

# On the Barnes function

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## Abstract

The multiple Barnes function, defined as a generalization of the Euler gamma function, is used in many applications of pure and applied mathematics and theoretical physics. This paper presents new integral representations as well as special values of the Barnes function. Moreover, the Barnes function is expressed in a closed form by means of the Hurwitz zeta function. These results can be used for numeric and symbolic computations of the Barnes function.

## 1 Preamble

In 1899, Barnes [5, 6, 7] introduced and studied the generalization of the Euler gamma function defined by the following functional equation:

$$G(z + 1) = \Gamma(z) G(z), \quad z \in \mathbf{C},$$

$$G(1) = 1$$

where  $\Gamma$  is the gamma function. For the integer positive values of  $z$ , the Barnes  $G$  function is simply a product of factorials:

$$G(n + 1) = \prod_{k=1}^{n-1} k!$$

From the above functional equation and the Weierstrass canonical product for the gamma function,

$$\Gamma(z) = \frac{1}{z} \exp(-z \gamma) \prod_{m=1}^{\infty} \frac{\exp(\frac{z}{m})}{1 + \frac{z}{m}}$$

Barnes derived

$$G(z+1) = (2\pi)^{z/2} \exp\left(-\frac{z+z^2(1+\gamma)}{2}\right) \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right) \quad (1)$$

where  $\gamma$  is the Euler-Mascheroni constant. The right hand side of (1) is an entire function in a whole complex plane and can serve as an explicit definition of the Barnes function. Originally, the  $G$  function was introduced (in a different form) by Kinkelin [12] (see also Glaisher [11]) in his research on the asymptotic behavior of the following product:

$$1^1 2^2 \dots n^n = \frac{n!^n}{G(n+1)} \quad (2)$$

when  $n \rightarrow \infty$ . Kinkeln also applied the theory of the  $G$  function to the class of trigonometric integrals

$$\int_0^z \log \text{trig}(x) dx$$

where  $\text{trig}(x)$  is any trigonometric or hyperbolic function. All such integrals can be expressed in finite terms of the  $G$  function. Alexeiewsky [4] generalized the Kinkelin product (2) to

$$1^{1^p} 2^{2^p} \dots n^{n^p} = \exp(\zeta'(-p, n+1) - \zeta'(-p)) \quad (3)$$

which is related to the multiple Barnes function. Here  $\zeta'(t, z)$  denotes  $\partial\zeta(t, z)/\partial t$ .

The theory of the  $G$  function remains the active topic of research (see [2, 9, 10, 13, 14, 17, 18]). The Barnes function has been related to certain spectral functions in mathematical physics, to the study of determinants of Laplacians, and to the Hecke L-functions. In [16] and [2] the  $G$  function is expressed in terms of the Hurwitz zeta function by

$$\log G(z+1) - z \log \Gamma(z+1) = \zeta'(-1) - \zeta'(-1, z) \quad (4)$$

for  $\operatorname{Re}(z) > 0$ .  $\zeta'(-1)$  is related to Glaisher's constant  $A$ :

$$\log A = \frac{1}{12} - \zeta'(-1) = \frac{\gamma + \log(2\pi)}{12} - \frac{\zeta'(2)}{2\pi^2}$$

In [2] Adamchik obtained the closed form representation for the class of integrals involving cyclotomic polynomials and nested logarithms in terms of the Hurwitz zeta function. In the view of (4) those results can be translated into the  $G$  function notation, for example

$$\int_0^1 \frac{x^4 - 6x^2 + 1}{(x^2 + 1)^3} \log \log\left(\frac{1}{x}\right) dx = 4 \log \frac{G(\frac{3}{4})}{G(\frac{1}{4})} - 3 \log \Gamma\left(\frac{1}{4}\right) + \log \Gamma\left(\frac{3}{4}\right)$$

Choi et al. [9, 10] and Adamchik [1, 2, 3] considered a class of sums involving the Riemann zeta function which can be evaluated by means of the  $G$  function. Here is a series representation of  $G$  function

$$2 \log G(z+1) = z \log(2\pi) - \gamma z^2 - z(z+1) + 2 \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^{k+1}}{k+1}, \quad |z| < 1$$

and Glaisher's constant

$$\log A = \frac{\log 2}{12} + \sum_{k=1}^{\infty} \left( \frac{\zeta(2k+1) - 1}{36} \right) \frac{(4k+7)(7k+8)}{(k+1)(k+2)}$$

Barnes [8] (followed by Vignéras [17] and Vardi [16]) generalized the  $G$  function to the multiple gamma function  $G_n$  by the recurrence formula

$$\begin{aligned} G_{n+1}(z+1) &= \frac{G_{n+1}(z)}{G_n(z)}, \quad z \in \mathbf{C}, n \in \mathbf{N} \\ G_2(z) &= G(z) \\ G_1(z) &= \frac{1}{\Gamma(z)} \end{aligned} \tag{5}$$

Vignéras and Vardi considered a slightly different form of the multiple gamma function defined as a reciprocal to the Barnes function  $\Gamma_n(z) = 1/G_n(z)$ . Vardi derived an implicit representation for  $\Gamma_n(z)$  in terms of the multiple zeta function. In this paper we obtain a closed form of  $G_n(z)$  in finite terms of

the Hurwitz function and special polynomials. Here are two particular cases rewritten in terms of the derivatives of the Hurwitz function when  $n = 3$

$$2 \log G_3(z+1) - z^2 \log \Gamma(z) = (1-2z) \log G_2(z+1) + \zeta'(-2) - \zeta'(-2, z)$$

and  $n = 4$ :

$$6 \log G_4(z+1) - z^3 \log \Gamma(z) = 6(1-z) \log G_3(z+1) - (3z^2 - 3z + 1) \log G_2(z+1) + \zeta'(-3) - \zeta'(-3, z)$$

We also present various integral and series representations as well as some special values of the Barnes function. These results can be used in numeric and symbolic computations of the  $G$  function. Because of its significant applications in physics, number theory, combinatorics and applied mathematics, the multiple gamma function is of direct interest to computer algebra researchers and users, and ought to be implemented in computer algebra systems.

## 2 Special Values

There are a few known special cases when the  $G$  function is expressible in a closed form. The first one is due to Barnes [5]:

$$\log G\left(\frac{1}{2}\right) = \frac{\log 2}{24} - \frac{\log \pi}{4} - \frac{3 \log A}{2} + \frac{1}{8}$$

the other two are due to Choi and Srivastava [9]:

$$\begin{aligned} \log G\left(\frac{1}{4}\right) &= -\frac{G}{4\pi} - \frac{3}{4} \log \Gamma\left(\frac{1}{4}\right) - \frac{9}{8} \log A + \frac{3}{32} \\ \log G\left(\frac{3}{4}\right) &= \log G\left(\frac{5}{4}\right) + \frac{G}{2\pi} - \frac{\log(2\pi^2)}{8} \end{aligned}$$

where  $G$  is Catalan's constant. In a view of closed form representation for the derivatives of the Hurwitz functions, obtained in [15], and the formula (4), it is readily to obtain the additional special cases of  $\log G(p)$  for  $p = \frac{1}{3}, \frac{1}{6}, \frac{2}{3}, \frac{5}{6}$ :

$$\log G\left(\frac{1}{3}\right) = \frac{\log 3}{72} + \frac{\pi}{18\sqrt{3}} - \frac{2}{3} \log \Gamma\left(\frac{1}{3}\right) - \frac{4}{3} \log A - \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{12\pi\sqrt{3}} + \frac{1}{9}$$

$$\log G\left(\frac{2}{3}\right) = \frac{\log 3}{72} + \frac{\pi}{18\sqrt{3}} - \frac{1}{3} \log \Gamma\left(\frac{2}{3}\right) - \frac{4}{3} \log A - \frac{\psi^{(1)}\left(\frac{2}{3}\right)}{12\pi\sqrt{3}} + \frac{1}{9}$$

$$\log G\left(\frac{1}{6}\right) = -\frac{\log 12}{144} + \frac{\pi}{20\sqrt{3}} - \frac{5}{6} \log \Gamma\left(\frac{1}{6}\right) - \frac{5}{6} \log A - \frac{\psi^{(1)}\left(\frac{1}{6}\right)}{40\pi\sqrt{3}} + \frac{5}{72}$$

$$\log G\left(\frac{5}{6}\right) = -\frac{\log 12}{144} + \frac{\pi}{20\sqrt{3}} - \frac{1}{6} \log \Gamma\left(\frac{5}{6}\right) - \frac{5}{6} \log A - \frac{\psi^{(1)}\left(\frac{5}{6}\right)}{40\pi\sqrt{3}} + \frac{5}{72}$$

where  $\psi^{(1)}(z) = \partial \log \Gamma(z) / \partial z^2$  is the polygamma function. Similar representations can be derived for the multiple  $G_n$  function.

### 3 Connection to L-series

From the Lerch functional equation for the Hurwitz zeta function and formula (4), we can derive

$$\log\left(\frac{G(1+z)}{G(1-z)}\right) = \frac{i}{2\pi} \text{Li}_2(e^{2\pi iz}) + z \log\left(\frac{\pi}{\sin(\pi z)}\right) - \frac{\pi i}{2} B_2(z), \quad 0 < z < 1$$

where  $B_2(z)$  is the second Bernoulli polynomial, and  $\text{Li}_2(z)$  is the dilogarithm. The identity can be rewritten in an alternative form by means of the Clausen function  $\text{Cl}_2(z)$

$$\log\left(\frac{G(1+z)}{G(1-z)}\right) = z \log\left(\frac{\pi}{\sin(\pi z)}\right) - \frac{1}{2\pi} \text{Cl}_2(2\pi z) \quad (6)$$

where

$$\text{Cl}_2(x) = -\text{Im}(\text{Li}_2(e^{-ix}))$$

Obviously enough, the  $G$  function is related to the Dirichlet  $L$ -series. Here is one of the formulas:

$$\log\left(\frac{G\left(\frac{7}{6}\right)}{G\left(\frac{5}{6}\right)}\right) = \frac{1}{6} \log(2\pi) - \frac{3\sqrt{3}}{8\pi} L_{-3}(2)$$

## 4 Binet-like representation

The Binet formula for the gamma function is

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \log \sqrt{2\pi} + \int_0^\infty \frac{e^{-zx}}{x} \left(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2}\right) dx$$

In this section we derive a similar representation for the Barnes function. Recall the well-known integral for  $\zeta(s, z)$ :

$$\zeta(s, z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{-zx}}{1-e^{-x}} dx, \quad R(s) > 1, R(z) > 0$$

The integral can be analytically continued to the domain  $R(s) > -1$ . To do so, we perform the standard procedure of removing the integrand singularity by subtracting the truncated Taylor series at  $x = 0$ :

$$\begin{aligned} \zeta(s, z) = & -\frac{z^{1-s}}{1-s} + \frac{z^{-s}}{2} + \frac{s}{12} z^{-s-1} + \\ & \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-zx} \left(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{x}{12} - \frac{1}{2}\right) dx \end{aligned}$$

Differentiating the above formula with respect to  $s$  and computing the limit at  $s = -1$ , we therefore arrive at

**Proposition 1** *The Barnes  $G$  function admits the Binet integral representation:*

$$\begin{aligned} \log G = & z \log \Gamma(z) - \log A + \frac{z^2}{4} - \frac{\log z}{2} B_2(z) - \\ & \int_0^\infty \frac{e^{-zx}}{x^2} \left(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{x}{12} - \frac{1}{2}\right) dx, \quad R(z) > 0 \end{aligned}$$

In a similar way we derive the representation for Glaisher's constant:

$$\log A = \frac{1 + \log(2\pi)}{12} - \frac{1}{2\pi^2} \int_0^\infty \frac{x \log x}{e^x - 1} dx$$

## 5 The multiple G-function

For the multiple  $G_n$  function defined by the functional equation (5), Vardi [16] obtained the following formula

$$\log G_n(z) = -\lim_{s \rightarrow 0} \frac{\partial \zeta_n(s, z)}{\partial s} - \sum_{k=1}^n (-1)^k \binom{z}{k-1} R_{n+1-k}$$

where

$$R_n(z) = \sum_{k=1}^n \lim_{s \rightarrow 0} \frac{\partial \zeta_k(s, 1)}{\partial s} \quad (7)$$

and  $\zeta_n(s, z)$  is the multiple zeta function:

$$\zeta_n(s, z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^s} \binom{k+n-1}{n-1} \quad (8)$$

In this section we find a closed form representation for  $G_n(z)$  in terms of the Hurwitz zeta function. For clarity of exposition, we first consider polynomials

$$P_{k,n}(z) = \sum_{i=k+1}^n (-z)^{i-k-1} \binom{i-1}{k} \left[ \begin{matrix} n \\ i \end{matrix} \right] \quad (9)$$

which can be rewritten in the alternative form

$$P_{k,n}(z) = \sum_{i=1}^n \binom{z}{n-i} \frac{(n-1)!}{(i-1)!} \left[ \begin{matrix} i \\ k+1 \end{matrix} \right] \quad (10)$$

where  $\left[ \begin{matrix} n \\ i \end{matrix} \right]$  are unsigned Stirling numbers of the first kind. The polynomials  $P_{k,n}(z)$  satisfy the functional equation

$$\begin{aligned} P_{k,n+1}(z) - n P_{k,n}(z) - P_{k,n+1}(z+1) &= 0, \\ P_{k,n}(z) &= 0, \quad k \geq n \end{aligned}$$

Here are the main properties of these polynomials:

$$\begin{aligned}
P_{n-1,n}(z) &= 1, & P_{k,n}(1) &= \left[ \begin{matrix} n-1 \\ k \end{matrix} \right], \\
P_{n-2,n}(z) &= z(1-n) + \frac{n(n-1)}{2}, \\
P_{0,n}(z) &= \binom{n-z-1}{n-1} (n-1)!, \\
P_{0,n}\left(\frac{1}{2}\right) &= \frac{(2n-3)!!}{2^{n-1}}
\end{aligned} \tag{11}$$

**Lemma 1** *The multiple zeta function  $\zeta_n(s, z)$  defined by (8) may be expressed by means of the Hurwitz function*

$$\zeta_n(s, z) = \frac{1}{(n-1)!} \sum_{j=0}^n P_{j,n}(z) \zeta(s-j, z) \tag{12}$$

*Proof.* Recall the definition of unsigned Stirling numbers of the first kind

$$\binom{k+n-1}{n-1} = \frac{1}{(n-1)!} \sum_{i=0}^n k^{i-1} \left[ \begin{matrix} n \\ i \end{matrix} \right]$$

Expanding  $k^{i-1} = ((k+z) - z)^{i-1}$  by the binomial theorem, interchanging the order of summation and making use of (9), we obtain

$$\binom{k+n-1}{n-1} = \frac{1}{(n-1)!} \sum_{j=0}^n P_{j,n}(z) (k+z)^j$$

We complete the proof by substituting this into (8).

**Lemma 2** *The coefficients  $R_n$  defined by (7) may be expressed as*

$$R_n = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \zeta'(-k) \left[ \begin{matrix} n \\ k+1 \end{matrix} \right] \tag{13}$$

*Proof.* First we make use of the Lemma 1 and the identity (11) for  $P_{j,n}(1)$

$$\lim_{s \rightarrow 0} \frac{\partial \zeta_k(s, 1)}{\partial x} = \frac{1}{(n-1)!} \sum_{j=0}^n P_{j,n}(1) \zeta'(-j) = \frac{1}{(n-1)!} \sum_{j=0}^n \zeta'(-j) \left[ \begin{matrix} n-1 \\ j \end{matrix} \right]$$

Further, in a view of (7), we have

$$R_n = \sum_{k=1}^n \frac{1}{(k-1)!} \sum_{j=0}^k \zeta'(-j) \left[ \begin{matrix} n-1 \\ j \end{matrix} \right] = \zeta'(0) + \sum_{j=1}^n \zeta'(-j) \sum_{k=j}^n \frac{1}{(k-1)!} \left[ \begin{matrix} k-1 \\ j \end{matrix} \right]$$

Taking into account that the inner sum on the right hand side is (it follows from (10) with  $z = 1$ )

$$\sum_{k=j}^n \frac{(n-1)!}{(k-1)!} \left[ \begin{matrix} k-1 \\ j \end{matrix} \right] = \left[ \begin{matrix} n \\ j+1 \end{matrix} \right]$$

we complete the proof.

**Proposition 2** *The multiple Barnes function  $G_n(z)$  may be expressed by means of the derivatives of the zeta functions*

$$\log G_n(z) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} P_{j,n}(z) (\zeta'(-j) - \zeta'(-j, z))$$

where the polynomials  $P_{j,n}(z)$  are defined by (9).

## 6 Integral representations

In [2] Adamchik expressed the integral involving the polygamma function in finite terms of the Hurwitz zeta function:

$$\int_0^z x^n \psi(x) dx = (-1)^n \left( \frac{B_{n+1} H_n}{n+1} - \zeta'(-n) \right) + \sum_{k=0}^n (-1)^k \binom{n}{k} \left( \zeta'(-k, z) - \frac{B_{k+1}(z) H_n}{k+1} \right) \quad (14)$$

where  $B_n$  and  $H_n$  are Bernoulli and Harmonic numbers respectively. If  $n = 1$  the integral (14) leads to the following representation for the Barnes  $G$  function:

$$\log G(z+1) = \frac{z(1-z)}{2} + z \log \sqrt{2\pi} + \int_0^z x \psi(x) dx \quad (15)$$

under assumption  $R(z) > -1$ . This representation demonstrates that the complexity of computing  $G(z)$  depends at most on the complexity of computing the polygamma function, which in its turn is reduced to integration of elementary functions:

$$\int_0^z x \psi(x) dx = \int_0^1 \left( \frac{z^2}{2t} - \frac{z}{t^2} - \frac{1 - (1-t)^z}{t^2 \log(1-t)} \right) dt + \frac{z(z-2)}{2} - \frac{\gamma z^2}{2} \quad (16)$$

The restriction  $R(z) > -1$  can be easily removed by analyticity of the polygamma. For example, by resolving the singularity of the integrand at the pole  $x = -1$ , we continue  $\log G(z+1)$  to the wider area  $R(z) > -2$ :

$$\log G(z+1) = \frac{z(1-z)}{2} + z \log \sqrt{2\pi} + \log(z+1) + \int_0^z x \psi(x) - \frac{1}{x+1} dx \quad (17)$$

Another way to continue (15) into the left half-plane is to use the identity

$$\psi(x) = \psi(-x) - \pi \cot(\pi x) - \frac{1}{x}$$

which upon substituting it into (15) yields

$$\log G(z+1) = \log G(1-z) + z \log(2\pi) - \int_0^z \pi x \cot(\pi x) dx \quad (18)$$

This identity holds everywhere in a complex plane of  $z$ , except the real axes, where the integrand has simple poles. Therefore, in a view of (18) we can continue (15) to  $\mathbf{C}/\mathbf{R}^-$ . Note, if  $z$  is negative we can still use the representation (15), but with a contour of integration deformed in such a way that it does not cross poles.

For  $n = 2$  and  $n = 3$ , the formula (14) yields

$$\int_0^z x^2 \psi(x) dx = -2 \log G_3(z+1) + \log G_2(z+1) + z^2 \zeta'(0) - 2z \zeta'(-1) + \frac{z}{12}(6z^2 - 3z - 1)$$

$$\int_0^z x^3 \psi(x) dx = 6 \log G_4(z+1) - 6 \log G_3(z+1) + \log G_2(z+1) + z^3 \zeta'(0) - 3z^2 \zeta'(-1) - 3z \zeta'(-2) + \frac{z^2}{24}(11z^2 - 4z - 1)$$

respectively. The general formula can also be derived. Skipping the technical details, we have

**Proposition 3** *Let  $n$  be a positive integer and  $R(z) > -1$ , then*

$$\int_0^z x^n \psi(x) dx = \sum_{k=0}^{n-1} (-1)^k z^{n-k} \binom{n}{k} \left( \zeta'(-k) - \frac{B_{k+1}(z)H_k}{k+1} \right) - \sum_{k=1}^n (-1)^k k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \log G_{k+1}(z+1) + (-1)^n H_n \frac{B_{n+1} - B_{n+1}(z)}{n+1} \quad (19)$$

are  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  Stirling numbers of the second kind.

We can resolve the equation (19) with respect to  $G_{k+1}(z+1)$ . In particular, this leads to the following integral representation for the triple Barnes function:

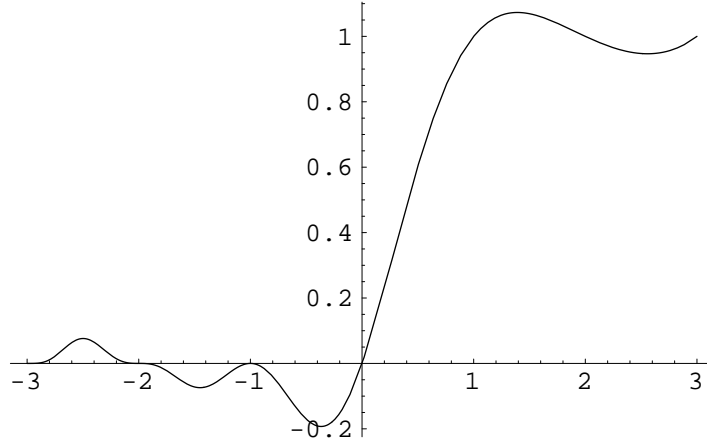
$$2 \log G_3(z+1) = \int_0^z x(1-x) \psi(x) dx + \frac{z}{12}(6z^2 - 9z + 5) - z(1-z) \zeta'(0) - 2z \zeta'(-1) \quad (20)$$

and for the quadruple Barnes function

$$6 \log G_4(z+1) = \int_0^z x(1-x)(2-x) \psi(x) dx - \frac{z}{24}(11z^3 - 40z^2 + 41z - 18) - z(1-z)(2-z) \zeta'(0) - 3z(2-z) \zeta'(-1) - 3z \zeta'(-2) \quad (21)$$

Formulas (20) and (21) are suitable for numeric computation of  $G_3(z)$  and  $G_4(z)$  for any  $z \in \mathbf{C}/\mathbf{R}^-$ .

Figure 1: The Barnes function



## 7 Implementation remarks

We have presented here many available results on the Barnes function as well as new results on numeric and symbolic computations. In this section we discuss a numeric computational scheme for the Barnes  $G$  function. The integral representation via polygamma functions considered in the previous section seems to be an efficient numeric procedure for evaluating  $G$  function. Based on (15) (or (16)), we define the double Barnes function  $G(z) = G_2(z)$  as

$$G(z) = (2\pi)^{\frac{z-1}{2}} \exp\left(-\frac{(z-1)(z-2)}{2} + \int_0^{z-1} x \psi(x) dx\right) \quad (22)$$

This representation is valid for  $z \in \mathbf{C}/\mathbf{R}^-$ . If  $z$  is a negative real, we have

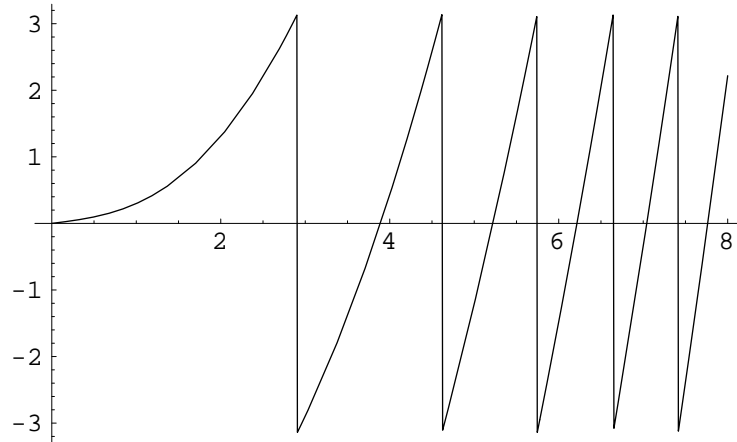
$$G(-n) = 0, \quad n \in \mathbf{N}$$

$$G(z) = (2\pi)^{\frac{z-1}{2}} \exp\left(-\frac{(z-1)(z-2)}{2} + \int_{\gamma} x \psi(x) dx\right), \quad n \in \mathbf{R}^- \quad (23)$$

where the contour of integration  $\gamma$  is a path between 0 and  $z-1$  that does not cross the negative real axis; for example,  $\gamma$  could be the following path:  $\{0, i, i+z-1, z-1\}$ . Another method of defining the Barnes  $G$  function for negative reals is to use the formula (6). This gives

$$G(-z) = (-1)^{\lfloor z/2 \rfloor - 1} G(z+2) \left| \frac{\sin(\pi z)}{\pi} \right|^{z+1} \exp\left(\frac{1}{2\pi} \text{Cl}_2(2\pi(z - \lfloor z \rfloor))\right) \quad (24)$$

Figure 2: The logarithm of G



where  $\lfloor z \rfloor$  is the floor function, and  $\text{Cl}_2(z)$  is the Clausen function. Combination (22) with (24), defines the Barnes function everywhere in  $\mathbf{C}$ . Figure 1 shows the graphic of  $G$  on the interval  $(-3, 3)$ .

In a similar manner, taking into consideration the formulas (20) and (21), we can define  $G_3$  and  $G_4$  respectively.

There are many applications, particularly in number theory, where the logarithm of the Barnes function often appears. However, because of a branch cut of logarithm, the function  $\log(G(z))$  includes spurious discontinuities for complex argument. The graphic of  $\text{Im}(\log(G(2 - iy)))$ , where  $y \in \{0, 8\}$  shown in Figure 2 indicates such discontinuities. Therefore, we shall define an additional function  $\log G(z)$  (the logarithm of the Barnes function, similar to *LogGamma*, in some systems) which is an analytic function throughout the complex  $z$  plane

$$\log G(z) = -\frac{(z-1)(z-2)}{2} + \frac{z-1}{2} \log(2\pi) + \int_0^{z-1} x \psi(x) dx$$

If  $z$  is a negative real, we understand the path of integration as in (23).

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