On the Barnes function

Victor Adamchik
Carnegie Mellon University
adamchik@cs.cmu.edu

Abstract. The multiple Barnes function, defined as a generalization of the Euler gamma function, is used in many applications of pure and applied mathematics and theoretical physics. This paper presents new integral representations as well as special values of the Barnes function. Moreover, the Barnes function is expressed in a closed form by means of the Hurwitz zeta function. These results can be used for numeric and symbolic computations of the Barnes function.
Preamble

In 1899, Barnes [1,2,3] introduced and studied the generalization of the Euler gamma function defined by the following functional equation:

$$ G(z + 1) = \Gamma(z) G(z), \quad z \in C $$

$$ G(1) = 1 $$

where $\Gamma$ is the gamma function. For the integer positive values of $z$, the Barnes G function is simply a product of factorials:

$$ G(n + 1) = \prod_{k=1}^{n-1} k! = \prod_{k=1}^{n} \Gamma(k) $$

From the above functional equation and the Weierstrass canonical product for the gamma function,

$$ \Gamma(z) = \frac{1}{z} \exp(-z \gamma) \prod_{m=1}^{\infty} \frac{\exp\left(\frac{z}{m}\right)}{1 + \frac{z}{m}} $$

Barnes derived

$$ G(z + 1) = (2\pi)^{\gamma/2} \exp\left(-\frac{z + z^2(1 + \gamma)}{2}\right) \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right) $$

where $\gamma$ is the Euler-Mascheroni constant. The above product is an entire function in a whole complex plane and can serve as an explicit definition of the Barnes function. Originally, the G function was introduced (in a different form) by Kinkelin [5] (see also Glaisher [6]) in his research on the asymptotics behavior of this product at $n \to \infty$:

$$ 1^1 2^2 \ldots n^n = \frac{n^1 n}{G(n + 1)} \quad (1) $$

Kinkelin also applied the theory of the G function to the class of trigonometric integrals

$$ \int_{0}^{\infty} \log \text{trig}(x) \, dx $$

where $\text{trig}(x)$ is any trigonometric or hyperbolic function. All such integrals can be expressed in finite terms of the G function.
Alexeiewsky [7] generalized the Kinkelin product (1) to

\[ 1^p \ 2^p \ \ldots \ n^p = \exp(\zeta'(-p, n+1) - \zeta'(-p)) \]  

which is related to the multiple Barnes function.

These days the G function remains the active topic of research (see [8--15].) The theory of the Barnes function has been related to certain spectral functions in mathematical physics, to the study of determinants of Laplacians, and to the Hecke L-functions. In [10] and [15] the G function is expressed in terms of the Hurwitz zeta function by

\[ \log G(z+1) - z \log \Gamma(z) = \zeta'(-1) - \zeta'(-1, z), \quad R(z) > 0 \]  

where \( \zeta'(t, z) = \frac{d}{dt} \zeta(t, z) \) and \( \zeta'(-1) \) is related to Glaisher’s constant \( A \):

\[ \log A = \frac{1}{12} - \zeta'(-1) = \frac{\gamma + \log(2\pi)}{12} - \frac{\zeta''(2)}{2\pi^2} \]

In [15] Adamchik obtained the closed form representation for the class of integrals involving cyclotomic polynomials and nested logarithms in terms of the Hurwitz zeta function. In the view of (3) the results can be translated into the G function notation, for example

\[ \int_0^1 \frac{(x^4 - 6x^2 + 1)}{(x^2 + 1)^3} \log \log \left( \frac{1}{x} \right) \, dx = 4 \log \left( \frac{G(\frac{3}{4})}{G(\frac{1}{4})} \right) - 3 \log \Gamma \left( \frac{1}{4} \right) + \log \Gamma \left( \frac{3}{4} \right) \]

Choi et al. [13, 14] and Adamchik [15-17] considered a class of sums involving the Riemann zeta function which can be evaluated by means of the G function. Here is a series representation for the G function for \(|z| < 1\),

\[ 2 \log G(z+1) = z \log(2\pi) - \gamma z^2 - z(z+1) + 2 \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{z^{k+1}}{k+1} \]

and for Glaisher’s constant

\[ \log A = \frac{\log 2}{12} + \sum_{k=1}^{\infty} \left( \frac{\zeta(2k+1) - 1}{36} \right) \frac{(4k+7)(7k+8)}{(k+1)(k+2)} \]
Barnes [4] (followed by Vigneras [8] and Vardi [10]) generalized the G function to the multiple gamma function $G_n$ by the recurrence formula

$$G_{n+1}(z+1) = \frac{G_{n+1}(z)}{G_n(z)}, \quad z \in \mathbb{C}, \ n \in \mathbb{N}$$

$$G_2(z) = G(z)$$

$$G_1(z) = \frac{1}{\Gamma(z)}$$

Here $G(z)$ is the Barnes function and $\Gamma(z)$ is the Euler gamma function. Vigneras and Vardi considered a slightly different form of the multiple gamma function defined as a reciprocal to the Barnes function $\Gamma_n(z) = \frac{1}{G_n(z)}$. Vardi derived an implicit representation for $\Gamma_n(z)$ in terms of the multiple zeta function. In this paper we derive a closed form of $G_n$ in finite terms of the Hurwitz function and special polynomials. Here are two particular cases rewritten in terms of the derivatives of the Hurwitz function when $n = 3$

$$2 \log G_3(z+1) - z^2 \log \Gamma(z) = (1 - 2z) \log G_2(z+1) + \zeta'(-2) - \zeta'(-2, z)$$

and $n = 4$

$$6 \log G_4(z+1) - z^3 \log \Gamma(z) = 6 (1 - z) \log G_3(z+1) -$$

$$3 z^2 - 3 z + 1 \log G_2(z+1) + \zeta'(-3) - \zeta'(-3, z)$$

We also present various integral and series representations as well as some special values of the Barnes function. These results can be used in numeric and symbolic computations of the G function. I believe the multiple gamma function is of direct interest for computer algebra researchers and users, because of its significant applications in physics, number theory, combinatorics and applied mathematics. The Barnes function deserves to be implemented in computer algebra systems.

1. G function of the rational argument

There are a few known special cases when the G function is expressible in a closed form. The first one is due to Barnes [1]:

$$\log G\left(\frac{1}{2}\right) = \frac{\log 2}{24} - \frac{\log \pi}{4} - \frac{3 \log A}{2} + \frac{1}{8}$$

the other two are due to Choi and Srivastava [14]:
\[
\log G\left(\frac{1}{4}\right) = -\frac{G}{4\pi} - \frac{3}{4} \log \Gamma\left(\frac{1}{4}\right) - \frac{9}{8} \log A + \frac{3}{32}
\]
\[
\log G\left(\frac{3}{4}\right) = \log G\left(\frac{5}{4}\right) + \frac{G}{2\pi} - \frac{\log (2\pi^2)}{8}
\]

where \(G\) is Catalan's constant. In a view of closed form representation for the derivatives of the Hurwitz functions, obtained in [18], and the formula (3), and it is easy to derive the additional special cases of \(\log G(p)\) for \(p = \frac{1}{3}, \frac{1}{6}, \frac{2}{3}, \frac{5}{6}\).

**Proposition 1.**

1. \[
\log G\left(\frac{1}{3}\right) = \frac{\log 3}{72} + \frac{\pi}{18\sqrt{3}} - \frac{2}{3} \log \Gamma\left(\frac{1}{3}\right) - \frac{4}{3} \log A - \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{12\sqrt{3}\pi} + \frac{1}{9} \tag{7}
\]
2. \[
\log G\left(\frac{2}{3}\right) = \frac{\log 3}{72} + \frac{\pi}{18\sqrt{3}} - \frac{1}{3} \log \Gamma\left(\frac{2}{3}\right) - \frac{4}{3} \log A - \frac{\psi^{(1)}\left(\frac{2}{3}\right)}{12\sqrt{3}\pi} + \frac{1}{9} \tag{8}
\]
3. \[
\log G\left(\frac{1}{6}\right) = -\frac{\log 12}{144} + \frac{\pi}{20\sqrt{3}} - \frac{5}{6} \log \Gamma\left(\frac{1}{6}\right) - \frac{5}{6} \log A - \frac{\psi^{(1)}\left(\frac{1}{6}\right)}{40\sqrt{3}\pi} + \frac{5}{72} \tag{9}
\]
4. \[
\log G\left(\frac{5}{6}\right) = -\frac{\log 12}{144} + \frac{\pi}{20\sqrt{3}} - \frac{1}{6} \log \Gamma\left(\frac{5}{6}\right) - \frac{5}{6} \log A - \frac{\psi^{(1)}\left(\frac{5}{6}\right)}{40\sqrt{3}\pi} + \frac{5}{72} \tag{10}
\]

where \(\psi^{(1)}(z) = \frac{d^2}{dz^2} \log \Gamma(z)\) is the polygamma function.

Similar representations can be derived for the multiple \(G\) function.

## 2. Connection to the polylogarithm

From the Lerch functional equation for the Hurwitz zeta function and formula (3), we can easily derive

\[
\log\left(\frac{G(1+z)}{G(1-z)}\right) = -\frac{\pi i}{2} B_2(z) + z \log\left(\frac{\pi}{\sin(\pi z)}\right) + \frac{i}{2\pi} \text{Li}_2(e^{2\pi iz}), \quad 0 < z < 1 \tag{11}
\]

where \(B_2(z)\) is the second Bernoulli polynomial, and \(\text{Li}_2(z)\) is the polylogarithm. The identity can be rewritten in an alternative form by means of the Clausen function \(\text{Cl}_2(z)\):

\[
\log\left(\frac{G(1+z)}{G(1-z)}\right) = z \log\left(\frac{\pi}{\sin(\pi z)}\right) - \frac{1}{2\pi} \text{Cl}_2(2\pi z), \quad 0 < z < 1 \tag{12}
\]

where

\[
\text{Cl}_2(x) = -\text{Im}(\text{Li}_2(e^{-ix}))
\]
Obviously enough, the G function is related to the Dirichlet L-series. Here is one of the formulas:

$$
\log \left( \frac{G\left(\frac{7}{6}\right)}{G\left(\frac{5}{6}\right)} \right) = \frac{1}{6} \log(2 \pi) - \frac{3 \sqrt{3}}{8 \pi} L_{-3}(2)
$$

\( (13) \)

### 3. Binet-like representation

The Binet formula for the gamma function is

$$
\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \log(\sqrt{2\pi}) + \int_{0}^{\infty} \frac{e^{-zx}}{x} \left( \frac{1}{1 - e^{-x}} - \frac{1}{x} - \frac{1}{2} \right) dx
$$

In this section we derive a similar representation for the Barnes function. Recall the well-known integral for \( \zeta(s, z) \):

$$
\zeta(s, z) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} x^{s-1} \frac{e^{-zx}}{1 - e^{-x}} dx, \quad R(s) > 1, \; R(z) > 0
$$

\( (14) \)

The integral can be analytically continued to the domain \( R(s) > -1 \). To do so, we perform the standard procedure of removing the integrand singularity by subtracting the truncated Taylor series at \( x = 0 \):

$$
\zeta(s, z) = -\frac{z^{1-s}}{1-s} + \frac{z^{-s}}{2} + \frac{s}{12} z^{-s+1} + \frac{1}{\Gamma(s)} \int_{0}^{\infty} x^{s-1} \frac{e^{-zx}}{1 - e^{-x}} dx - \frac{1}{x} \frac{x}{12} \frac{1}{2}
$$

Differentiating the above formula with respect to \( s \) and computing the limit at \( s = -1 \), we get

$$
\zeta'(-1, z) = \frac{1}{12} - \frac{z^{2}}{4} - \frac{z}{2} B_{2}(z) + \int_{0}^{\infty} \frac{e^{-zx}}{x^{2}} \left( \frac{1}{1 - e^{-x}} - \frac{1}{t} - \frac{t}{12} - \frac{1}{2} \right) dx
$$

where \( B_{2}(z) \) is the second Bernoulli polynomial. From here it immediately follows

**Proposition 2.** The Barnes G function admits the Binet integral representation:

\[
\log G(z + 1) = z \log \Gamma(z) - \log A + \frac{z^{2}}{4} - \frac{\log z}{2} B_{2}(z) - \int_{0}^{\infty} \frac{e^{-zx}}{x^{2}} \left( \frac{1}{1 - e^{-x}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12} \right) dx, \quad R(z) > 0
\]

\( (15) \)

In a similar way we derive the representation for Glaisher’s constant:

\[
\log A = \frac{1 + \log(2 \pi)}{12} - \frac{1}{2 \pi^{2}} \int_{0}^{\infty} \frac{x \log x}{e^{x} - 1} dx
\]

\( (16) \)
4. The multiple G function

For the multiple G function defined by the functional equation (4), Vardi [10] obtained the following formula

\[
\log G_n(z) = -\lim_{s \to 0} \left( \frac{\partial \zeta_n(s, z)}{\partial s} \right) - \sum_{k=1}^{n} (-1)^k \left( \frac{z}{k-1} \right) R_{n+1-k}
\]  

(17)

where

\[
R_n = \sum_{k=1}^{n} \lim_{s \to 0} \left( \frac{\partial \zeta_k(s, 1)}{\partial s} \right)
\]

(18)

and \(\zeta_n(s, z)\) is the multiple zeta function:

\[
\zeta_n(s, z) = \sum_{k_1=0}^{\infty} \ldots \sum_{k_n=0}^{\infty} \frac{1}{(k_1 + k_2 + \ldots + k_n + z)^s} = \sum_{k=0}^{\infty} \frac{1}{(k+z)^n} \left( \begin{array}{c} k+n-1 \\ n-1 \end{array} \right)
\]

(19)

The aim of this section is to find a closed form representation for \(G_n(z)\) in terms of the Hurwitz zeta function. For clarity of exposition, we first consider polynomials

\[
P_{k,n}(z) = \sum_{i=k+1}^{n} (-z)^{i-k-1} \left( \begin{array}{c} i-1 \\ k \end{array} \right) \left[ \begin{array}{c} n \\ i \end{array} \right]
\]

(20)

which can be rewritten in the alternative form

\[
P_{k,n}(z) = \sum_{i=1}^{n} \left( \begin{array}{c} z \\ n-i \end{array} \right) \frac{(n-1)!}{(i-1)!} \left[ \begin{array}{c} i \\ k+1 \end{array} \right]
\]

(21)

where \(\left[ \begin{array}{c} n \\ i \end{array} \right]\) are unsigned Stirling numbers of the first kind. The polynomials \(P_{k,n}(z)\) satisfy the functional equation

\[
P_{k,n+1}(z) - n P_{k,n}(z) - P_{k,n+1}(z+1) = 0
\]

\[
P_{k,n}(z) = 0, \ k \geq n
\]
Here are the main properties of these polynomials:

\[ P_{n-1, n}(z) = 1 \quad \quad P_{k, n}(1) = \binom{n-1}{k} \]
\[ P_{n-2, n}(z) = z(1-n) + \frac{n(n-1)}{2} \quad \quad P_{0, n}(z) = \binom{n-z-1}{n-1}(n-1)! \quad \quad (22) \]
\[ P_{0, n}\left(\frac{1}{2}\right) = \frac{(2n-3)!!}{2^{n-1}} \]

Lemma 1. The multiple zeta function \( \zeta_n(s, z) \) defined by (19) may be expressed by means of the Hurwitz function

\[ \zeta_n(s, z) = \frac{1}{(n-1)!} \sum_{j=0}^{n} P_{j, n}(z) \zeta(s - j, z) \quad \quad (23) \]

Proof. Recall the definition of unsigned Stirling numbers of the first kind

\[ \binom{k+n-1}{n-1} = \frac{1}{(n-1)!} \sum_{i=0}^{n} k^{i-1} \left[ \begin{array}{c} n \\\ i \end{array} \right] \]

Expanding \( k^{i-1} = ((k + z) - z)^{i-1} \) by the binomial theorem, implies that

\[ \binom{k+n-1}{n-1} = \frac{1}{(n-1)!} \sum_{i=0}^{n} \sum_{j=0}^{i-1} (i-1) \left( \binom{k+z}{j} \right) (-z)^{i-j-1} \left[ \begin{array}{c} n \\\ i \end{array} \right] \]

Interchanging the order of summation and making use of (20), we obtain

\[ \binom{k+n-1}{n-1} = \frac{1}{(n-1)!} \sum_{j=0}^{n} P_{j, n}(z) (k + z)^{j} \]

We complete the proof by substituting this into (19).

Lemma 2. The multiple zeta function \( R_n \) defined by (18) may be expressed by means of the derivatives of the zeta function

\[ R_n = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \zeta'(-k) \left[ \begin{array}{c} n \\\ k+1 \end{array} \right] \quad \quad (24) \]

Proof. First we make use of the Lemma 1 and the identity (22) for \( P_{j, n}(1) \)

\[ \lim_{s \to 0} \left( \frac{\partial \zeta(s, 1)}{\partial s} \right) = \frac{1}{(n-1)!} \sum_{j=0}^{n} P_{j, n}(1) \zeta'(-j) = \frac{1}{(n-1)!} \sum_{j=0}^{n} \zeta'(-j) \left[ \begin{array}{c} n-1 \\\ j \end{array} \right] \]
Further, in a view of (18), we have

\[
R_n = \sum_{k=1}^{n} \frac{1}{(k-1)!} \sum_{j=0}^{k} \zeta'(-j) \left[ \frac{k-1}{j} \right] = \zeta'(0) + \sum_{j=1}^{n} \zeta'(-j) \sum_{k=j}^{n} \frac{1}{(k-1)!} \left[ \frac{k-1}{j} \right]
\]

Taking into account that the inner sum on the right hand side is (it follows from (21) with \( z = 1 \))

\[
\sum_{k=j}^{n} \frac{(n-1)!}{(k-1)!} \left[ \frac{k-1}{j} \right] = \left[ \frac{n}{j+1} \right]
\]

we complete the proof.

**Proposition 3.** The multiple Barnes function \( G_n(z) \) may be expressed by means of the derivatives of the zeta functions

\[
\log G_n(z) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} P_{j,n}(z) (\zeta'(-j) - \zeta'(-j, z))
\]

(25)

where the polynomials \( P_{j,n}(z) \) are defined by (20).

### 5. Integral representations via polygammas

In [15] Adamchik derived a closed form solution to the integral

\[
\int_{0}^{\infty} x^n \psi(x) \, dx, \quad R(z) > 0, \ n \in \mathbb{N}
\]

in finite terms of the Hurwitz zeta function:

\[
\int_{0}^{\infty} x^n \psi(x) \, dx = (-1)^{n-1} \zeta'(-n) + \frac{(-1)^n}{n+1} B_{n+1} H_n + \\
\sum_{k=0}^{n} (-1)^k \binom{n}{k} z^{n-k} \left( \zeta'(-k, z) - \frac{B_{k+1}(z) H_k}{k+1} \right)
\]

(26)

where \( B_n \) and \( H_n \) are Bernoulli and Harmonic numbers respectively. If \( n = 1 \) the integral (26) leads to the following representation for the Barnes G function:

\[
\log G(z+1) = \frac{z - z^2}{2} + z \log(\sqrt{2\pi}) + \int_{0}^{\infty} x \psi(x) \, dx, \quad R(z) > -1
\]

(27)

This representation demonstrates that the complexity of computing \( G(z) \) depends at most on the complexity of computing the polygamma function. The restriction \( R(z) > -1 \) can be easily
removed by analyticity of the polygamma. For example, by resolving the singularity of the integrand at the pole \( x = -1 \), we continue \( \log G(z + 1) \) to the wider area \( R(z) > -2 \):

\[
\log G(z + 1) \approx \frac{z - z^2}{2} + z \log(\sqrt{2\pi}) + \log(z + 1) + \int_0^z x \psi(x) - \frac{1}{x + 1} \, dx \quad (28)
\]

Another way to continue (27) into the left half-plane is to use the identity

\[
\psi(x) = \psi(-x) - \pi \cot(\pi x) - \frac{1}{x}
\]

which upon substituting it into (27) yields

\[
\log G(1 + z) = \log G(1 - z) + z \log(2\pi) - \int_0^z \pi x \cot(\pi x) \, dx \quad (29)
\]

The identity (29) holds everywhere in a complex plane of \( z \), except the real axes, where the integrand has simple poles. Therefore, in a view of (29) we can continue (27) to \( \mathbb{C} / \mathbb{R}^- \). Note, if \( z \) is negative we can still use the representation (27), but with a contour of integration deformed in such a way that it does not cross poles.

For \( n = 2 \), the formula (26) yields

\[
\int_0^z x^2 \psi(x) \, dx = -2 \log G_3(z + 1) + \log G_2(z + 1) + z^2 \zeta'(0) - 2 z \zeta'(-1) + \frac{z}{12} (6 z^2 - 3 z - 1) \quad (30)
\]

and for \( n = 3 \):

\[
\int_0^z x^3 \psi(x) \, dx = 6 \log G_4(z + 1) - 6 \log G_3(z + 1) + \log G_2(z + 1) + z^3 \zeta'(0) - 3 z^2 \zeta'(-1) + 3 z \zeta'(-2) + \frac{z^2}{24} (11 z^2 - 4 z - 1) \quad (31)
\]

The general formula can also be derived. Skipping the technical details, we have

**Proposition 4.** Let \( n \) be a positive integer and and \( R(z) > -1 \), then

\[
\int_0^z x^n \psi(x) \, dx = -\sum_{k=1}^{n} (-1)^k k! \left\{ \begin{array}{c} n \\ k \end{array} \right\} \log G_{k+1}(z + 1) - \sum_{k=0}^{n-1} (-1)^k \left\{ \begin{array}{c} n \\ k \end{array} \right\} z^{n-k} \left( \zeta'(-k) - \frac{B_{k+1}(z) H_k}{k + 1} \right) + (-1)^n H_n \frac{B_{n+1} - B_{n+1}(z)}{n+1} \quad (32)
\]

where \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) are Stirling numbers of the second kind.
We can resolve the equation (32) with respect to $G_{k+1}(z+1)$. In particular, combining (27) with (30) leads us to the following integral representation for the triple Barnes function:

$$2 \log G_3(z + 1) = \int_0^z x(1 - x)\psi(x) \, dx - z(1 - z)\zeta'(0) - 2 z \zeta'(-1) + \frac{z}{12} (6 z^2 - 9 z + 5)$$

(33)

and combining (27), (30) and (31) yields:

$$6 \log G_4(z + 1) = \int_0^z x(x - 1)(x - 2)\psi(x) \, dx - z(z - 1)(z - 2)\zeta'(0) + 3 z(z - 2)\zeta'(-1) - 3 z\zeta'(-2) - \frac{z}{24} (11 z^3 - 40 z^2 + 41 z - 18)$$

(34)

Formulas (33) and (34) are suitable for numeric computation of $G_3(z)$ and $G_4(z)$ for any $z \in \mathbb{C}/\mathbb{R}^-$.

6. Implementation remarks

We have presented here many available results on the Barnes function as well as new results on numeric and symbolic computations. In this section we discuss a numeric computational scheme for the Barnes $G$ function. The integral representation via polygamma functions considered in the previous section seems to be an efficient numeric procedure for evaluating $G$ function. Based on (27), we define the double Barnes function $G(z) = G_2(z)$ as

$$G(z) = (2\pi)^{-\frac{z-1}{2}} \exp \left( -\frac{(z - 1)(z - 2)}{2} + \int_0^{\gamma} x\psi(x) \, dx \right), \quad \arg(z) \neq \pi$$

(35)

This representation is valid for $z \in \mathbb{C}/\mathbb{R}^-$.

If $z$ is a negative real, we have

$$G(-n) = 0, \quad n \in \mathbb{N}$$

$$G(z) = (2\pi)^{-\frac{z-1}{2}} \exp \left( -\frac{(z - 1)(z - 2)}{2} + \int_0^{\gamma} x\psi(x) \, dx \right), \quad \arg(z) = \pi$$

(36)

where the contour of integration $\gamma$ is a line between 0 and $z - 1$ that does not cross the negative real axis; for example, $\gamma$ could be the following path: $[0, i, i + z - 1, z - 1]$. 
Another method of defining the Barnes G function for negative reals is to use the formula (12). This gives

\[
G(-z) = (-1)^{|z|} G(z + 2) \left( \frac{\sin(\pi z)}{\pi} \right)^{z+1} \exp \left( -\frac{1}{2\pi} \text{Cl}_2(2\pi (z - \lfloor z \rfloor)) \right)
\]

(37)

where \(\lfloor z \rfloor\) is the floor function, and \(\text{Cl}_2(z)\) is the Clausen function. Here is a picture of the G function over the interval (-3,3):

In a similar manner, taking into consideration the formulas (33) and (34), we can define \(G_3\) and \(G_4\) respectively.

There are many applications, particularly in number theory, where the logarithm of the Barnes function often appears. However, because of a branch cut of logarithm, the function \(\log G(z)\) (where \(G\) is defined by (35)) includes spurious discontinuities for complex argument. The following the picture of \(\text{Im}(\log G(2 - i y))\), where \(y \in [1, 10]\) demonstrates this:
Therefore, we define an additional function LogG(z) (the logarithm of the Barnes function) which is an analytic function throughout the complex z plane

\[ \text{LogG}(z) = -\frac{(z-1)(z-2)}{2} + \frac{z-1}{2} \log(2\pi) + \int_0^{\frac{z}{2}} x\psi(x) \, dx \]  

(38)

If \( z \) is a negative real, we understand the path of integration as in (36). Here is the picture of Im(LogG(2 – i y)), where \( y \in \{1, 10\} \):

![Graph of Im(LogG(2 – i y))](image)

**References**


