : \mathbb{S}^1}$$

HITs in cubicaltt: the circle, $\mathbb{S}^1$

Elimination:

$$\frac{\Gamma, x : \mathbb{S}^1 \vdash C \qquad \Gamma \vdash b : C(\text{base}) \qquad \Gamma \vdash l : \text{Path}^i\, C(\text{loop}\, i)\, b\, b \qquad \Gamma \vdash u : \mathbb{S}^1}{\Gamma \vdash \mathbb{S}^1\text{-elim}_{x.C}\ b\, l\, u : C(u)}$$

The judgmental computation rules for $\mathbb{S}^1\text{-elim}_{x.C}\ b\, l\, u$ by cases on u

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The judgmental computation rules for $\mathbb{S}^1\text{-elim}_{x.C}\ b\, l\, u$ by cases on u

$$\Gamma \vdash \mathbb{S}^1\text{-elim}_{x.C}\ b\, l\, \mathsf{base} = b : C(\mathsf{base})$$
$$\Gamma \vdash \mathbb{S}^1\text{-elim}_{x.C}\ b\, l\, (\mathsf{loop}\, r) = l\, r : C(\mathsf{loop}\, r)$$

The last equation looks nicer than in HoTT because of the builtin "path-over" types (i.e. no need for $\mathsf{apd}_f(\mathsf{loop}) = l$)

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What about transport for $\mathbb{S}^1$?

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What about transport for $\mathbb{S}^1$?

In order to explain transport in general in cubicaltt we need to introduce a more general operation called "Kan composition":

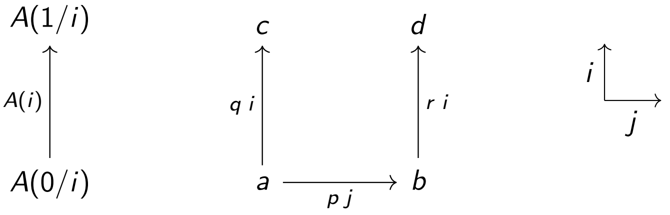
$$\frac{\begin{array}{cc} \Gamma, i : \mathbb{I} \vdash A(i) & \Gamma \vdash \varphi : \mathbb{F} \\ \Gamma, \varphi, i : \mathbb{I} \vdash u : A(i) & \Gamma \vdash u_0 : A(0/i)[\varphi \mapsto u(0/i)] \end{array}}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ u_0 : A(1/i)[\varphi \mapsto u(1/i)]}$$

We recover transport in the case when $\varphi = 0_{\mathbb{F}}$

Composition and filling

$i : \mathbb{I} \vdash A \qquad p : \text{Path } A(0/i)\ a\ b \qquad q : \text{Path}^i\ A(i)\ a\ c \qquad r : \text{Path}^i A(i)\ b\ d$

corresponds to the following diagram:



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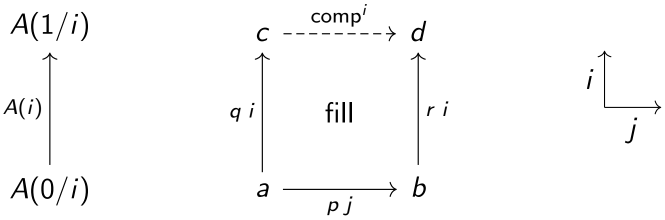

We get the dashed line and interior/filler by:

$$j : \mathbb{I} \vdash \text{comp}^i\ A\ [(j=0) \mapsto q\, i, (j=1) \mapsto r\, i]\ (p\, j) : A(1/i)$$
$$i, j : \mathbb{I} \vdash \text{fill}\ A\ [(j=0) \mapsto q\, i, (j=1) \mapsto r\, i]\ (p\, j) : A$$

HITs in cubicaltt: the circle, $\mathbb{S}^1$

How can we define $\text{comp}^i\ \mathbb{S}^1\ [\varphi \mapsto u]\ u_0$?

HITs in cubicaltt: the circle, $\mathbb{S}^1$

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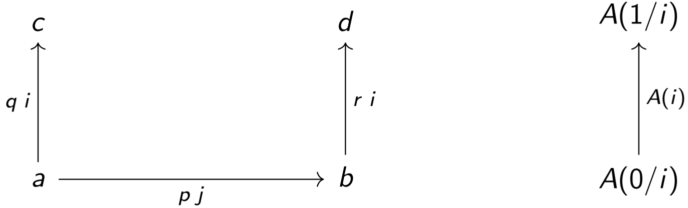
Key idea: decompose composition into **homogeneous composition** and **generalized transport**

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash u : A \qquad \Gamma \vdash u_0 : A[\varphi \mapsto u(0/i)]}{\Gamma \vdash \text{hcomp}^i_A\,[\varphi \mapsto u]\,u_0 : A[\varphi \mapsto u(1/i)]}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash A = A(0/i) \qquad \Gamma \vdash u_0 : A(0/i)}{\Gamma \vdash \text{trans}^i\,A\,\varphi\,u_0 : A(1/i)[\varphi \mapsto u_0]}$$

We recover transport in the case when $\varphi = 0_{\mathbb{F}}$

Generalized transport



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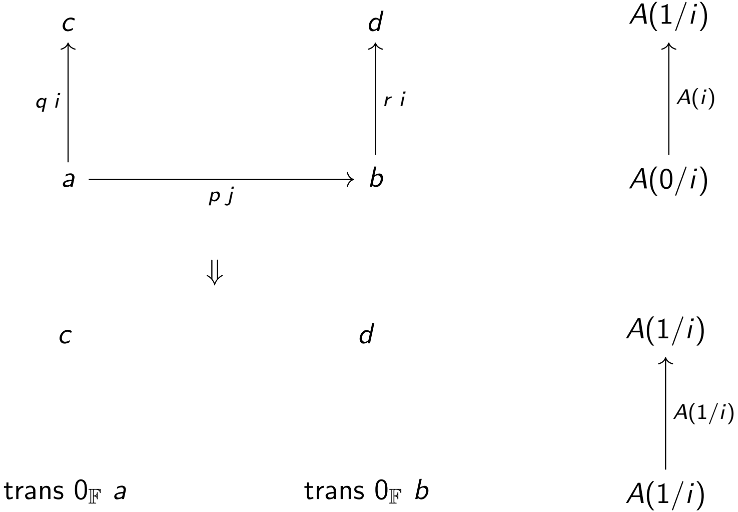

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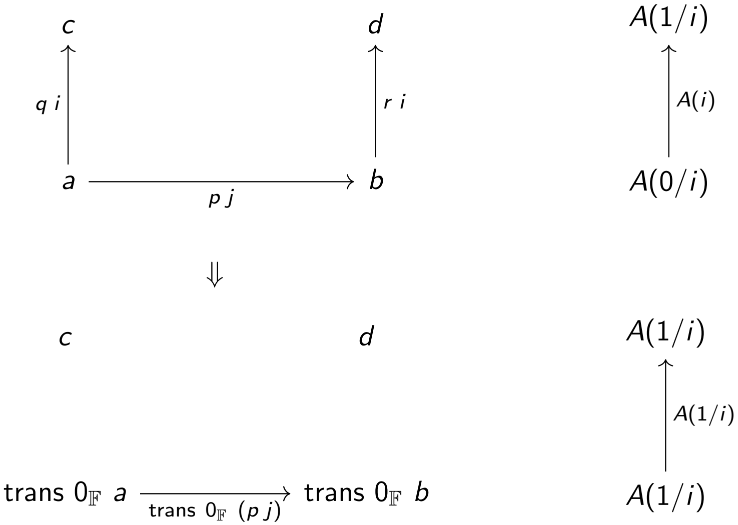

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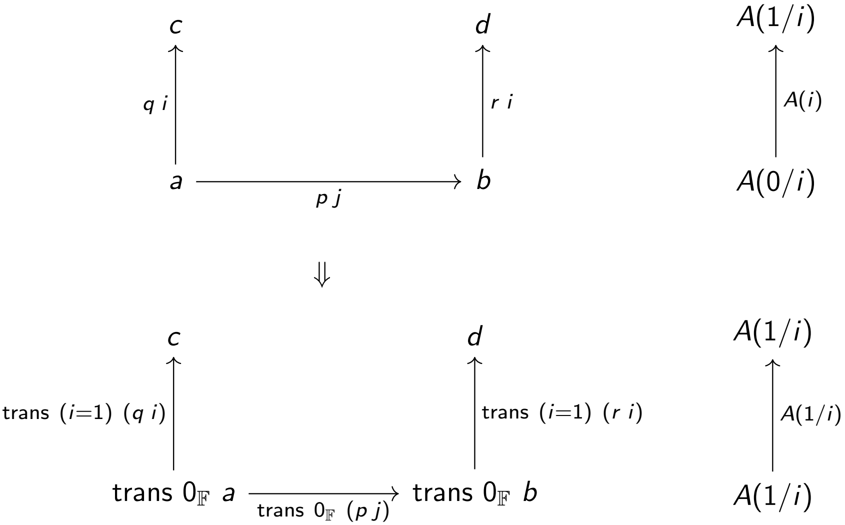

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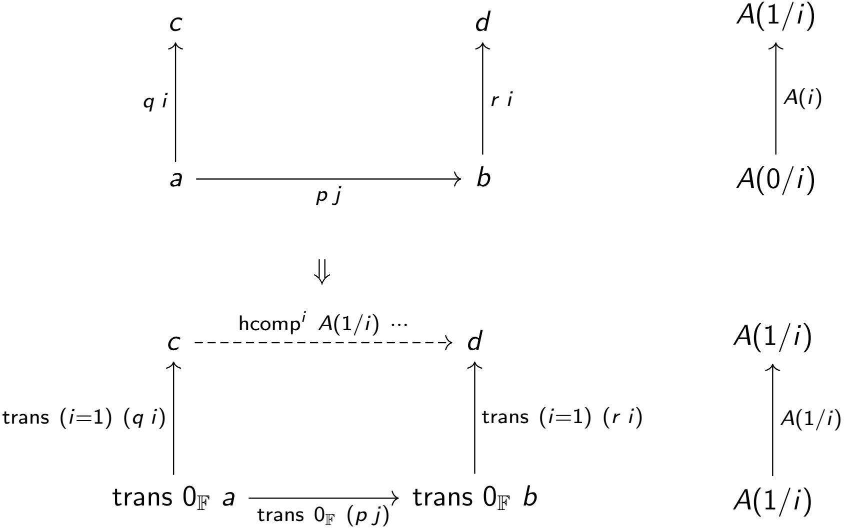

HITs in cubicaltt: the circle, $\mathbb{S}^1$

We now add hcomp for $\mathbb{S}^1$ as a constructor:

Introduction: ...

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : \mathbb{S}^1 \qquad \Gamma \vdash u_0 : \mathbb{S}^1[\varphi \mapsto u(0/i)]}{\Gamma \vdash \mathsf{hcomp}^i_{\mathbb{S}^1}\ [\varphi \mapsto u]\ u_0 : \mathbb{S}^1[\varphi \mapsto u(1/i)]}$$

Elimination: ...

$\Gamma \vdash \mathbb{S}^1\text{-elim}_{x.C}\ b\, l\, (\mathsf{hcomp}^i_{\mathbb{S}^1}\, [\varphi \mapsto u]\, u_0) =$

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Elimination: ...

$$\begin{aligned}\Gamma \vdash \mathbb{S}^1\text{-elim}_{x.C}\ b\,l\,(\mathsf{hcomp}^i_{\mathbb{S}^1}\,[\varphi \mapsto u]\,u_0) = \\ \mathsf{comp}^i\ C(v)\,[\varphi \mapsto \mathbb{S}^1\text{-elim}_{x.C}\ b\,l\,u]\,(\mathbb{S}^1\text{-elim}_{x.C}\ b\,l\,u_0) : C(v(1/i))\end{aligned}$$

where $v = \mathsf{hfill}^i\,\mathbb{S}^1\,[\varphi \mapsto u]\,u_0$, so that

$$v(1/i) = \mathsf{hcomp}^i_{\mathbb{S}^1}\,[\varphi \mapsto u]\,u_0$$

HITs in cubicaltt: the circle, $\mathbb{S}^1$

Composition: we can now define transport for $\mathbb{S}^1$ by:

$$\Gamma \vdash \mathsf{trans}^i\, \mathbb{S}^1\, \varphi\, u = u : \mathbb{S}^1$$

so that

$$\Gamma \vdash \mathsf{comp}^i\, \mathbb{S}^1\, [\varphi \mapsto u]\, u_0 = \mathsf{hcomp}^i_{\mathbb{S}^1}\, [\varphi \mapsto u]\, u_0 : \mathbb{S}^1$$

For parametrized HITs this simple definition does not work and we have to do something slightly more complicated

HITs in cubicaltt: suspensions, ΣA

Formation:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Sigma A}$$

Introduction:

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{N} : \Sigma A} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{S} : \Sigma A} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{merid}\, a\, r : \Sigma A}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{merid}\, a\, 0 = \mathsf{N} : \Sigma A} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{merid}\, a\, 1 = \mathsf{S} : \Sigma A}$$

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : \Sigma A \qquad \Gamma \vdash u_0 : \Sigma A[\varphi \mapsto u(0/i)]}{\Gamma \vdash \mathsf{hcomp}^i_{\Sigma A}\ [\varphi \mapsto u]\ u_0 : \Sigma A[\varphi \mapsto u(1/i)]}$$

HITs in cubicaltt: suspensions, ΣA

Elimination:

$$\frac{\Gamma, x : \Sigma A \vdash C \qquad \Gamma \vdash n : C(\mathsf{N}) \qquad \Gamma \vdash s : C(\mathsf{S}) \qquad \Gamma \vdash m : (a : A) \to \mathsf{Path}^i\, C(\mathsf{merid}\, a\, i)\, \mathsf{N}\, \mathsf{S} \qquad \Gamma \vdash u : \Sigma A}{\Gamma \vdash \Sigma\text{-}\mathsf{elim}^A_{x.C}\ n\, s\, m\, u : C(u)}$$

The judgmental computation rules are defined by cases on u:

$$\Sigma\text{-}\mathsf{elim}^A_{x.C}\ n\, s\, m\, \mathsf{N} = n$$

$$\Sigma\text{-}\mathsf{elim}^A_{x.C}\ n\, s\, m\, \mathsf{S} = s$$

$$\Sigma\text{-}\mathsf{elim}^A_{x.C}\ n\, s\, m\, (\mathsf{merid}\, a\, r) = m\, a\, r$$

HITs in cubicaltt: suspensions, ΣA

Composition: The $\mathsf{trans}^i\,\Sigma A\,\varphi\,u$ operation is defined by cases on u:

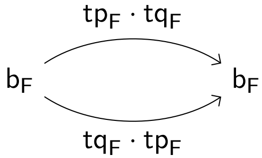
$$\begin{aligned}
\mathsf{trans}^i\,\Sigma A\,\varphi\,\mathsf{N} &= \mathsf{N}\\
\mathsf{trans}^i\,\Sigma A\,\varphi\,\mathsf{S} &= \mathsf{S}\\
\mathsf{trans}^i\,\Sigma A\,\varphi\,(\mathsf{merid}\,a\,r) &= \mathsf{merid}\,(\mathsf{trans}^i\,A\,\varphi\,a)\,r\\
\mathsf{trans}^i\,\Sigma A\,\varphi\,(\mathsf{hcomp}^j_{\Sigma A(0/i)}\,[\psi \mapsto u]\,u_0) &=\\
&\mathsf{hcomp}^j_{\Sigma A(1/i)}\,[\psi \mapsto \mathsf{trans}^i\,\Sigma A\,\varphi\,u]\,(\mathsf{trans}^i\,\Sigma A\,\varphi\,u_0)
\end{aligned}$$

Note: directly structurally recursive!

More HITs in cubicaltt

The other HITs follow the same pattern, but with slight differences:

- $\mathbb{T}_\mathsf{F}$: Torus à la HoTT book with "fibrant structure" in endpoints:



  with $\mathsf{tp_F} \cdot \mathsf{tq_F}$ represented using $\mathsf{hcomp}_{\mathbb{T}_\mathsf{F}}$

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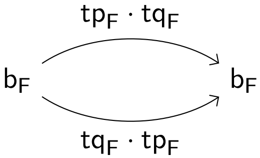

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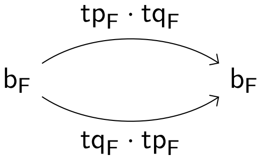

  with $\mathsf{tp_F} \cdot \mathsf{tq_F}$ represented using $\mathsf{hcomp}_{\mathbb{T}_\mathsf{F}}$
- Propositional truncation: exactly like ΣA, except the eliminator needs to make recursive calls for recursive arguments
- Pushouts: more complicated trans^i operation because of "endpoint corrections"

Cubical semantics of HITs

$\mathcal{C}$ = category of CCHM cubical sets: symmetries, connections, reversals, diagonals ("de Morgan algebra")[1]

We work in the internal language of the presheaf topos $\widehat{\mathcal{C}}$: "extensional" Martin-Löf type theory (equality reflection, UIP...) extended with:

- Formal interval type $\mathbb{I}$
- Type of cofibrant propositions $\mathbb{F} \hookrightarrow \Omega$

[1] Weaker axiomatization suffices, see *Axioms for Modelling Cubical Type Theory in a Topos*, Ian Orton and Andrew Pitts, CSL 2016

Cubical semantics of HITs

Using the interval we define the **Path type**: $\mathrm{Path}(A) := \mathbb{I} \to A$

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A **dependent type** A **over** Γ is given by $A : \Gamma \to \mathcal{U}$, i.e. by a family of types $A\,\rho$ for $\rho : \Gamma$

Composition structures

Definition (CCHM fibrations (Orton-Pitts))

The type of composition structures on a type A over Γ is:

$$\begin{aligned}\mathsf{Comp}(\Gamma, A) := \Pi(\gamma : \mathsf{Path}(\Gamma))\,(\varphi : \mathbb{F})\,(u : [\varphi] \to \Pi(i : \mathbb{I})A\gamma(i))\\ (u_0 : A\gamma(0)[\varphi \mapsto u\ \mathsf{tt}\ 0]) \to A\gamma(1)[\varphi \mapsto u\ \mathsf{tt}\ 1]\end{aligned}$$

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We write $\mathsf{Fib}(\Gamma) := \Sigma(A : \Gamma \to \mathcal{U})\,\mathsf{Comp}(\Gamma, A)$ for the type of CCHM fibrations over Γ

Given $\sigma : \Delta \to \Gamma$ and $(A, c_A) : \mathsf{Fib}(\Gamma)$ then we get $\sigma^*(A, c_A) : \mathsf{Fib}(\Delta)$. This is functorial and Fib has the structure of a CwF with $\mathsf{Fib}(\Gamma)$ as families

Homogeneous composition structures

A is a **fibrant object** if we have an element of $\mathsf{Comp}(1, A)$

A over Γ has a **homogeneous composition structure** if every fiber $A\rho$ for $\rho : \Gamma$ is a fibrant object:

$$\begin{aligned}\mathsf{HComp}(\Gamma, A) :=\, &\Pi(\rho : \Gamma)\,(\varphi : \mathbb{F})\,(u : [\varphi] \to \mathsf{Path}(A\rho)) \\ &(u_0 : A\rho[\varphi \mapsto u \;\mathsf{tt}\; 0]) \to A\rho[\varphi \mapsto u \;\mathsf{tt}\; 1]\end{aligned}$$

Having an element of $\mathsf{HComp}(\Gamma, A)$ does **not** imply that we have an element of $\mathsf{Comp}(\Gamma, A)$

$\mathsf{Comp} = \mathsf{HComp} + \mathsf{Trans}$

Definition (Generalized transport (CHM))

The type of generalized transport structures is:

$$\mathsf{Trans}(\Gamma, A) := \Pi(\varphi : \mathbb{F})\,(\gamma : \{p : \mathsf{Path}(\Gamma) \mid \varphi \Rightarrow \forall(i : \mathbb{I}).p\,i = p\,0\})$$
$$(u_0 : A\,\gamma(0)) \to A\,\gamma(1)[\varphi \mapsto u_0]$$

Comp = HComp + Trans

Definition (Generalized transport (CHM))

The type of generalized transport structures is:

$$\mathsf{Trans}(\Gamma, A) := \Pi(\varphi : \mathbb{F})\,(\gamma : \{p : \mathsf{Path}(\Gamma) \mid \varphi \Rightarrow \forall (i : \mathbb{I}).p\,i = p\,0\})\\ (u_0 : A\,\gamma(0)) \to A\,\gamma(1)[\varphi \mapsto u_0]$$

Theorem (CHM)

A family of types A over Γ has a composition structure iff it has a homogeneous composition structure and a transport structure

Key idea: decompose composition and freely add homogeneous composition to HITs ("fiberwise fibrant replacement")

HITs in $\widehat{\mathcal{C}}$

For every HIT T that we consider we:

1. Define T-algebra structure on a type A
2. Assume the existence of an initial T-algebra α and
    1. Construct $\mathsf{Comp}(\Gamma, \alpha)$
    2. Prove dependent elimination principle
3. Construct the initial T-algebra structure α (externally)

The circle $\mathbb{S}^1$

An S^1**-algebra structure on** A consists of $h_A : \mathrm{Comp}(1, A)$, a base point $b_A : A$ and a path $l_A : \mathrm{Path}(A)$ s.t. $l_A\, 0 = l_A\, 1 = b_A$

Assume that there is an initial S^1-algebra ($\mathbb{S}^1$,hcomp,base,loop), then $\mathbb{S}^1$ is fibrant by definition (as it has no parameters and an hcomp operation)

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Theorem (CHM)

$\mathbb{S}^1$ satisfies the dependent elimination rule for the circle: given a family of types P over $\mathbb{S}^1$ with a composition structure, and a in P base *and l i in P* (loop i) *such that $l\ 0 = l\ 1 = a$ there exists a map* $\text{elim} : \Pi(x : \mathbb{S}^1) P\ x$ *such that* elim base $= a$ *and* elim (loop i) $= l\ i$.

External construction of $\mathbb{S}^1$

Theorem (CHM)

There exists an initial S^1-algebra: $(\mathbb{S}^1, \mathsf{hcomp}, \mathsf{base}, \mathsf{loop})$

The construction is done in three steps:

1. Given $I \in \mathcal{C}$ construct a family of sets $\mathbb{S}^1_{\text{pre}}(I)$ which is an "upper approximation" of the circle
2. Define maps $\mathbb{S}^1_{\text{pre}}(I) \to \mathbb{S}^1_{\text{pre}}(J)$, $u \mapsto uf$ for $f : J \to I$
3. Define a cubical set $\mathbb{S}^1$, such that $\mathbb{S}^1(I)$ is a subset of $\mathbb{S}^1_{\text{pre}}(I)$

This construction is analogous for all HITs we consider

The suspension semantically

Given a type X, a $\mathsf{S}(X)$-**algebra structure on** A consists of $h_A : \mathsf{Comp}(1, A)$ together with two points n_A, s_A, and a family of paths l_A in $X \to \mathsf{Path}(A)$ connecting n_A to s_A (i.e., $l_A\ x\ 0 = n_A$ and $l_A\ x\ 1 = s_A$ for all x in X)

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As for the circle we can show using external reasoning:

Theorem (CHM)

There exists an initial $\mathsf{S}(X)$*-algebra, which will be denoted by* $(\Sigma X, \mathsf{hcomp}, \mathsf{N}, \mathsf{S}, \mathsf{merid}_X)$

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Σ is functorial: given $u : X \to Y$ we get a $\mathsf{S}(X)$-structure on ΣY by taking $l_{\Sigma Y}\ x\ i = \mathsf{merid}_Y\ (u\ x)\ i$ and hence a map $\Sigma u : \Sigma X \to \Sigma Y$

The suspension

Given a type A over Γ, we define a new family of types Σ_A over Γ by taking $(\Sigma_A)\rho := \Sigma(A\rho)$. By construction Σ_A *always* has a homogeneous composition structure

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Theorem (CHM)

If A has a transport structure $\mathsf{Trans}(\Gamma, A)$, *then Σ_A has a transport structure* $\mathsf{Trans}(\Gamma, \Sigma_A)$

Proof.

Given γ in $\Gamma^{\mathbb{I}}$ and φ such that γ is constant on φ we have a map $t_A\ \gamma\ \varphi : A\gamma(0) \to A\gamma(1)$ which is the identity on φ. Using the functoriality of Σ we get a map

$$\Sigma(t_A\ \gamma\ \varphi) : \Sigma(A\gamma(0)) \to \Sigma(A\gamma(1))$$

□

Further HITs

Many other HITs follow the same pattern: torus (and other CW-complexes), truncations, James construction, join construction...

For some HITs (pushout, W-suspensions, f-localizations...) the transport structure is more involved because of complicated endpoints of higher constructors. However, what we do for the pushout directly generalizes to these more involved HITs

HITs in cubicaltt

We have implemented a very general schema in cubicaltt (see the "hcomptrans" branch):

```
data S1 = base
        | loop <i> [ (i=0) → base, (i=1) → base]

helix : S1 → U = split
  base → Z
  loop @ i → sucPathZ @ i

winding (p : Path S1 base base) : Z = trans Z Z (<i> helix (p @ i)) zeroZ
```

Computing the Brunerie number

Guillaume Brunerie constructed in 2013 a number $n \in \mathbb{N}$ such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$. This construction uses univalence and a variety of HITs

His thesis proves, using sophisticated techniques from algebraic topology, that $n = 2$. From the Appendix:

"*If you wrote a proof assistant for homotopy type theory giving a computational interpretation of univalence and higher inductive types, please try to implement the following computation and check that you do get 2 as the result.*"

Computing the Brunerie number

The number n is defined as the absolute value of the image of 1 by the following composition of maps:

$$\mathbb{Z} \xrightarrow{n \mapsto \mathsf{loop}^n} \Omega\mathbb{S}^1 \xrightarrow{\Omega\varphi_{\mathbb{S}^1}} \Omega^2\mathbb{S}^2 \xrightarrow{\Omega^2\varphi_{\mathbb{S}^2}} \Omega^3\mathbb{S}^3 \xrightarrow{\Omega^3 e} \Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow{\Omega^3\alpha} \Omega^3\mathbb{S}^2$$

$$\xrightarrow{h}$$

$$\Omega^3(\mathbb{S}^1 * \mathbb{S}^1) \xrightarrow[\Omega^3(e^{-1})]{} \Omega^3\mathbb{S}^3 \xrightarrow[e_3]{} \Omega^2\|\mathbb{S}^2\|_2 \xrightarrow[\Omega\kappa_{2,\mathbb{S}^2}]{} \Omega\|\Omega\mathbb{S}^2\|_1 \xrightarrow[\kappa_{1,\Omega\mathbb{S}^2}]{} \|\Omega^2\mathbb{S}^2\|_0 \xrightarrow[e_2]{} \Omega\mathbb{S}^1 \xrightarrow[e_1]{} \mathbb{Z}$$

Attempts to compute the Brunerie number

2015: first attempt of Guillaume Brunerie, Thierry Coquand, Simon Huber using the predecessor of cubicaltt (called "cubical"). Ran out of memory on h

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End of 2017: Guillaume and I optimized the construction and cubicaltt a lot. Still running out of memory (but a lot faster)

Attempts to compute the Brunerie number

Lessons learned:

- It matters **a lot** how you prove things as any equality proof can be unfolded during computation with higher structures
- Avoid using J and stick to cubical primitives

But why are we stuck?

Attempts to compute the Brunerie number

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But why are we stuck?

My theory: combinatorial explosion due to very large terms and complicated algorithm for composition in the universe and Glue types

Yet Another Cartesian Cubical Type Theory: yacctt

Recently started project with Carlo Angiuli to implement a “formal” version of CHTT

We have implemented everything up to V-types and the proof of univalence (on a private repository, email me if you want access)

My goal is to understand the CHTT algorithms and to compare with what we do in cubicaltt, hopefully yacctt will compute better and we can get further on $\pi_4(\mathbb{S}^3)$

Yet Another Cartesian Cubical Type Theory: yacctt

Some things are simpler in yacctt:

```
trans (A B : U) (p : Path U A B) (a : A) : B = coe 0->1 p a
transNeg (A B : U) (p : Path U A B) (b : B) : A = coe 1->0 p b

transK (A B : U) (p : Path U A B) (a : A) : Path A a (transNeg A B p (trans A B p a)) =
  <i> coe i->0 p (coe 0->i p a)
```

The same theorem in cubicaltt:

```
trans (A B : U) (p : Path U A B) (a : A) : B = comp p a []
transNeg (A B : U) (p : Path U A B) (b : B) : A = comp (<i> p @ -i) b []

transK (A B : U) (p : Path U A B) (a : A) : Path A a (transNeg A B p (trans A B p a)) =
  <i> comp (<j> p @ -j) (trans A B p a) [(i=0) -> rem1, (i=1) -> rem2 (trans A B p a) ]
  where
  rem1 : PathP (<i> p @ -i) (trans A B p a) a =
    <i> comp (<j> p @ -i /\ j) a [(i=1) -> <j>a]
  rem2 (b : B) : PathP (<i> p @ -i) b (transNeg A B p b) =
    <i> comp (<j> p @ -i \/ -j) b [(i=0) -> <j> b]
```

Yet Another Cartesian Cubical Type Theory: yacctt

Proof of ua and uabeta in yacctt:

```
ua (A B : U) (e : equiv A B) : Path U A B = <i> V i A B e

uabeta (A B : U) (e : equiv A B) (a : A) : Path B (coe 0→1 (ua A B e) a) (e.1 a) =
  <i> coe i→1 (<_> B) (e.1 a)
```

and in cubicaltt:

```
ua (A B : U) (e : equiv A B) : Path U A B =
 <i> Glue B [ (i=0) → (A,e), (i=1) → (B,idEquiv B) ]

uabeta (A B : U) (e : equiv A B) (a : A) : Path B (trans A B (ua A B e) a) (e.1 a) =
  <i> hfill B (hfill B (e.1 a) [] @ ¬i) [] @ ¬i
```

This looks promising, but some things get more involved because of the lack of reversals and connections (e.g. Path “computation rule” for J)

Conclusions

Key idea: decomposition of composition into homogeneous composition and generalized transport, freely add homogeneous composition as a constructor to HITs

Open problem: can these ideas be translated to simplicial sets in order to give a model of HITs with universes in sSet?

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Open problem: can these ideas be translated to simplicial sets in order to give a model of HITs with universes in sSet?

Open problem: even though we have a constructive type theory we haven't succeeded in computing $\pi_4(\mathbb{S}^3)$. Maybe some completely new idea for how to compute with univalence and HITs not based on cubes is necessary?

Thank you for your attention!