Penrose: From Mathematical Notation to Beautiful Diagrams

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Set $A, B, C, D, E, F, G$
$B \subset A$
$C \subset A$
$D \subset B$
$E \subset B$
$F \subset C$
$G \subset C$
$E \cap D = \emptyset$
$F \cap G = \emptyset$
$B \cap C = \emptyset$
Diagrams are useful, but too rare

• Diagrams are useful…
  • Diagrams help in solving math problems [Larkin&Simon]
  • High-impact papers have many figures [Lee et al.]

• But rare: just 39% of arXiv math papers contain diagrams
  • And even those contain only 1 figure for every 10 pages

“People have very powerful facilities for taking in information visually... On the other hand, they do not have a good built-in facility for turning an internal spatial understanding back into a two-dimensional image. [So] mathematicians usually have fewer and poorer figures in their papers and books than in their heads.”

- Fields medalist William Thurston
The Penrose Vision

You write this:

\[ a, b, d, e, i, j, k, \inf \in \mathbb{R} \]
\[ b \text{ IsLessThan } k \]
\[ A := [a, b] \subseteq \mathbb{R} \]
\[ J := [d, e] \subseteq \mathbb{R} \]
\[ I := (i, j) \subseteq \mathbb{R} \]
\[ K := (k, \inf) \subseteq \mathbb{R} \]

\[ f : A \to \mathbb{R} \]
\[ f \text{ IsDifferentiable} \]

\[ U := (J \cup I) \subseteq \mathbb{R} \]
\[ h : U \to \mathbb{R} \]
\[ h \text{ IsDiscontinuous} \]

\[ l \in A \]
\[ p2 := \text{Pt}(l, f(l)) \]
\[ df1 := f'(l) \]
\[ ih := \int_J h \]

Penrose generates this:

Or, if you prefer, this:
Penrose in Action

• Linear algebra – simple intro
• Linear algebra – sugar and direct manipulation
• SIGGRAPH teaser video
Can we create a LaTeX for Diagrams?

**LaTeX**

- Describe document content (.tex) separate from layout
- Extensible formatting styles (.sty)
- Extensible with new document structuring concepts (macros)
- Optimizes (mostly textual) layout of documents

**Penrose**

- Describe mathematical content (Substance) separate from visual representation
- Extensible rendering (Style)
- Extensible with new math domains (Domain)
- Optimizes (graphical) layout of diagrams
Existing tools are inadequate

- **Graphing calculators** (e.g. Wolfram Alpha)
  - Visualize concrete data or functions
  - Don’t understand, can’t visualize mathematical abstractions

- **Drawing tools** (e.g. Adobe Illustrator, TikZ)
  - Require laborious specification of low-level details
  - Don’t understand semantics

- **Domain-specific visualizations** (e.g. Group Explorer)
  - Work well for a particular domain, but are not extensible
The Penrose Architecture and Users

Compilation
- Lexing & parsing
- Pattern matching & cascading
- Search & substitution

Optimization
- Objective graph
- Constraint graph
- Final values

Rendering
- Final diagram

Input source files
- .DSL
- SUB
- .STY

ASTs
- Computation graph

Substance
- Typical users
- Domain/Style
- Package developers
Anatomy of a Substance Program

Object U of type VectorSpace

\[
\text{VectorSpace } U \\
\text{Vector } u_1, u_2, u_3, u_4, u_5 \in U \\
u_3 \ := \ u_1 + u_2 \\
u_5 \ := \ u_3 + u_4
\]

Syntactic sugar for \(AddV(u_3, u_4)\)

Syntactic sugar declares variables and relationships \(In(u_i, U)\) for each \(u_i\)

Declares that \(u_5\) is equal to \(AddV(u_3, u_4)\)
The Domain Language

type Vector

type VectorSpace

predicate In: Vector * VectorSpace V

function addV: Vector * Vector -> Vector

notation "v1 + v2" ~ "addV(v1, v2)"

notation "Vector a ∈ U" ~ "Vector a; In(a, U)"

Declares type VectorSpace. Constructors may eventually have arguments.

Declares a predicate and its type

Declares an operator and its type

Declares syntactic sugar
Substance and Domain Design Features

• Separate, reusable domain extensions
  • New types, predicates, operators
  • New domain-specific notation
    • Similar to Coq notation extension

• Generic, typed object model
  • Check that substance programs are well-formed
  • Match on types in style programs
The Style Language

```plaintext
Vector v
with VectorSpace U
where v ∈ U { v.shape = Arrow { start = U.shape.center } encourage nearHead(v.shape, v.text) ensure contains(U.shape, v.shape) }

Vector u
with Vector v, w; VectorSpace U
where u := v + w; u, v, w ∈ U { u.shape.end = v.shape.end + w.shape.end

u.slider_v = Arrow { start = w.shape.end end = u.shape.end style = "dashed" }

u.slider_w = Arrow { ... }
```
The Style Language

Vector v
with VectorSpace U
where v ∈ U { v.shape = Arrow { start = U.shape.center }
encourage nearHead(v.shape, v.text) ensure contains(U.shape, v.shape) }

Vector u
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u.slider_v = Arrow { start = w.shape.end end = u.shape.end style = "dashed"
}
u.slider_w = Arrow { ... }
Substance, Style, and Output

VectorSpace $U$

Vector $u_1, u_2, u_3, u_4, u_5 \in U$

$u_4 := u_1 + u_2$

$u_5 := u_4 + u_3$

Vector $v$

with VectorSpace $U$

where $v \in U$

```
    v.shape = Arrow {
        start = U.shape.center
    }
```

courage nearHead(v.shape, v.text)

ensure contains(U.shape, v.shape)

Vector $u$

with Vector $v, w$; VectorSpace $U$

where $u := v + w; u, v, w \in U$

```
    u.shape.end = v.shape.end + w.shape.end
    u.slider_v = Arrow {
        start = w.shape.end
        end = u.shape.end
        style = "dashed"
    }
    u.slider_w = Arrow { ... }
```
Style Design Characteristics

• Extensible and reusable
  • Many styles per domain
  • Use different styles with the same substance program
  • Typical end-users need not understand style programs
    • But expert users can edit them or write new ones if they want to

• Provides a *visual semantics* for substance programs
  • Pattern matches over logical objects, generates graphical objects
  • Generates objectives and constraints for later optimization
  • Later matches can refine the semantics provided by earlier ones
Optimization

• Basically hill-climbing to solve constraints and maximize objectives

• All Penrose functions are end-to-end differentiable
  • Can take the derivative and modify the input(s) in the direction(s) that improve the composite objective function

• Can run multiple times
  \[\rightarrow\] multiple diagrams
Mathematics Underlying the Constraints

- **Intersection** Energy
  \[ \min_{a \in \partial A} \min_{b \in \partial B} |a - b|^2 \]

- **Inside** Energy
  \[ \int_{b \in \partial B \setminus A} \min_{a \in \partial A} |a - b|^2 dL \]

- **Outside** Energy
  \[ \int_{b \in \partial B \setminus A} \min_{a \in \partial A} |a - b|^2 dL \]

- **Boundary Intersection**
- **Containment**
- **Disjoint**
- **Inside Tangent**
- **Outside Tangent**
Euclidean Geometry

How do these relationships look if we assume that two parallel lines never meet?

Point \( p, q, r, s \)
Segment \( a := p, q \)
Segment \( b := p, r \)
Point \( m := \text{Midpoint}(a) \)
Angle \( \theta := \angle(q, p, r) \)
Triangle \( t := p, r, s \)
Ray \( w := \text{Bisector}(\theta) \)
Ray \( h := \text{PerpendicularBisector}(a) \)
Euclidean Geometry

drawn in euclidean geometry
(assuming parallel postulate)
What if the parallel postulate doesn’t hold?
How would we visualize these relationships on, say, a sphere?
Non-Euclidean Geometry

Point p, q, r, s
Segment a := p, q
Segment b := p, r
Point m := Midpoint(a)
Angle theta := ∠(q, p, r)
Triangle t := p, r, s
Ray w := Bisector(theta)
Ray h := PerpendicularBisector(a)

(different samples of the same Substance program, not a rotated sphere of the same diagram)

Here's the style program
Non-Euclidean Geometry

Point \( p, q, r, s \)
Segment \( a := p, q \)
Segment \( b := p, r \)
Point \( m := \text{Midpoint}(a) \)
Angle \( \theta := \angle(q, p, r) \)
Triangle \( t := p, r, s \)
Ray \( w := \text{Bisector}(\theta) \)
Ray \( h := \text{PerpendicularBisector}(a) \)

(different samples of the same Substance program)
More Penrose Demonstrations

- Set theory
  - tree style
  - Venn style

- Real analysis
  - parallel axis style
  - perpendicular axis style

- Any live requests?
  - Set theory
  - Linear algebra
  - Real analysis
Penrose: customizable visual semantics for concept-level expressions in an extensible set of domains

```plaintext
-- regex for a caustic
pathType = [L, S, D, S, E]
path1 = sample(pathType)
path2 = sample(pathType)
```

**VectorSpace** $U$

**Vector** $u_1, u_2, u_3, u_4, u_5 \in U$

\[ u_3 := u_1 + u_2 \]
\[ u_5 := u_3 + u_4 \]
More examples

Set A, B, C, D, E, F, G  Subset F C
Subset B A             Subset G C
Subset C A             NoIntersect E D
Subset D B             NoIntersect F G
Subset E B             NoIntersect B C

Set x {
    shape = Circle { }  
    ensure x contains x.label 
}  
NoIntersect x y {
    ensure x notOverlap y 
}
Subset x y {
    ensure y contains x 
    ensure x smallerThan y 
    ensure y.label outside x 
}

Set x { shape = Text{ } }  
Subset x y {
    encourage y above x 
    encourage x sameX y 
    shape = Arrow {
        start = x.shape 
        end = y.shape 
    )
}
Set x, Set y {
    encourage x repel y 
}