NOTES ON SHATTERING COEFFICIENTS

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1. Definition

(Taken directly from Larry Wasserman's 10-705 notes)

Let $F = \{x_1, ..., x_n\}$ be a finite set, and $\mathcal{A}$ be some class of sets. Let G be a subset of F. We say that $\mathcal{A}$ picks out G if $A \cup F = G$ for some $A \in \mathcal{A}$.

For example, let $\mathcal{A} = \{(a, b) : a \leq b\}$. Suppose that $F = \{1, 2, 7\}$ and $G = \{2, 7\}$. Then $\mathcal{A}$ picks out G since $A \cup F = G$ if we choose $A = (1.5, 7.5)$ for example.

Let $S(A, F)$ be the number of these subsets picked out by $\mathcal{A}$. Then $S(\mathcal{A}, F) \leq 2^n$.

Now let F_n denote all finite sets with n elements. Define the shatter coefficient

$$s_n(\mathcal{A}) = \sup_{F \in F_n} S(\mathcal{A}, F).$$

I like to think of $\mathcal{A}$ as a toolbox, where each tool is an $A \in \mathcal{A}$. Each tool can be used to pick out one (and only one) subset of F. In the example above we chose the tool $A = (1.5, 7.5)$ from the infinite toolbox $\mathcal{A} = \{(a, b) : a \leq b\}$. To find the shatter coefficient of $\mathcal{A}$ we must find the set $F \in F_n$ that has the most subsets that can be picked out using tools in $\mathcal{A}$. Here, with $n = 3$, $s_n(\mathcal{A}) = 7$ which is $2^3 - 1$. The one set that cannot be picked out of F is $\{1, 7\}$ because every tool in our toolbox ($A \in \mathcal{A}$), that can pick out 1 and 7 will also pick out 2.

2. Examples

1. Show that

$$s_n(\mathcal{A}_1 \cup \mathcal{A}_2) \leq s_n(\mathcal{A}_1) + s_n(\mathcal{A}_2)$$

Here we have the case where our toolbox is actually two toolboxes. When picking out a set $F \in F_n$ we can choose any tool from $\mathcal{A}_1$ or $\mathcal{A}_2$. Thus, every set picked out by $\mathcal{A}_1$ and every set picked out by $\mathcal{A}_2$ must appear in $s_n(\mathcal{A}_1 \cup \mathcal{A}_2)$. But, there may be duplicates; some sets will be picked out by both $\mathcal{A}_1$ and $\mathcal{A}_2$. If there are duplicates, $s_n(\mathcal{A}_1 \cup \mathcal{A}_2) < s_n(\mathcal{A}_1) + s_n(\mathcal{A}_2)$, and if there are no duplicates $s_n(\mathcal{A}_1 \cup \mathcal{A}_2) = s_n(\mathcal{A}_1) + s_n(\mathcal{A}_2)$.

2. Show that, if $\mathcal{A} = \{A_1 \cup A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$

$$s_n(\mathcal{A}) \leq s_n(\mathcal{A}_1)s_n(\mathcal{A}_2).$$

In this case, not only do we have two toolboxes, but we have the ability to use two tools at the same time. For example, if we have $\mathcal{A}_1$ = all axis-aligned rectangles, and $\mathcal{A}_2$ = all circles in $\mathbb{R}^2$. To pick out subsets of F we can use a circle and a rectangle at the same time. Thus, $s_n(\mathcal{A})$ is less than or equal to the size of the cartesian product $\mathcal{A}_1 \times \mathcal{A}_2$, again with duplicates removed.

3. Let $\mathcal{A} = A_1, ..., A_m$. Show that $s_n(\mathcal{A}) \leq m$ for all n.

Here, we have a finite toolbox of m tools. Because each tool may pick out only one $F \in F_n$, we cannot possibly pick out more than m sets, so $s_n(\mathcal{A})$ cannot be larger than m. In fact, when $m < 2^n$ we cannot even pick out m sets, since the total number of sets to pick out is bounded by the permutations of F_n. Thus $s_n(\mathcal{A}) \leq m$.