MULTIPLE GAMMA AND RELATED FUNCTIONS

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ABSTRACT. The authors give several new (and potentially useful) relationships between the multiple Gamma functions and other mathematical functions and constants. As by-products of some of these relationships, a classical definite integral due to Euler and other definite integrals are also considered together with closed-form evaluations of some series involving the Riemann and Hurwitz Zeta functions.

1. Introduction and Preliminaries

The multiple Gamma functions were defined and studied by Barnes (cf. [7] and [8]) and others in about 1900. Although these functions did not appear in the tables of the most well-known special functions, yet the double Gamma function was cited in the exercises by Whittaker and Watson [42, p. 264] and recorded also by Gradshteyn and Ryzhik [24, p. 661, Entry 6.441(4); p. 937, Entry 8.333]. Recently, these functions were revived in the study of the determinants of the Laplacians on the $n$-dimensional unit sphere $S^n$ (see [11], [17], [18], [30], [39], and [41]), and in evaluations of specific classes of definite integrals and infinite series involving, for example, the Riemann and Hurwitz Zeta functions (see [4], [15], [16], [18], and [19]). The subject of some of these developments can be traced back to an over two-century old theorem of Christian Goldbach (1690–1764), as noted in the work of Srivastava [32, p. 1] who investigated this subject in a systematic and unified manner (see also [35], [36, Chapter 3], and [37]). More recently, Adamchik ([2] and [3]) presented new integral representations for the $G$-function, its asymptotic expansions, its various relations with other special functions, as well as a procedure for its efficient numerical computation (see also the works of Choi [12], Choi and Seo [14], and Vardi [39] for several closely-related results). The theory of the double Gamma function has indeed found interesting applications in many other recent investigations (see, for details, [36]).

In this paper we aim at presenting several new (and potentially useful) relationships between other mathematical functions and the multiple Gamma functions. As by-products of some of these relationships, we shall derive a classical definite integral due to Euler and evaluate some other related integrals and series involving the Riemann and Hurwitz Zeta functions.

2000 Mathematics Subject Classification. Primary 33B99; Secondary 11M06, 11M99.

Key words and phrases. Multiple Gamma functions, Riemann’s $\zeta$-function, Hurwitz Zeta function, determinants of Laplacians, Clausen integrals, series involving the Zeta function.

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We begin by recalling the Barnes $G$-function ($1/G = \Gamma_2$ being the so-called double Gamma function) which has several equivalent forms including (for example) the Weierstrass canonical product:

\[
\{\Gamma_2(z+1)\}^{-1} = G(z+1)
\]

(1.1)

\[
= (2\pi)^{\frac{1}{2}} e^{-\frac{1}{2}(1+\gamma)z^2} \prod_{k=1}^{\infty} \left\{ 1 + \frac{z^2}{k^2} \right\} e^{-z^2/k^2},
\]

where $\gamma$ denotes the Euler-Mascheroni constant given by

\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \approx 0.577 215 664 901 532 860 606 512 \ldots.
\]

For sufficiently large real $x$ and $a \in \mathbb{C}$, we have the Stirling formula for the $G$-function:

\[
\log G(x + a + 1) = \frac{x + a}{2} \log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - \frac{ax}{2} \left( \frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax \right) \log x + O(1/x) \quad (x \to \infty),
\]

(1.3)

where $A$ is the Glaisher-Kinkelin constant given by (cf. [39] and [41])

\[
\log A = \frac{1}{12} - \zeta'(-1).
\]

(1.4)


The $G$-function satisfies the following fundamental functional relationships:

\[
G(1) = 1 \quad \text{and} \quad G(z + 1) = \Gamma(z) G(z) \quad (z \in \mathbb{C}),
\]

(1.5)

where $\Gamma$ denotes the familiar Gamma function. Barnes [7] generalized the functional relationships in (1.5) to the case of the multiple Gamma functions which are denoted usually by $G_n(z)$ or $\Gamma_n(z)$. Throughout this paper, we choose to follow the notations used (for example) by Vignéras [40] (see also Srivastava and Choi [36]). Thus we have

\[
\Gamma_n(z) := \{G_n(z)\}^{(-1)^{n-1}} \quad (n \in \mathbb{N}),
\]

(1.6)

so that

\[
G_1(z) = \Gamma_1(z) = \Gamma(z), \quad G_2(z) = \frac{1}{\Gamma_2(z)} = G(z), \quad G_3(z) = \Gamma_3(z),
\]

(1.7)

and so on. In terms of the multiple Gamma functions $G_n(z)$ of order $n$ ($n \in \mathbb{N}$), the aforementioned functional relationships of Barnes [7] can be rewritten as follows (cf. [40, p. 239]; see also [36, p. 14, Theorem 1.4]):
(1.8) \[ G_n(1) = 1 \quad \text{and} \quad G_{n+1}(z + 1) = G_n(z)G_{n+1}(z) \]
\[(z \in \mathbb{C}; \ n \in \mathbb{N})\]

or, equivalently,

(1.9) \[ \Gamma_n(1) = 1 \quad \text{and} \quad \Gamma_{n+1}(z + 1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)} \]
\[(z \in \mathbb{C}; \ n \in \mathbb{N}),\]

which may be used to define the multiple Gamma functions \( G_n(z) \) and \( \Gamma_n(z) \) \((n \in \mathbb{N})\).

From (1.8) and (1.2) one can easily derive the Weierstrass canonical product form of the triple Gamma function \( \Gamma_3 \) (see [18]):

(1.10) \[ \Gamma_3(1 + z) = G_3(1 + z) = \]
\[ \exp \left[ -\frac{z^3}{6} \left( \gamma + \frac{\pi^2}{6} + \frac{3}{2} \right) + \frac{1}{4} \left( \gamma + \log(2\pi) + \frac{1}{2} \right) z^2 + \left( \frac{3}{8} - \frac{\log(2\pi)}{4} - \log A \right) z \right] \]
\[ \cdot \prod_{k=1}^{\infty} \left( \frac{1}{1 + \frac{z}{k}} \right)^{-\frac{1}{2}(k+1)} \exp \left[ \frac{1}{2}(k+1)z - \frac{1}{4} \left( 1 + \frac{1}{k} \right) z^2 + \frac{1}{6k} \left( 1 + \frac{1}{k} \right) z^3 \right]. \]

Another form of the triple Gamma function \( \Gamma_3 \) appeared in the work of Choi [11] who expressed, in terms of the double and triple Gamma functions, the analogous Weierstrass canonical product of the shifted sequence of the eigenvalues of the Laplacian on the unit sphere \( S^3 \) with standard metric which was indispensable to evaluate the determinant of the Laplacian on \( S^3 \) there.

The Riemann Zeta function \( \zeta(s) \) is defined by

(1.11) \[ \zeta(s) := \begin{cases} \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1 - 2^{-s}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} & (\Re(s) > 1) \\ (1 - 2^{1-s})^{-1} \sum_{k=1}^{\infty} \frac{(-1)^{s+1}}{k^s} & (\Re(s) > 0; \ s \neq 1), \end{cases} \]

which can, except for a simple pole only at \( s = 1 \) with its residue 1, be continued analytically to the whole complex \( s \)-plane by means of the contour integral representation (cf. Whittaker and Watson [42, p. 266]) or many other integral representations (cf. Erdélyi et al. [21, p. 33]). It satisfies the functional equation (see [38, p. 13]):

(1.12) \[ \zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \zeta(1-s) \sin \left( \frac{\pi s}{2} \right). \]

The generalized (or Hurwitz) Zeta function \( \zeta(s, a) \) is defined by

(1.13) \[ \zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}), \]
which can, just as $\zeta(s)$, be continued analytically to the whole complex $s$-plane except for a simple pole only at $s = 1$. Indeed, from the definitions (1.11) and (1.13), it is easily observed that

\[(1.14) \quad \zeta(s, 1) = \zeta(s) = (2^s - 1)^{-1} \zeta(s, 1) \quad \text{and} \quad \zeta(s, 2) = \zeta(s) - 1,\]

\[(1.15) \quad \zeta(s, ma) = \frac{1}{m^s} \sum_{j=0}^{m-1} \zeta\left(s, a + \frac{j}{m}\right) \quad (m \in \mathbb{N}),\]

and

\[(1.16) \quad \zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right) \quad (m \in \mathbb{N} \setminus \{1\}).\]

Many elementary identities such as those derivable especially from (1.15) and (1.16) will be required in our present investigation.

2. A Family of Generalized Glaisher-Kinkelin Constants

Following the works of Glaisher [22] and Alexeievsky [5] on the asymptotic behavior of the product:

\[1^{1^p} \cdot 2^{2^p} \cdot 3^{3^p} \cdots n^{n^p}\]

when $n \to \infty$, Bendersky [9] considered the limit:

\[(2.1) \quad \log A_k = \lim_{n \to \infty} \left( \sum_{m=1}^{n} m^k \log m - p(n, k) \right),\]

where

\[(2.2) \quad p(n, k) = \frac{n^k}{2} \log n + \frac{n^{k+1}}{k+1} \left( \log n - \frac{1}{k+1} \right) + k! \sum_{j=1}^{k} \frac{n^{k-j} B_{j+1}}{(j+1)! (k-j)!} \left( \log n + (1 - \delta_{j,k}) \sum_{l=1}^{k} \frac{1}{k-l+1} \right),\]

and $\delta_{j,k}$ is the Kronecker delta. For $n = 0$, (2.1) yields

\[(2.3) \quad \log A_0 = \lim_{n \to \infty} \left[ \sum_{m=1}^{n} \log m - \left( \frac{1}{2} + n \right) \log n + n \right].\]

This limit is fairly well-known. The finite sum on the right-hand side of (2.3) is equal to $\log n!$, and thus, using the Stirling formula, we immediately obtain

\[(2.4) \quad \log A_0 = \frac{\log (2\pi)}{2}.\]
For $n = 1$, the limit (2.1) defines the Glaisher-Kinkelin constant $A$ [cf. Eq. (1.4) above]:

\[(2.5) \quad \log A_1 = \log A = \lim_{n \to \infty} \left[ \sum_{m=1}^{n} m \log m - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \log n + \frac{n^2}{4} \right].\]

Using the functional relation (1.5), we express the finite sum in (2.5) in terms of the Barnes $G$-function:

\[(2.6) \quad \sum_{m=1}^{n} m \log m = \log \left( \prod_{m=1}^{n} m^m \right) = n \log n! - \log G(n + 1).\]

By combining (2.5), (2.6), and the Stirling formula, we can establish the relationship:

\[(2.7) \quad \log A = \lim_{n \to \infty} \left[ -\log G(n + 1) + \left( \frac{n^2}{2} - \frac{1}{12} \right) \log n - \frac{3n^2}{4} + \frac{n}{2} \log(2\pi) + \frac{1}{12} \right].\]

In a similar way, we can demonstrate that the higher-order Glaisher-Kinkelin constant $A_n$ is related to the multiple Barnes function. Upon setting $n = 3$ in (2.1) and observing that

\[(2.8) \quad \sum_{m=1}^{n} m^2 \log m = n^2 \log n! - (2n - 1) \log G(n + 1) - 2 \log G_3(n + 1),\]

we find that

\[(2.9) \quad \log A_2 = \lim_{n \to \infty} \left[ -2 \log G_3(n + 1) - \left( \frac{n^3}{3} - \frac{n^2}{2} + \frac{1}{12} \right) \log n \right. \]

\[- \left. + \left( \frac{11n^2}{18} - \frac{3n^2}{4} - \frac{n}{6} + \frac{1}{12} \right) - \left( \frac{n^2}{2} - \frac{n}{2} \right) \log(2\pi) + (2n - 1) \log A \right].\]

The generalized Glaisher-Kinkelin constants $A_n$ appear naturally in the asymptotic expansion of the multiple Barnes function. With a view to circumventing the problem of evaluation of the limits in (2.7) and (2.9), Adamchik [1] found the following closed-form representation for the generalized Glaisher-Kinkelin constant $A_n$:

\[(2.10) \quad \log A_n = \frac{B_{n+1} H_n}{n + 1} - \zeta'(-n) \quad (n \in \mathbb{N}_0),\]

where $B_k$ and $H_k$ are the Bernoulli and harmonic numbers, respectively. The proof is based on the following analytical property of the Hurwitz Zeta function $\zeta(s, a)$:

\[(2.11) \quad \sum_{m=1}^{n} m^k \log m = \zeta'(-k, n + 1) - \zeta'(-k)\]

and its known asymptotic expansion at infinity. In fact, in light of the following consequence of the functional equation (1.12) [33, p. 387, Eq. (1.15)]:

\[(2.12) \quad \zeta(2n + 1) = (-1)^n \frac{2 \cdot (2\pi)^{2n}}{(2n)!} \zeta'(-2n) \quad (n \in \mathbb{N}),\]
it is easily seen from (2.10) that

\begin{equation}
(2.13) \quad \log A_{2n} = (-1)^{n+1} \frac{(2n)!}{2 \cdot (2\pi)^{2n}} \zeta(2n+1) \quad (n \in \mathbb{N}).
\end{equation}

The constants $A_1$, $A_2$, and $A_3$ (denoted commonly by $A$, $B$, and $C$, respectively) were also considered by Voros [41], Vardi [39], Choi and Srivastava (cf. [16] and [18]; see also [36, Chapter 1]).

3. Special cases of the multiple Barnes function

The Barnes $G$-function, given by the functional equation in (1.5), is a generalization of the Euler Gamma function. Thus it is not surprising to anticipate that the $G$-function would have closed-form representations for particular values of its argument. As a matter of fact, the $G$-function can be expressed in finite terms by means of the generalized Glaisher-Kinkelin constants $A_n$, as well as other known constants.

Barnes [7, p. 283] evaluated the following integral (see also [36, p. 207, Eq. 3.4(444)]):

\begin{equation}
(3.1) \quad \int_0^z \log \Gamma(t+a) \, dt = \frac{1}{2} \left[ \log(2\pi) + 1 - 2a \right] z - \frac{z^2}{2} + (z + a - 1) \log \Gamma(z + a) - \log G(z + a) + (1 - a) \log \Gamma(a) + \log G(a),
\end{equation}

which, in the special case when $a = 1$, reduces immediately to Alexeiewsky’s theorem (cf. [5] and [36, p. 32, Eq. 1.3 (42)]):

\begin{equation}
(3.2) \quad \int_0^z \log \Gamma(t+1) \, dt = \frac{1}{2} \left[ \log(2\pi) - 1 \right] z - \frac{z^2}{2} + z \log \Gamma(z + 1) - \log G(z + 1)
\end{equation}

or, equivalently,

\begin{equation}
(3.3) \quad \int_0^z \log \Gamma(t) \, dt = \frac{z(1 - z)}{2} + \frac{z}{2} \log(2\pi) - (1 - z) \log \Gamma(z) - \log G(z),
\end{equation}

since

\begin{equation}
(3.4) \quad \int_0^z \log t \, dt = z \log z - z.
\end{equation}

The integral formula (3.3), evaluated recently by Gosper [23] and Adamchik [1], happens to be a source for many closed-form representations of the Barnes $G$-function. The simplest non-trivial case $z = \frac{1}{2}$ is due to Barnes [7, p. 288, Section 7]:

\begin{equation}
(3.5) \quad G\left(\frac{1}{2}\right) = 2\pi^{\frac{3}{2}} \cdot \pi^{-\frac{1}{4}} \cdot e^\frac{\pi}{2} \cdot A^{-\frac{3}{2}}.
\end{equation}

By recalling the following special value of the Gamma function (see Spiegel [31, p. 1]):

\begin{equation}
(3.6) \quad \Gamma\left(\frac{1}{4}\right) \cong 3.625609908221908 \ldots
\end{equation}
and making use of a duplication formula for the $G$-function (cf. Barnes [7, p. 291] for the general case; see also Vardi [39]), Choi and Srivastava [16] showed that

$$(3.7) \quad G \left( \frac{1}{4} \right) = e^\frac{\pi}{2} \cdot A^{-\frac{5}{8}} \cdot \left\{ \Gamma \left( \frac{1}{4} \right) \right\}^{-\frac{3}{4}} \approx 0.293756 \ldots$$

or, equivalently, that

$$(3.8) \quad G \left( \frac{3}{4} \right) = 2^{-\frac{8}{5}} \cdot \pi^{-\frac{1}{4}} \cdot e^\frac{\pi}{2} \cdot A^{-\frac{5}{8}} \cdot \left\{ \Gamma \left( \frac{1}{4} \right) \right\}^{\frac{3}{4}} \approx 0.848718 \ldots,$$

where $G$ is the Catalan constant defined by

$$(3.9) \quad G := \frac{1}{2} \int_0^1 K(k) \, dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \approx 0.915965594177219015 \ldots,$$

and $K$ is the complete elliptic integral of the first kind, given by

$$(3.10) \quad K(k) := \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}.$$

The Catalan constant $G$ becomes a special case of many other functions (one of which will be considered in the next section) and has, among other things, been used to evaluate integrals, such as (see [24, p. 526, Entry 4.224]):

$$(3.11) \quad \int_0^{\pi/4} \log(\sin t) \, dt = -\frac{\pi}{4} \log 2 - \frac{1}{2} G,$$

and to give closed-form evaluations of a certain class of series involving the Zeta function (cf. Choi and Srivastava [16]).

Other particular cases of the Barnes function $G(z)$ when

$$(3.12) \quad z = \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \text{ and } \frac{5}{6}$$

were given by Adamchik (see [2] and [3]).

The identity (3.5) was generalized by Vardi [39] to the multiple Barnes function. The formula is much too complicated to be reproduced here. Here is one particular case when $n = 3$ and $z = \frac{1}{2}$ (see also [18, p. 95, Eq. (4.8)]):

$$(3.13) \quad G_3 \left( \frac{1}{2} \right) = 2^{-\frac{8}{5}} \cdot \pi^{-\frac{1}{4}} \cdot e^{-\frac{\pi}{4}} \cdot A_1^\frac{3}{5} \cdot A_2^\frac{3}{5} = \Gamma_3 \left( \frac{1}{2} \right),$$

where $A_k$ denotes the generalized Glaisher-Kinkelin constants given by (2.1) with, of course,

$$A_1 = A \quad \text{ and } \quad A_2 = B.$$
4. RELATIONSHIPS BETWEEN MULTIPLE GAMMA AND OTHER FUNCTIONS

The Clausen function (or the Clausen integral) $\text{Cl}_2(t)$ (see Lewin [28, p. 101]; see also Chen and Srivastava [10, p. 184]) defined by

\begin{equation}
\text{Cl}_2(t) := - \int_0^t \log \left[ 2 \sin \left( \frac{1}{2} u \right) \right] \, du = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^2},
\end{equation}

stems actually from the imaginary part of the Dilogarithm function $\text{Li}_2(e^{it})$ defined by

\begin{equation}
\text{Li}_2(z) := - \int_0^z \frac{\log(1-u)}{u} \, du = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|z| \leq 1).
\end{equation}

Now, by employing integration by parts in the following well-known integral formula (see Barnes [7, p. 279]; see also Choi and Srivastava [16, p. 94, Eq. (2.1)]) due originally to Kinkel:

\begin{equation}
\int_0^t \pi u \cot(\pi u) \, du = t \log(2\pi) + \log \left( \frac{G(1-t)}{G(1+t)} \right),
\end{equation}

we have (cf. Choi and Srivastava [16, p. 95, Eq. (2.2)]):

\begin{equation}
\int_0^t \log \sin(\pi u) \, du = t \log \left( \frac{\sin(\pi t)}{2\pi} \right) + \log \left( \frac{G(1+t)}{G(1-t)} \right).
\end{equation}

If we combine (4.1) and (4.4), we obtain the following relationship between the Clausen function and the $G$-function:

\begin{equation}
\text{Cl}_2(2\pi t) = \sum_{n=1}^{\infty} \frac{\sin(2\pi nt)}{n^2} = -2\pi t \log \left( \frac{\sin(\pi t)}{\pi} \right) - 2\pi \log \left( \frac{G(1+t)}{G(1-t)} \right),
\end{equation}

which, for $t = \frac{1}{4}$, readily yields the Catalan constant:

\begin{equation}
\text{Cl}_2 \left( \frac{\pi}{4} \right) = G
\end{equation}

by just observing, for the third member of (4.5), the following identity derivable from Eqs. (3.7), (3.8), and (1.5):

\begin{equation}
\frac{G \left( \frac{5}{2} \right)}{G \left( \frac{3}{2} \right)} = 2^\frac{1}{2} \cdot \pi^\frac{1}{2} \cdot \exp \left( -\frac{G}{2\pi} \right).
\end{equation}

Next we recall a relationship (cf. Cvijović and Klinowski [20, p. 210]):

\begin{equation}
S_2(\alpha) = \text{Cl}_2(\alpha) - \frac{1}{4} \text{Cl}_2(2\alpha),
\end{equation}

where $S_\nu(\alpha)$ denotes the classical trigonometric series defined by (cf. [20] and [34])

\begin{equation}
S_\nu(\alpha) := \sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^\nu}.
\end{equation}
A combination of (4.5) and (4.8) gives the following relationship between $S_2(\alpha)$ and the $G$-function:

$$S_2(2\pi t) = \pi t \log[2 \cot(\pi t)] + \frac{\pi}{2} \log \left( \frac{G(1 + 2t)}{G(1 - 2t)} \right) - 2\pi \log \left( \frac{G(1 + t)}{G(1 - t)} \right).$$

Integrating both sides of the second equality of (4.5) from 0 to $t$, we find an interesting relationship among a series involving a cosine function, a definite integral, and integrals of $G$-function:

$$
\frac{1}{4} \pi^2 \sum_{n=1}^{\infty} \frac{\cos(2n\pi t) - 1}{n^3} = \int_0^t u \log[\sin(\pi u)] \, du - \frac{t^2}{2} \log \pi
$$

$$
+ \int_0^t \log G(1 + u) \, du + \int_{-t}^0 \log G(1 + u) \, du.
$$

(4.11)

Now we recall an integral formula related to the multiple Gamma functions (see Choi and Srivastava [18, p. 96, Eq. (5.7)]) in the form:

$$
\int_0^t \log G(u + a) \, du = \left[ \frac{1}{2}(a - 1) \log(2\pi) - 2 \log A - \frac{a^2}{2} + a - \frac{1}{4} \right] t
$$

$$
+ \frac{1}{4} [\log(2\pi) + 2 - 2a] t^2 - \frac{1}{6} t^3
$$

$$
+ (t + a - 2) \log G(t + a) - 2 \log \Gamma_3(t + a)
$$

$$
+ (2 - a) \log G(a) + 2 \log \Gamma_3(a),
$$

which, for $a = 1$, yields [cf. Eq. (3.14) above]

$$
\int_0^t \log G(u + 1) \, du = \left( \frac{1}{4} - 2 \log A \right) t + \frac{t^2}{4} \log(2\pi)
$$

$$
- \frac{t^3}{6} + (t - 1) \log G(t + 1) - 2 \log \Gamma_3(t + 1),
$$

(4.12)

by virtue of Eqs. (1.5) and (1.8) with $n = 2$. If we apply (4.13) in (4.11), we obtain an interesting identity involving a series and an integral through multiple Gamma functions:

$$
\frac{1}{4} \pi^2 \sum_{n=1}^{\infty} \frac{\cos(2n\pi t) - 1}{n^3} = \frac{1}{\pi^2} \int_0^{\pi t} u \log(\sin u) \, du + \frac{t^2}{2} \log 2
$$

$$
+ (t - 1) \log G(t + 1) - (1 + t) \log G(1 - t) - 2 \log [\Gamma_3(1 + t) \Gamma_3(1 - t)],
$$

(4.14)

the left-hand summation part of which can, in terms of the higher-order Clausen function $\text{Cl}_n(t)$ defined, for all $n \in \mathbb{N} \setminus \{1\}$, by (cf. Lewin [28, p. 191])

$$
(4.15)
\text{Cl}_n(t) := \begin{cases} 
\sum_{k=1}^{\infty} \frac{\sin(kt)}{k^n} & \text{if } n \text{ is even}, \\
\sum_{k=1}^{\infty} \frac{\cos(kt)}{k^n} & \text{if } n \text{ is odd}, 
\end{cases}
$$
be expressed as follows:

\[
\sum_{n=1}^{\infty} \frac{\cos(2n\pi t) - 1}{n^3} = \text{Cl}_3(2\pi t) - \zeta(3).
\]

(4.16)

It is known that \((\text{cf., e.g.,} \) Hansen [26, p. 356, Entry (54.5.4)]; see also Chen and Srivastava [10, p. 184, Eq. (2.23)])

\[
\frac{1}{4\pi t} \text{Cl}_2(2\pi t) = \frac{1}{2} - \frac{1}{2} \log[2 \sin(\pi t)] - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} t^{2k}.
\]

(4.17)

A combination of (4.5) and (4.17) gives another proof of a special case of the following known identity for series involving the Riemann Zeta function (see Choi and Srivastava [16, p. 107, Eq. (4.10)]):

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} t^{2k+1} = \frac{1}{2} \left[ 1 - \log(2\pi) \right] t + \frac{1}{2} \log \left( \frac{G(1+t)}{G(1-t)} \right) \quad (|t| < 1).
\]

(4.18)

The special case of (4.14) when \(t = \frac{1}{2}\), with the aid of (1.5), (1.6) \textit{with} \(n = 2, (1.8), (2.13) \textit{with} n = 1, (3.2), \text{and} (3.10), \) yields a known integral formula:

\[
\int_{0}^{\pi/2} u \log(\sin u) \, du = \frac{7}{16} \zeta(3) - \frac{\pi^2}{8} \log 2,
\]

(4.19)

which is due to Euler (1772) who proved it through a striking and elaborate scheme as noted by Ayoub [6, p. 1084].

We now give here another short proof of (4.19). Indeed, if we multiply the well-known formula:

\[
u \cot u = -2 \sum_{k=0}^{\infty} \zeta(2k) \left( \frac{u}{\pi} \right)^{2k} \quad (|u| < \pi)
\]

(4.20)

by \(u\) and integrate the resulting equation with respect to \(u\) from 0 to \(\pi/2\), we find that

\[
\int_{0}^{\pi/2} u^2 \cot u \, du = -2 \int_{0}^{\pi/2} u \log(\sin u) \, du = -\frac{\pi^2}{4} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1) \cdot 2^{2k}},
\]

(4.21)

which, when combined with a known result \((\text{cf., e.g.,} \) [10, p. 191, Eq. (3.19)]), proves (4.19).

From (4.17) and a result of Grosjean [25, p. 334, Eq. (12)] we readily obtain a closed-form evaluation of a class of series involving the Riemann Zeta function:

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} \left( \frac{p}{2q} \right)^{2k} = \frac{1}{2} - \frac{1}{2} \log \left[ 2 \sin \left( \frac{p\pi}{2q} \right) \right]
\]

\[
- \frac{1}{8pq\pi} \sum_{k=1}^{q-1} \left[ \zeta \left( 2, \frac{k}{2q} \right) - \zeta \left( 2, 1 - \frac{k}{2q} \right) \right] \sin \left( \frac{pk\pi}{q} \right)
\]

\[
(p, q \in \mathbb{N}; 1 \leq p \leq 2q - 1; (p, q) = 1),
\]

(4.22)
which, in the special cases when (respectively) \( p = q = 1 \), \( p = q - 1 = 1 \), and \( p = q - 2 = 1 \), yields

\[
(3.23) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1) \cdot 2^{2k}} = \frac{1}{2} - \frac{1}{2} \log 2,
\]

which is equivalent to a known result [11, p. 109, Eq. (4.22)];

\[
(4.24) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1) \cdot 4^{2k}} = \frac{1}{2} - \frac{1}{4} \log 2 - \frac{G}{\pi},
\]

which is precisely the same as the known result [16, p. 114, Eq. (5.9)];

\[
(4.25) \quad \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1) \cdot 6^{2k}} = \frac{1}{2} + \frac{\pi}{2\sqrt{3}} - \frac{\sqrt{3}}{4\pi} \zeta\left(2, \frac{1}{3}\right),
\]

which corresponds to the special case of the known result (4.17) above when \( t = \frac{1}{6} \) (see also [36, Problem 2 (Chapter 3) and Problem 13 (Chapter 4)]).

In fact, closed-form evaluations of series involving the Zeta function has an over two-century old history as commented in Section 1 and has attracted many mathematicians ever since then (cf. Srivastava [32]; see also Srivastava and Choi [36, Chapter 3]). Here we also give a general formula for this subject by recalling (see Choi and Nash [13, p. 224, Eq. (1.11)])

\[
(4.26) \quad \sum_{k=2}^{\infty} \frac{(-1)^k a^{k+\beta}}{k+\beta} \zeta(k, \alpha) = \int_0^a t^\beta \left[ \psi(t + \alpha) - \psi(\alpha) \right] dt \quad (\beta \geq 0),
\]

where the Digamma function (or the \( \psi \)-function) is the logarithmic derivative of the Gamma function \( \Gamma \), and a closed-form solution to the integral (see Adamchik [1] and [2]):

\[
(4.27) \quad \int_0^a t^n \psi(t) dt = (-1)^{n-1} \zeta'(-n) + \frac{(-1)^n}{n+1} B_{n+1} H_n + \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} \left( \zeta'(-k, a) - \frac{B_{k+1}(a) H_k}{k+1} \right),
\]

where \( B_n \) and \( B_n(a) \) are the Bernoulli numbers and polynomials (see [21, pp. 35-36]), respectively, and \( H_n \) are the harmonic numbers defined by [cf. Eq. (2.10) above]

\[
(4.28) \quad H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N}).
\]

Combining (4.26) and (4.28), we obtain the desired result:

\[
(4.29) \quad \sum_{k=2}^{\infty} \frac{(-1)^k a^{k+n}}{k+n} \zeta(k) = \frac{a^n}{n} + \frac{\gamma a^{n+1}}{n+1} + (-1)^{n-1} \zeta'(-n) + \frac{(-1)^n}{n+1} B_{n+1} H_n + \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} \left( \zeta'(-k, a) - \frac{B_{k+1}(a) H_k}{k+1} \right)
\]

\[\Re(a) > 0; \quad |a| < 1; \quad n \in \mathbb{N},\]

where \( \gamma \) is the Euler-Mascheroni constant given by (1.2).
5. Evaluations of some definite integrals

We begin by combining the formulas (4.14) and (4.16) in the form:

\begin{equation}
\frac{1}{4 \pi^2} \left[ \text{Cl}_3(2\pi t) - \zeta(3) \right] = \frac{1}{\pi^2} I(t) + \frac{t^2}{2} \log 2 \\
+ (t - 1) \log G(t + 1) - (1 + t) \log G(1 - t) - 2 \log A(t),
\end{equation}

where, for convenience, \( I(t) \) and \( A(t) \) are defined by

\begin{equation}
I(t) := \int_0^{\pi t} u \log(\sin u) \, du
\end{equation}

and

\begin{equation}
A(t) := \Gamma_3(1 + t) \Gamma_3(1 - t),
\end{equation}

so that, obviously, Euler’s integral is precisely \( I \left( \frac{1}{4} \right) \). Now we evaluate the integral \( I \left( \frac{1}{4} \right) \). First of all, it follows from (1.7) that

\begin{equation}
\log A(t) = \frac{1}{2} \left[ \gamma + \log(2\pi) + \frac{1}{2} \right] t^2 \\
+ \frac{1}{2} \left( \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k+1} t^{2k+2} + \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} t^{2k+2} \right).
\end{equation}

Furthermore, the special case of a known result [16, p. 107, Eq. (4.11)] when \( a = 1 \) gives

\begin{equation}
\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} t^{2k+2} = -\left( \gamma + 1 \right) t^2 - \log \left[ G(1 + t) G(1 - t) \right] \quad (|t| < 1),
\end{equation}

which is a companion formula of (5.4) for the \( G \)-function. We thus find from (1.5), (3.7), and (3.8) that

\begin{equation}
\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{k+1} \cdot 4^{2k} = -4 - \gamma + \log \left[ 4 \cdot \pi^4 \cdot A^{36} \cdot \{ \Gamma \left( \frac{1}{4} \right) \}^{-8} \right].
\end{equation}

The identity of Chen and Srivastava [10, p. 191, Eq. (3.20)] is rewritten here in the following form:

\begin{equation}
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1) \cdot 4^{2k}} = \frac{1}{2} - \frac{1}{2} \log 2 - \frac{4 \log G}{\pi} + \frac{35}{4\pi^2} \zeta(3),
\end{equation}

Next, the special case of another known result [16, p. 108, Eq. (4.14)] when \( a = 1 \) yields

\begin{equation}
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k+1} z^{2k+2} = \left[ 1 - \log(2\pi) \right] \frac{z^2}{2} + z \log \left[ \frac{G(1 + z)}{G(1 - z)} \right] \\
- \int_0^z \log G(t + 1) \, dt - \int_0^{-z} \log G(t + 1) \, dt \quad (|t| < 1),
\end{equation}
which, upon setting \( z = \frac{1}{4} \) and using (5.7), gives the following interesting identity:

\[
(5.9) \quad \int_0^{\frac{1}{4}} \log G(t+1) \, dt + \int_0^{-\frac{1}{4}} \log G(t+1) \, dt = \frac{1}{32} \left( \log(2\pi) + \frac{4\mathbf{G}}{\pi} - \frac{35}{2\pi^2} \zeta(3) \right).
\]

In order to illustrate the usefulness of (5.9), we integrate the special case of [16, p. 107, Eq. (4.10)] when \( a = 1 \) from 0 to \( \frac{1}{4} \), and we readily find from (5.9) that

\[
(5.10) \quad \zeta(3) = \frac{4\pi^2}{35} \left( \frac{1}{2} + \frac{2\mathbf{G}}{\pi} - \sum_{k=1}^\infty \frac{\zeta(2k)}{(k+1)(2k+1) \cdot 4^{2k}} \right),
\]

which is an interesting addition to the results recorded by Chen and Srivastava [10] who treated the subject of series representations for \( \zeta(3) \) in a rather systematic manner and presented new results and some relevant connections with other functions.

From the definition (4.15) and (1.11), it is easy to see that (cf. Lewin [28, p. 300])

\[
(5.11) \quad \text{Cl}_{2n+1} \left( \frac{\pi}{2} \right) = \frac{1 - 2^n}{2^{4n+1}} \zeta(2n+1) \quad (n \in \mathbb{N}).
\]

Finally, upon setting \( t = \frac{1}{4} \) in (5.1), and making use of (5.4), (5.6), (5.7), and (5.11) (with \( n = 1 \)), we obtain the desired integral evaluation:

\[
(5.12) \quad \int_0^{\frac{1}{4}} u \log(\sin u) \, du = \mathcal{I} \left( \frac{1}{4} \right) = \frac{35}{128} \zeta(3) - \frac{\pi \mathbf{G}}{8} - \frac{\pi^2}{32} \log 2,
\]

which, upon employing integration by parts, yields

\[
(5.13) \quad \int_0^{\frac{1}{4}} u^2 \cot u \, du = -\frac{35}{64} \zeta(3) + \frac{\pi \mathbf{G}}{4} + \frac{\pi^2}{32} \log 2.
\]

We evaluate some more definite integrals and series involving the Zeta function. By applying the known result [20, p. 210, Eq. (13c)]:

\[
(5.14) \quad S_2 \left( \frac{\pi}{4} \right) = \frac{1}{32} \left[ \sqrt{2} \zeta \left( 2, \frac{1}{8} \right) - 2 \left( \sqrt{2} + 1 \right) \pi^2 - 16 \sqrt{2} \mathbf{G} \right]
\]

in (4.10) with \( t = \frac{1}{8} \), and using the relationship (4.7), we obtain

\[
(5.15) \quad \log \left( \frac{G \left( \frac{9}{8} \right)}{G \left( \frac{5}{8} \right)} \right) = \frac{1}{16} \log \left[ 2 \left( 2 + \sqrt{2} \right) \pi \right] + \frac{1}{8\pi} \left( 2\sqrt{2} - 1 \right) \mathbf{G}
+
\frac{1}{64\pi} \left[ 2 \left( \sqrt{2} + 1 \right) \pi^2 - \sqrt{2} \zeta \left( 2, \frac{1}{8} \right) \right].
\]

If we set \( t = \frac{1}{8} \) in (4.3), (4.4), and (4.18), and make use of (5.15), we find that

\[
(5.16) \quad \int_0^{\frac{\pi}{8}} u \cot u \, du = \frac{\pi}{16} \log \left[ \left( 2 - \sqrt{2} \right) \pi \right] + \frac{1}{8} \left( 1 - 2\sqrt{2} \right) \mathbf{G}
+
\frac{1}{64} \left[ \sqrt{2} \zeta \left( 2, \frac{1}{8} \right) - 2 \left( \sqrt{2} + 1 \right) \pi^2 \right],
\]
$$\int_0^{\pi/8} \log \sin u \, du = -\frac{\pi}{16} \log(4\pi) + \frac{1}{8} \left( 2\sqrt{2} - 1 \right) G$$

$$+ \frac{1}{64} \left[ 2 \left( \sqrt{2} + 1 \right) \pi^2 - \sqrt{2} \zeta \left( 2, \frac{1}{8} \right) \right],$$

and

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1) \cdot 2^{2k}} = \frac{1}{2} - \frac{1}{4} \log \left[ \left( 2 - \sqrt{2} \right) \pi \right] + \frac{1}{2\pi} \left( 2\sqrt{2} - 1 \right) G$$

$$+ \frac{1}{16\pi} \left[ 2 \left( \sqrt{2} + 1 \right) \pi^2 - \sqrt{2} \zeta \left( 2, \frac{1}{8} \right) \right].$$

By setting \( t = \frac{1}{6} \) in (4.18) and applying (4.25), we obtain

$$\log \left( G \left( \frac{\pi}{6} \right) \right) = \frac{\sqrt{3}}{18} \pi + \frac{1}{6} \log(2\pi) - \frac{\sqrt{3}}{12\pi} \zeta \left( 2, \frac{1}{3} \right).$$

In view of (5.19) and the relationship (cf. [20, p. 210, Eq. (13e)]):

$$S_2 \left( \frac{\pi}{6} \right) = \frac{2G}{3},$$

(4.10) with \( t = \frac{1}{12} \) yields

$$\log \left( G \left( \frac{13}{12} \right) \right) = -\frac{G}{3\pi} + \frac{1}{24} \log \left[ 4 \left( 2 + \sqrt{3} \right) \pi \right] + \frac{\sqrt{3}}{24} \left[ \frac{\pi}{3} - \frac{1}{2\pi} \zeta \left( 2, \frac{1}{3} \right) \right].$$

If we set \( t = \frac{1}{12} \) in (4.3), (4.4), and (4.18), and apply (5.21), we see that

$$\int_0^{\pi/12} u \cot u \, du = \frac{G}{3} - \frac{\pi}{24} \log \left( \frac{2 + \sqrt{3}}{\pi} \right) + \frac{\sqrt{3}}{24} \left[ \frac{\pi}{3} - \frac{1}{2\pi} \zeta \left( 2, \frac{1}{3} \right) - \frac{\pi^2}{3} \right],$$

$$\int_0^{\pi/12} \log \sin u \, du = -\frac{G}{3} - \frac{\pi}{24} \log(4\pi) + \frac{\sqrt{3}}{24} \left[ \frac{\pi^2}{3} - \frac{1}{2\pi} \zeta \left( 2, \frac{1}{3} \right) \right],$$

and

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1) \cdot 12^{2k}} = \frac{1}{2} - \frac{2G}{\pi} + \frac{1}{4} \log \left( \frac{2 + \sqrt{3}}{\pi} \right) + \frac{\sqrt{3}}{4} \left[ \frac{\pi}{3} - \frac{1}{2\pi} \zeta \left( 2, \frac{1}{3} \right) \right].$$

Just as in getting (4.23), (4.24), and (4.25), the special cases of (4.22) when

\( p = q - 3 = 1 \) and \( p = q - 5 = 1 \)
can also be seen to yield (5.18) and (5.24), respectively. In view of the rôles played by (5.15) and (5.21) in closed-form evaluations of definite integrals, we choose to record here the following general identity:

$$\log \left( \frac{G \left( 1 + \frac{p}{2q} \right)}{G \left( 1 - \frac{p}{2q} \right)} \right) = \frac{p}{2q} \log \left[ \frac{1}{\pi} \sin \left( \frac{p\pi}{q} \right) \right]$$

$$+ \frac{1}{8q^2 \pi} \sum_{k=1}^{q-1} \left[ \zeta \left( 2, \frac{k}{2q} \right) - \zeta \left( 2, 1 - \frac{k}{2q} \right) \right] \sin \left( \frac{pk\pi}{q} \right)$$

\( (p, q \in \mathbb{N}; 1 \leq p \leq 2q - 1; (p, q) = 1), \)

which follows immediately from (4.18) with \( t = \frac{p}{2q} \) and (4.22).

We conclude this paper by remarking that closed-form evaluation of \( I \left( \frac{p}{q} \right) \) \( 0 \leq p \leq q; \ p, q \in \mathbb{N} \) can be done in terms of the Hurwitz function:

$$\int_{0}^{\frac{\pi}{q}} x \log \sin(x) dx = -\frac{p^2 \pi^2}{2q^2} \log 2 + \frac{\zeta(3)}{4} - \frac{\zeta(3)}{4q^3} - \frac{p \pi}{2q^3} \sum_{n=1}^{q-1} \sin \left( \frac{2np\pi}{q} \right) \zeta (2, \frac{n}{q}) - \frac{1}{4q^3} \sum_{n=1}^{q-1} \cos \left( \frac{2np\pi}{q} \right) \zeta (3, \frac{n}{q})$$

ACKNOWLEDGEMENTS

For the first-named author, this work was supported by grant No. 2001-1-10200-004-2 from the Basic Research Program of the Korea Science and Engineering Foundation. The second-named author was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

REFERENCES

22. J. W. L. Glaisher, *On the product $1^5 \cdot 2^5 \cdot 3^5 \cdots n^5$*, Messenger Math. 7 (1877), 43–47.
23. R. W. Gosper, Jr., \( \int_{n/4}^{m/6} \ln \Gamma(z) \, dz \), Fields Inst. Comm. 14 (1997), 71–76.


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