A Class of Logarithmic Integrals

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Abstract. A class of definite integrals involving cyclotomic polynomials and nested logarithms is considered. The results are given in terms of derivatives of the Hurwitz Zeta function. Some special cases for which such derivatives can be expressed in closed form are also considered. The integration procedure is implemented in *Mathematica* V3.1.

1 Introduction

The aim of the paper is to develop an approach for evaluating a class of integrals involved cyclotomic polynomials and the nested logarithms $\log\log x$. This class of integrals arose from the research regarding the Potts model on the triangular lattice (see [1], [2]). The Potts model encompasses a number of problems in statistical physics and lattice theory. It generalizes the Ising model so that each spin can have more than two values. It includes the ice-vertex and bond percolation models as special cases. It is also related to graph-coloring problems. Baxter, Temperley and Ashley (see [3]) derived the following generating function for the Potts model on the triangular lattice:

$$P_3(y) = 3\int_0^\infty \frac{\sinh((\pi - y)x)\sinh\left(\frac{2yx}{3}\right)}{x\sinh(\pi x)\cosh(yx)}dx \tag{1}$$

Performing a logarithmic substitution, the integral can be rewritten in the "algebraic" form:

$$P_3(y) = 3\int_0^1 \frac{z^{-y-1}(1-z^{2y})(z^{3\pi}-z^{3y})}{(1-z^{3\pi})(1+z^{3y})\log(z)}dz$$

Although it isn't known whether the function $P_3(y)$ has a closed-form expression for all values of y, it can be evaluated explicitly for any y that is a rational multiple of π. Let $y = \frac{\pi p}{q}$, where p and q are positive integers, then

$$P_3(y) = 3\int_0^1 \frac{z^{-p-1}(1-z^{2p})(z^{3q}-z^{3p})}{(1+z^{3p})(1-z^{3q})\log(z)}dz$$

This integral belongs to the more common class of integrals

$$\int_0^1 \frac{R(z)}{\log(z)}dz$$

where R(z) is a rational function. We assume that the integral is convergent. The above integral can be envisaged in an alternative form. Performing an integration by parts, we obtain

$$\int_0^1 Q(z)\log\log\Big(\frac{1}{z}\Big)dz \tag{2}$$

where Q(z) is a rational function. It is not known whether the above integral is doable for any Q(z). However, if the denominator of Q(z) is a cyclotomic polynomial then the integral can be always expressed in terms of derivatives of the Hurwitz Zeta function. Using the Graeffe procedure for determining if a given polynomial is cyclotomic (see [4]), and then converting a cyclotomic polynomial to the form $1 \pm x^n$, allows us to reduce the problem of integration of (2) to the following two classes of integrals:

$$\int_0^1 \frac{x^{p-1}}{(1+x^n)^q}\log\log\Big(\frac{1}{x}\Big)dx \quad \text{and} \quad \int_0^1 x^{p-1}\Big(\frac{1-x}{1-x^n}\Big)^q\log\log\Big(\frac{1}{x}\Big)dx$$

assuming that p, q, and n provide the convergence of the integrals. A few such integrals (with $p = q = 1$ and $n = 2, 3$) can be found in Gradzhteyn and Ryzhyk's handbook (see [5], pp. 532, 571-572) and in [6].

2 Derivatives of the Hurwitz Zeta Function

It is well-known (see [7]) that

$$\frac{\partial}{\partial s}\zeta(s,z)\bigg|_{s=0} = \log\Big(\frac{,\,(z)}{\sqrt{2\pi}}\Big) \tag{3}$$

However, if the first argument of $\frac{\partial}{\partial s}\zeta(s,z)$ is not zero no exact formulas were developed. In this section we consider the difference of derivatives of the Zeta functions

$$\zeta'\Big(s,\frac{p}{q}\Big) - \zeta'\Big(s,1-\frac{p}{q}\Big) \tag{4}$$

where $\zeta'(s,z)$ for ease of notation denotes $\frac{\partial}{\partial s}\zeta(s,z)$ and p and q are positive integers, and show that (4) can be represented in finite terms of other functions. Throughout the paper we will freely use the notation

$$\zeta'\Big(1,\frac{p}{q}\Big) - \zeta'\Big(1,1-\frac{p}{q}\Big)$$

for the limit of (4) when $s \to 1$.

Proposition 1 *Let p and q be positive integers and $p < q$, then*

$$\zeta'\Big(1, \frac{p}{q}\Big) - \zeta'\Big(1, 1 - \frac{p}{q}\Big) =$$
$$\pi \cot\Big(\frac{\pi p}{q}\Big)(\log(2\pi q) + \gamma) - 2\pi \sum_{j=1}^{q-1} \log\Big(,\Big(\frac{j}{q}\Big)\Big) \sin\Big(\frac{2\pi j p}{q}\Big) \tag{5}$$

Proof. The identity (5) follows straightforwardly from Rademacher's formula (see [8]):

$$\zeta\Big(z, \frac{p}{q}\Big) = 2,(1-z)(2\pi q)^{z-1} \sum_{j=1}^{q} \sin\Big(\frac{\pi z}{2} + \frac{2jp\pi}{q}\Big)\zeta\Big(1 - z, \frac{j}{q}\Big) \tag{6}$$

by differentiating it with respect to z and then setting z to 1. We have

$$\zeta'\Big(1, \frac{p}{q}\Big) - \zeta'\Big(1, 1 - \frac{p}{q}\Big) =$$
$$2\pi(\log(2\pi q) + \gamma) \sum_{j=1}^{q} \sin\Big(\frac{2jp\pi}{q}\Big)\zeta\Big(0, \frac{j}{q}\Big) - 2\pi \sum_{j=1}^{q} \sin\Big(\frac{2jp\pi}{q}\Big)\zeta'\Big(0, \frac{j}{q}\Big)$$

Taking into account (3) along with

$$\zeta(0, z) = \frac{1}{2} - z$$

$$\sum_{j=1}^{q} \sin\Big(\frac{2jp\pi}{q}\Big) = 0$$

$$\sum_{j=1}^{q} j \sin\Big(\frac{2jp\pi}{q}\Big) = -\frac{q}{2}\cot\Big(\frac{p\pi}{q}\Big), \quad p < q$$

we arrive at the identity (5). QED.

The proposition was first proved by G. Almkvist and A. Meurman [9].

Let us consider several particular cases:

$$\zeta'\Big(1, \frac{1}{4}\Big) - \zeta'\Big(1, \frac{3}{4}\Big) = \pi\Big(\gamma + 4\log(2) + 3\log(\pi) - 4\log\Big(,\Big(\frac{1}{4}\Big)\Big)\Big) \tag{7}$$

$$\zeta'\Big(1, \frac{1}{3}\Big) - \zeta'\Big(1, \frac{2}{3}\Big) = \frac{\pi(2\gamma - \log(3) + 8\log(2\pi) - 12\log(,(\frac{1}{3})))}{2\sqrt{3}} \tag{8}$$

$$\zeta'\Big(1, \frac{1}{6}\Big) - \zeta'\Big(1, \frac{5}{6}\Big) = \frac{\pi(6\gamma + \log(314928) - 12\log(,(\frac{1}{3})) + 24\log(,(\frac{2}{3})))}{2\sqrt{3}} \tag{9}$$

$$\zeta'\Big(1, \frac{1}{6}\Big) - \zeta'\Big(1, \frac{2}{3}\Big) = \log(2)(2\gamma + \log(2) + 3\log(3)) +$$
$$\frac{\pi}{\sqrt{3}}\Big(2\gamma + \log\Big(\frac{2}{3}\Big) + 8\log(2\pi) - 12\log\Big(,\Big(\frac{1}{3}\Big)\Big)\Big) \tag{10}$$

Proposition 2 *Let n be a positive integer and $0 < x < 1$, then*

$$\zeta'(-n,x) + (-1)^n \zeta'(-n,1-x) = \pi i \frac{B_{n+1}(x)}{n+1} + e^{-\frac{\pi i n}{2}} \frac{n!}{(2\pi)^n} \mathrm{Li}_{n+1}(e^{2\pi i x}) \qquad (11)$$

Proof. From Lerch's transformation formula for the function $\Phi(z,s,v)$ (see [7]):

$$\Phi(z,s,v) = i z^{-v} (2\pi)^{s-1} , (1-s)$$
$$\left(e^{-\frac{1}{2}\pi i s} \Phi\left(e^{-2\pi i v}, 1-s, \tfrac{\log(z)}{2\pi i}\right) - e^{\pi i \left(\frac{s}{2}+2v\right)} \Phi\left(e^{2\pi i v}, 1-s, 1-\tfrac{\log(z)}{2\pi i}\right)\right)$$

with $v = 0$, $s = 1-s$, and $z = e^{2\pi i x}$, we obtain

$$\zeta(s,1-x) + e^{\pi i s} \zeta(s,x) = e^{\frac{\pi i s}{2}} \frac{(2\pi)^s}{, (s)} \mathrm{Li}_{1-s}(e^{2\pi i x}) \qquad (12)$$

where we assume that $0 < x < 1$ and s is real. Differentiating the functional equation (12) with respect to s, setting s to $-n$, where n is a positive integer, and making use of

$$\lim_{x \to -n} \frac{\psi(x)}{, (x)} = (-1)^{n+1} n!$$

$$\zeta(-n,x) = -\frac{B_{n+1}(x)}{n+1}$$

where $B_{n+1}(x)$ denotes the Bernoulli polynomials, we complete the proof. QED.

The identity (11) can be rewritten in the alternative form by means of the Clausen function that is defined by

$$\mathrm{Cl}_n(z) = \begin{cases} \Re(\mathrm{Li}_n(e^{-iz})), & n \,\mathrm{is\,odd} \\ -\Im(\mathrm{Li}_n(e^{-iz})), & n \,\mathrm{is\,even} \end{cases}$$

Hence we have the following

Corollary 1 *Let n be a positive integer and $0 < x < 1$, then*

$$\zeta'(-n,x) + (-1)^n \zeta'(-n,1-x) = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} n!}{(2\pi)^n} \mathrm{Cl}_{n+1}(2\pi x) \qquad (13)$$

where $\lfloor\rfloor$ is the floor function.

The following identities pop up immediately from (11):

$$\zeta'\Big(-1,\frac{1}{4}\Big) - \zeta'\Big(-1,\frac{3}{4}\Big) = \frac{G}{2\pi}$$

$$\zeta'\Big(-2,\frac{1}{4}\Big) + \zeta'\Big(-2,\frac{3}{4}\Big) = \frac{3\zeta(3)}{64\pi^2}$$

where G is Catalan's constant (see [7] and [10]). Moreover, using the multiplication property of the Zeta function

$$\zeta(s,kz) = k^{-s}\sum_{i=0}^{k-1}\zeta\Big(s,z+\frac{i}{k}\Big)$$

and the Proposition 2 (or Rademacher's formula (6)), one can easily deduce that

$$\zeta'\Big(-1,\frac{1}{6}\Big) = \frac{\log(12)}{144} - \frac{\pi}{12\sqrt{3}} + \frac{\psi'(\frac{1}{3})}{8\sqrt{3}\pi} + \frac{1}{6}\zeta'(-1) \tag{14}$$

$$\zeta'\Big(-1,\frac{1}{4}\Big) = \frac{G}{4\pi} - \frac{1}{8}\zeta'(-1) \tag{15}$$

$$\zeta'\Big(-1,\frac{1}{3}\Big) = -\frac{\log(3)}{72} - \frac{\pi}{18\sqrt{3}} + \frac{\psi'(\frac{1}{3})}{12\sqrt{3}\pi} - \frac{1}{3}\zeta'(-1) \tag{16}$$

$$\zeta'\Big(-1,\frac{1}{2}\Big) = -\frac{\log(2)}{24} - \frac{1}{2}\zeta'(-1) \tag{17}$$

$$\zeta'\Big(-1,\frac{2}{3}\Big) = -\frac{\log(3)}{72} + \frac{\pi}{18\sqrt{3}} - \frac{\psi'(\frac{1}{3})}{12\sqrt{3}\pi} - \frac{1}{3}\zeta'(-1) \tag{18}$$

$$\zeta'\Big(-1,\frac{3}{4}\Big) = -\frac{G}{4\pi} - \frac{1}{8}\zeta'(-1) \tag{19}$$

$$\zeta'\Big(-1,\frac{5}{6}\Big) = \frac{\log(12)}{144} + \frac{\pi}{12\sqrt{3}} - \frac{\psi'(\frac{1}{3})}{8\sqrt{3}\pi} + \frac{1}{6}\zeta'(-1) \tag{20}$$

Remark1. The special case of (11) when $n=1$ was also obtained by W. Gosper (see [11]).

Remark2. Notice that, $\zeta'(-1)$ is related to Glaisher's constant A (see [12]) as

$$\zeta'(-1) = \frac{1}{12} - \log(A)$$

3 Integrals

Proposition 3 *Let $\Re(p)>0$ and $\Re(n)>0$, then*

$$\begin{aligned}&\int_0^1 \frac{x^{p-1}}{1+x^n}\log\log\Big(\frac{1}{x}\Big)dx = \\ &\frac{\gamma+\log(2n)}{2n}(\psi(\frac{p}{2n})-\psi(\frac{n+p}{2n})) + \frac{1}{2n}(\zeta'(1,\frac{p}{2n})-\zeta'(1,\frac{n+p}{2n}))\end{aligned} \tag{21}$$

where $\zeta'(s,z)$ denotes $\frac{\partial}{\partial s}\zeta(s,z)$.

Proof. We shall proceed with the well-known identity

$$\lambda \int_0^1 \frac{x^{p-1}}{\lambda + x^n} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda^k (nk+p)} \tag{22}$$

Differentiating both sides of (22) q times with respect to p, we obtain

$$\int_0^1 \frac{x^{p-1}}{\lambda + x^n} \log^{q-1}\Big(\frac{1}{x}\Big) dx = \frac{,\,(q)}{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda^k (kn+p)^q} \tag{23}$$

Differentiating both sides of the new identity (23) with respect to q and then setting $q = 1$, we find that

$$\int_0^1 \frac{x^{p-1}}{\lambda + x^n} \log\log\Big(\frac{1}{x}\Big) dx = -\gamma \int_0^1 \frac{x^{p-1}}{\lambda + x^n} dx - \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k \log(kn+p)}{\lambda^k (kn+p)} \tag{24}$$

Now we consider the limiting case $\lambda \to 1$. It is clear that the infinite sum in the right-hand side of (24) is divergent when $\lambda = 1$. However, the sum can be analytically continued to the whole complex plane of the parameter λ by means of the Lerch function. We have

$$\sum_{k=0}^{\infty} \frac{(-1)^k \log(kn+p)}{\lambda^k (kn+p)} = -\lim_{s\to 1} \frac{\partial}{\partial s} \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda^k (kn+p)^s} = -\lim_{s\to 1} \frac{\partial}{\partial s} \frac{\Phi(-\frac{1}{\lambda}, s, \frac{p}{n})}{n^s}$$

Hence, when λ tends to 1, we obtain

$$\lim_{\lambda\to 1} \sum_{k=0}^{\infty} \frac{(-1)^k \log(kn+p)}{\lambda^k (kn+p)} = -\lim_{s\to 1} \frac{\partial}{\partial s} \left(\frac{\zeta(s, \frac{p}{2n}) - \zeta(s, \frac{1}{2} + \frac{p}{2n})}{(2n)^s} \right)$$

since

$$\Phi(-1, s, \frac{p}{n}) = 2^{-s}\Big(\zeta(s, \frac{p}{2n}) - \zeta(s, \frac{1}{2} + \frac{p}{2n})\Big)$$

Thus,

$$\int_0^1 \frac{x^{p-1}}{1 + x^n} \log\log\Big(\frac{1}{x}\Big) dx = -\gamma \int_0^1 \frac{x^{p-1}}{1 + x^n} dx + \lim_{s\to 1} \frac{\partial}{\partial s} \left(\frac{\zeta(s, \frac{p}{2n}) - \zeta(s, \frac{1}{2} + \frac{p}{2n})}{(2n)^s} \right)$$

Performing further evaluations and taking into account the asymptotic expansion

$$\zeta(s, n) \to -\gamma - \psi(n) + \zeta(s), s \to 1$$

we finally arrive at (21). QED.

Note if $\frac{p}{n}$ is a positive integer, then the right-hand side of (21) can be expressed in terms of elementary functions. Let $p = nr$, $r \in \mathsf{N}$, then

$$\int_0^1 \frac{x^{nr-1}}{1 + x^n} \log\log\Big(\frac{1}{x}\Big) dx =$$
$$\frac{1}{2n}\Big((\log(2n) + \gamma)\Big(\psi(\frac{r}{2}) - \psi\Big(\frac{r+1}{2}\Big)\Big) + \zeta'(1, \frac{r}{2}) - \zeta'\Big(1, \frac{r+1}{2}\Big)\Big) \tag{25}$$

Now making use of the following reduction formulas

$$\zeta'(s,v) = \zeta'(s,v-1) + \frac{\log(v-1)}{(v-1)^s}$$

$$\psi(s) = \psi(s-1) + \frac{1}{s-1}$$

and the identity

$$\zeta'\Big(1,\frac{1}{2}\Big) - \zeta'(1,1) = (\log(2)+\gamma)\log(4) - \log^2(2)$$

it is easy to see that the right-hand side of (25) can be transformed to the combination of logarithmic functions. Thus,

Corollary 2 *If $p = n$ and $\Re(n) > 0$, then*

$$\int_0^1 \frac{x^{n-1}}{1+x^n} \log\log\Big(\frac{1}{x}\Big)dx = -\frac{\log(2)\log(2n^2)}{2n} \tag{26}$$

Corollary 3 *If $p = 2n$ and $\Re(n) > 0$, then*

$$\int_0^1 \frac{x^{2n-1}}{1+x^n} \log\log\Big(\frac{1}{x}\Big)dx = \frac{\log^2(2) + 2(\log(2)-1)\log(n) - 2\gamma}{2n} \tag{27}$$

Here we consider several particular integrals.

3.1 $\int_0^1 \frac{1}{1+x^2} \log\log\Big(\frac{1}{x}\Big)dx$

From (21) with $p = 1$ and $n = 2$, we obtain

$$\int_0^1 \frac{1}{1+x^2} \log\log\Big(\frac{1}{x}\Big)dx =$$
$$\frac{\gamma+\log(4)}{4}\Big(\psi\Big(\frac{1}{4}\Big) - \psi\Big(\frac{3}{4}\Big)\Big) + \frac{1}{4}\Big(\zeta'\Big(1,\frac{1}{4}\Big) - \zeta'\Big(1,\frac{3}{4}\Big)\Big)$$

By means of (7), we have

$$\int_0^1 \frac{1}{1+x^2} \log\log\Big(\frac{1}{x}\Big)dx = \frac{\pi}{2}\log\left(\frac{\sqrt{2\pi},\ (\frac{3}{4})}{,\ (\frac{1}{4})}\right) \tag{28}$$

3.2 $\int_0^1 \frac{x}{1+x^4} \log\log\Big(\frac{1}{x}\Big)dx$

Taking $p=2$ and $n=4$ in (21), we have

$$\int_0^1 \frac{x}{1+x^4} \log\log\left(\frac{1}{x}\right) dx =$$
$$\frac{1}{8}(\log(8)+\gamma)\left(\psi\left(\frac{1}{4}\right)-\psi\left(\frac{3}{4}\right)\right)+\frac{1}{8}\left(\zeta'\left(1,\frac{1}{4}\right)-\zeta'\left(1,\frac{3}{4}\right)\right)$$

Using the identity (7), we obtain

$$\int_0^1 \frac{x}{1+x^4} \log\log\left(\frac{1}{x}\right) dx = \frac{\pi}{4}\log\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right) \tag{29}$$

3.3 $\int_0^1 \frac{1}{1+x^3} \log\log\left(\frac{1}{x}\right) dx$

From (21) with $p=1$ and $n=3$, we obtain

$$\int_0^1 \frac{1}{1+x^3} \log\log\left(\frac{1}{x}\right) dx =$$
$$\frac{1}{6}(\log(6)+\gamma)\left(\psi\left(\frac{1}{6}\right)-\psi\left(\frac{2}{3}\right)\right)+\frac{1}{6}\left(\zeta'\left(1,\frac{1}{6}\right)-\zeta'\left(1,\frac{2}{3}\right)\right)$$

Performing further simplifications and using the formula (10), we find that

$$\int_0^1 \frac{1}{1+x^3} \log\log\left(\frac{1}{x}\right) dx = \frac{\log(2)}{6}\log\left(\frac{3}{2}\right)-\frac{\pi\left(\log(54)-8\log(2\pi)+12\log\left(\Gamma\left(\frac{1}{3}\right)\right)\right)}{6\sqrt{3}} \tag{30}$$

3.4 $\int_0^1 \frac{1}{1-x+x^2} \log\log\left(\frac{1}{x}\right) dx$

We observe that a given integral can be rewritten as

$$\int_0^1 \frac{1}{1-x+x^2} \log\log\left(\frac{1}{x}\right) dx =$$
$$\int_0^1 \frac{1}{1+x^3} \log\log\left(\frac{1}{x}\right) dx + \int_0^1 \frac{x}{1+x^3} \log\log\left(\frac{1}{x}\right) dx$$

Applying Proposition 2 twice, we obtain

$$\int_0^1 \frac{1}{1-x+x^2} \log\log\left(\frac{1}{x}\right) dx =$$
$$\frac{1}{6}\left(\zeta'\left(1,\frac{1}{3}\right)-\zeta'\left(1,\frac{2}{3}\right)+\zeta'\left(1,\frac{1}{6}\right)-\zeta'\left(1,\frac{5}{6}\right)-\frac{4\pi(\log(6)+\gamma)}{\sqrt{3}}\right)$$

Then taking into account formulas (8) and (9), we finally find that

$$\int_0^1 \frac{1}{1-x+x^2} \log\log\left(\frac{1}{x}\right) dx = \frac{\pi\left(5\log(2\pi)-6\log\left(\Gamma\left(\frac{1}{6}\right)\right)\right)}{3\sqrt{3}} \tag{31}$$

Proposition 4 *Let $\Re(p) > 0$ and $\Re(n) \geq 1$, then*

$$\int_0^1 x^{p-1}\frac{1-x}{1-x^n}\log\log\Big(\frac{1}{x}\Big)dx = \frac{1}{n}(\log(n)+\gamma)(\psi(\frac{p}{n}) - \psi(\frac{p+1}{n}))+ \frac{1}{n}\Big(\zeta'(1,\frac{p}{n}) - \zeta'(1,\frac{p+1}{n})\Big) \tag{32}$$

where $\zeta'(s,z)$ denotes $\frac{\partial}{\partial s}\zeta(s,z)$.
If $n=1$, then the formula (32) simplifies to

$$\int_0^1 x^{p-1}\log\log\Big(\frac{1}{x}\Big)dx = -\frac{\log(p)+\gamma}{p} \tag{33}$$

Proof. We observe that

$$\lambda\int_0^1 x^{p-1}\frac{1-x}{\lambda-x^n}dx = \sum_{k=0}^{\infty}\frac{1}{\lambda^k(nk+p)} - \sum_{k=0}^{\infty}\frac{1}{\lambda^k(nk+p+1)} \tag{34}$$

since

$$\lambda\int_0^1 \frac{x^{p-1}}{\lambda-x^n}dx = \sum_{k=0}^{\infty}\frac{1}{\lambda^k(nk+p)}$$

Differentiating both sides of (34) q times with respect to p, we obtain

$$\int_0^1 x^{p-1}\log^{q-1}\Big(\frac{1}{x}\Big)\frac{1-x}{\lambda-x^n}dx = \frac{\Gamma(q)}{\lambda}\sum_{k=0}^{\infty}\frac{1}{\lambda^k(kn+p)^q} - \frac{\Gamma(q)}{\lambda}\sum_{k=0}^{\infty}\frac{1}{\lambda^k(kn+p+1)^q} \tag{35}$$

Differentiating (35) with respect to q and setting $q=1$, we find that

$$\int_0^1 x^{p-1}\frac{1-x}{\lambda-x^n}\log\log\Big(\frac{1}{x}\Big)dx = -\gamma\int_0^1 x^{p-1}\frac{1-x}{\lambda-x^n}dx - \frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{\log(kn+p)}{\lambda^k(kn+p)} + \frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{\log(kn+p+1)}{\lambda^k(kn+p+1)} \tag{36}$$

The infinite sums in the right side of (36) can be represented in terms of the Lerch function. We have

$$\sum_{k=0}^{\infty}\frac{\log(kn+p)}{\lambda^k(kn+p)} = -\lim_{s\to 1}\frac{\partial}{\partial s}\sum_{k=0}^{\infty}\frac{1}{\lambda^k(kn+p)^s} = -\lim_{s\to 1}\frac{\partial}{\partial s}\frac{\Phi(\frac{1}{\lambda},s,\frac{p}{n})}{n^s}$$

Therefore, setting $\lambda=1$, we obtain

$$\sum_{k=0}^{\infty}\Big(\frac{\log(kn+p+1)}{kn+p+1} - \frac{\log(kn+p)}{kn+p}\Big) = -\lim_{s\to 1}\frac{\partial}{\partial s}\Bigg(\frac{\zeta(s,\frac{p+1}{n})}{n^s} - \frac{\zeta(s,\frac{p}{n})}{n^s}\Bigg) = \frac{1}{n}\Big(\log(n)\Big(\psi(\frac{p}{n}) - \psi\Big(\frac{p+1}{n}\Big)\Big) + \zeta'(1,\frac{p}{n}) - \zeta'\Big(1,\frac{p+1}{n}\Big)\Big)$$

since

$$\Phi(1,s,\frac{p}{n})=\zeta(s,\frac{p}{n})$$

and

$$\zeta(s,n)\to -\gamma-\psi(n)+\zeta(s), s\to 1$$

Thus,

$$\int_0^1 x^{p-1}\frac{1-x}{1-x^n}\log\log\Big(\frac{1}{x}\Big)dx=-\gamma\int_0^1\frac{x^{p-1}}{1-x^n}(1-x)dx+$$
$$\frac{1}{n}\Big(\log(n)\Big(\psi(\frac{p}{n})-\psi\Big(\frac{p+1}{n}\Big)\Big)+\zeta'(1,\frac{p}{n})-\zeta'\Big(1,\frac{p+1}{n}\Big)\Big)$$

Evaluating the elementary integral,

$$\int_0^1\frac{x^{p-1}}{1-x^n}(1-x)dx=\frac{1}{n}\Big(\psi(\frac{p+1}{n})-\psi(\frac{p}{n})\Big)$$

we complete the proof. QED.

Here are some particular cases.

3.5 $\int_0^1\frac{1}{1+x+x^2}\log\log\Big(\frac{1}{x}\Big)dx$

Since

$$\int_0^1\frac{1}{1+x+x^2}\log\log\Big(\frac{1}{x}\Big)dx=\int_0^1\frac{1-x}{1-x^3}\log\log\Big(\frac{1}{x}\Big)dx$$

then from (32) with $p=1$ and $n=3$, we have

$$\int_0^1\frac{1}{1+x+x^2}\log\log\Big(\frac{1}{x}\Big)dx=-\frac{\pi(\log(3)+\gamma)}{3\sqrt{3}}+\frac{\zeta'(1,\frac{1}{3})-\zeta'(1,\frac{2}{3})}{3}$$

Hence, by virtue of (8)

$$\int_0^1\frac{1}{1+x+x^2}\log\log\Big(\frac{1}{x}\Big)dx=\frac{\pi(-3\log(3)+8\log(2\pi)-12\log(,(\frac{1}{3})))}{6\sqrt{3}}$$

3.6 $\int_0^1\frac{x}{1+x+x^2+x^3+x^4+x^5}\log\log\Big(\frac{1}{x}\Big)dx$

In view of

$$\int_0^1\frac{x}{1+x+x^2+x^3+x^4}\log\log\Big(\frac{1}{x}\Big)dx=\int_0^1\frac{(1-x)x}{1-x^5}\log\log\Big(\frac{1}{x}\Big)dx$$

from (32) with $p=2$ and $n=5$, it follows that

$$\int_0^1\frac{x}{1+x+x^2+x^3+x^4}\log\log\Big(\frac{1}{x}\Big)dx=$$
$$\frac{1}{5}\Big((\log(5)+\gamma)\Big(\psi\Big(\frac{2}{5}\Big)-\psi\Big(\frac{3}{5}\Big)\Big)+\zeta'\Big(1,\frac{2}{5}\Big)-\zeta'\Big(1,\frac{3}{5}\Big)\Big)$$

Applying Proposition 1, we obtain

$$\begin{gathered}\int_0^1 \frac{x}{1+x+x^2+x^3+x^4} \log\log\Big(\frac{1}{x}\Big) dx = \\ \frac{\pi}{5}\Bigg(\cot\Big(\frac{2\pi}{5}\Big)\log(2\pi) - 2\log\Bigg(\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{4}{5})}\Bigg)\sin\big(\frac{\pi}{5}\big) + 2\log\Bigg(\frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{3}{5})}\Bigg)\sin\Big(\frac{2\pi}{5}\Big)\Bigg)\end{gathered} \tag{37}$$

Proposition 5 *Let $\Re(p) > 0$ and $\Re(n) > 0$, then*

$$\begin{gathered}\int_0^1 \frac{x^{p-1}}{(1+x^n)^2} \log\log\Big(\frac{1}{x}\Big) dx = \frac{(n-p)(\log(2n)+\gamma)(\psi(\frac{p}{2n}) - \psi(\frac{n+p}{2n}))}{2n^2} - \\ \frac{1}{2n}\Bigg(\gamma + \log(2n) - 2\log\Bigg(\frac{\Gamma(\frac{p}{2n})}{\Gamma(\frac{n+p}{2n})}\Bigg)\Bigg) + \frac{(n-p)}{2n^2}(\zeta'(1,\frac{p}{2n}) - \zeta'(1,\frac{n+p}{2n}))\end{gathered} \tag{38}$$

where $\zeta'(s,z)$ denotes $\frac{\partial}{\partial s}\zeta(s,z)$

If $p = n$, then

$$\int_0^1 \frac{x^{n-1}}{(1+x^n)^2} \log\log\Big(\frac{1}{x}\Big) dx = -\frac{1}{2n}\Big(\gamma + \log\Big(\frac{2n}{\pi}\Big)\Big) \tag{39}$$

Proof. Differentiating both sides of (24) with respect to λ, we obtain

$$\begin{gathered}\int_0^1 \frac{x^{p-1}}{(\lambda+x^n)^2} \log\log\Big(\frac{1}{x}\Big) dx = \\ -\gamma\int_0^1 \frac{x^{p-1}}{(\lambda+x^n)^2} dx - \frac{1}{\lambda^2}\sum_{k=0}^{\infty} \frac{(-1)^k(k+1)\log(kn+p)}{\lambda^k(kn+p)}\end{gathered} \tag{40}$$

We express the infinite sum in the left-hand side of (40) by means of the Lerch function

$$\begin{gathered}\sum_{k=0}^{\infty} \frac{(-1)^k(k+1)\log(kn+p)}{\lambda^k(kn+p)} = -\lim_{s\to 1}\frac{\partial}{\partial s}\sum_{k=0}^{\infty}\frac{(-1)^k(k+1)}{\lambda^k(kn+p)^s} = \\ \lim_{s\to 1}\frac{\partial}{\partial s}\frac{1}{n^s}\Big((\frac{p}{n}-1)\Phi\Big(-\frac{1}{\lambda}, s, \frac{p}{n}\Big) - \Phi\Big(-\frac{1}{\lambda}, s-1, \frac{p}{n}\Big)\Big)\end{gathered}$$

Therefore, the limiting value of the sum when λ approaches 1 is

$$\begin{gathered}\lim_{\lambda\to 1}\sum_{k=0}^{\infty} \frac{(-1)^k(k+1)\log(kn+p)}{\lambda^k(kn+p)} = \\ \frac{\log(2n)}{2n} + \frac{(n-p)\log(2n)}{2n^2}(\psi(\frac{n+p}{2n}) - \psi(\frac{p}{2n})) + \\ \frac{1}{n}(\zeta'(0,\frac{n+p}{2n}) - \zeta'(0,\frac{p}{2n})) + \frac{n-p}{2n^2}(\zeta'(1,\frac{n+p}{2n}) - \zeta'(1,\frac{p}{2n}))\end{gathered}$$

Then taking into account the identity (3) and

$$\int_0^1 \frac{x^{p-1}}{(1+x^n)^2}dx = \frac{1}{2n} + \frac{(n-p)}{2n^2}(\psi(\frac{n+p}{2n}) - \psi(\frac{p}{2n}))$$

we arrive at (38). QED.

Here are a few nice-looking integrals that follow immediately from the above proposition

$$\begin{aligned} &\int_0^1 \frac{\sqrt{x}}{(1+x)^2} \log\log\left(\frac{1}{x}\right)dx = \\ &\frac{\gamma}{2} - \frac{3}{2}\log(2) + \log\left(\frac{4, (\frac{3}{4})}{, (\frac{1}{4})}\right) + \frac{\pi}{2}\log\left(\frac{2\sqrt{\pi}, (\frac{3}{4})}{, (\frac{1}{4})}\right) \end{aligned} \tag{41}$$

$$\begin{aligned} &\int_0^1 \frac{\sqrt{x}}{(1+x^3)^2} \log\log\left(\frac{1}{x}\right)dx = \\ &-\frac{\gamma}{6} - \frac{1}{3}\log\left(\frac{\sqrt{6}, (\frac{3}{4})}{, (\frac{1}{4})}\right) + \frac{1}{6}\pi\log\left(\frac{2\sqrt{\frac{\pi}{3}}, (\frac{3}{4})}{, (\frac{1}{4})}\right) \end{aligned} \tag{42}$$

$$\begin{aligned} &\int_0^1 \frac{x}{(1-x+x^2)^2} \log\log\left(\frac{1}{x}\right)dx = \\ &-\frac{\gamma}{3} - \frac{1}{3}\log\left(\frac{6\sqrt{3}}{\pi}\right) + \frac{\pi\sqrt{3}}{27}\left(5\log(2\pi) - 6\log\left(, \left(\frac{1}{6}\right)\right)\right) \end{aligned} \tag{43}$$

Proposition 6 *Let $\Re(p) > 0$ and $\Re(n) > 0$, then*

$$\begin{aligned} &\int_0^1 \frac{x^{p-1}}{(1+x^n)^3} \log\log\left(\frac{1}{x}\right)dx = \\ &-\frac{(5n-2p)(\log(2n)+\gamma)}{8n^2} + \frac{3n-2p}{2n^2}\log\left(\frac{, (\frac{p}{2n})}{, (\frac{n+p}{2n})}\right) + \\ &\frac{(n-p)(2n-p)(\log(2n)+\gamma)}{4n^3}(\psi(\frac{p}{2n}) - \psi(\frac{n+p}{2n})) + \\ &\frac{1}{n}(\zeta'(-1, \frac{p}{2n}) - \zeta'(-1, \frac{n+p}{2n})) + \\ &\frac{(n-p)(2n-p)}{4n^3}(\zeta'(1, \frac{p}{2n}) - \zeta'(1, \frac{n+p}{2n})) \end{aligned} \tag{44}$$

If $p = n$, then

$$\int_0^1 \frac{x^{n-1}}{(1+x^n)^3} \log\log\left(\frac{1}{x}\right)dx = \frac{-10\log(2) - 9\log(n) + 6\log(\pi) - 9\gamma - 36\zeta'(-1)}{24n} \tag{45}$$

Proof. The proof is similar to that for Proposition 5. QED.

The following integrals follow immediately from (44):

$$\int_0^1 \frac{\sqrt{x}}{(1+x)^3} \log\log\Big(\frac{1}{x}\Big) dx = -\frac{G}{2\pi} + \frac{\pi}{8}\log\left(\frac{2\sqrt{\pi}\,,\,(\frac{3}{4})}{,\,(\frac{1}{4})}\right) \tag{46}$$

$$\begin{aligned} &\int_0^1 \frac{1}{(1+x^2)^3} \log\log\Big(\frac{1}{x}\Big) dx = \\ &\frac{1}{32}\left(-16\log(2) + 3\pi\log(2\pi) + \frac{8G}{\pi} - 8\gamma + 2(3\pi - 8)\log\left(\frac{,\,(\frac{3}{4})}{,\,(\frac{1}{4})}\right)\right) \end{aligned} \tag{47}$$

$$\frac{\pi}{2}\int_0^1 \frac{(1-6x^2+x^4)}{(1+x^2)^3} \log\log\Big(\frac{1}{x}\Big) dx = \frac{\pi}{4}\int_0^\infty \frac{\operatorname{sech}(z)\tanh(z)}{z}\, dz = G \tag{48}$$

Let us consider the particular case of the generating function for the Potts model on the triangular lattice when $y = \frac{\pi}{2}$. Other special cases of the integral (1) are described in [2]. From (1) we have

$$P_3(\frac{\pi}{2}) = 3\int_0^\infty \frac{\tanh(x)}{x\,(1 - 2\,\cosh(2\,x))^2} dx \tag{49}$$

Originally the integral was calculated by R.J. Baxter (see [3]) by using the Fourier analysis and the residue theorem.

Performing an exponential substitution and then integrating a correspondent integral one time by parts, we transform (49) to

$$P_3(\frac{\pi}{2}) = 6\int_0^1 \frac{x\ (1+x)\ (1-x-x^2-x^3+x^4)}{(1+x^3)^3}\ \log\log(\frac{1}{x})\, dx$$

Finally, applying Proposition 6, we find that

$$P_3(\frac{\pi}{2}) = \frac{-2\sqrt{3}\,\pi^2 - 3\,\pi\ \log(3) + \sqrt{3}\,\psi'(\frac{1}{6})}{6\,\pi}$$

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