Logistic Map

The logistic map (first published in 1845 by the Belgian P.-F. Verhulst) is defined by

\[ x_{n+1} = r x_n (1 - x_n) \]
\[ x_0 \in [0, 1] \]

where \( r \) is a positive constant. The logistic map models the population growth, and the parameter \( r \) is determined by the birth rate.

Let us find appropriate conditions on \( r \) which keep points in the interval \([0, 1]\). The maximum is

\[
\text{Solve}[D[r x (1 - x), x] = 0, x]
\]
\[
\{ \{x \to \frac{1}{2} \} \}
\]

The largest value for the map is

\[
x \cdot (1 - x) \bigg/ \bigg. x \to 1/2
\]
\[
\frac{r}{4}
\]

So, to keep \( x_n \) inside the interval \([0, 1]\), the parameter \( r \) must be within the interval \([0, 4]\).

The next task we perform is to figure out the behavior of the logistic map depending on various values of the parameter \( r \). First we choose a small \( r \)

\[
r = 0.9; \text{NestList}[r \# (1 - \#) &, 0.1, 50];
\]
We see that for small $r$ the population decreases, and soon would disappear, like Indians Mayan.

Now we increase the birth rate:

$$ r = 1.4; \text{NestList}[r \# (1 - \#) &, 0.1, 50]; $$

$$ \text{ListPlot}[% , \text{PlotJoined} \rightarrow \text{True}]; $$

There is a range of $r$ where the population grows and then approaches a stable equilibrium number.

We increase the birth rate a little bit more
\[ r = 2.7 ; \text{NestList}[r \# (1 - \#) & , 0.1, 30] ; \]
\[ \text{ListPlot}[% , \text{PlotJoined} \to \text{True} , \text{PlotRange} \to \text{All}] ; \]

and more (you might need to add an option \text{PlotRange} \to \text{All})

\[ r = 3.05 ; \text{NestList}[r \# (1 - \#) & , 0.1, 30] ; \]
\[ \text{ListPlot}[% , \text{PlotJoined} \to \text{True} , \text{PlotRange} \to \text{All}] ; \]

We see cycles here. The population settles down to a steady state and cycles between the two values shown in the graph above. Is this behavior stable? Change the initial value from 0.1 to 0.9
$r = 3.05; \text{NestList}\left[r \# (1 - \#) \& , 0.9, 30\right];$

ListPlot[% , PlotJoined -> True, PlotRange -> All];

The population never settles down and is said to be chaotic. One of the key components of chaos is sensitivity to initial conditions. We run two tests, one with the initial value 0.1 and the second 0.1001:
\[
\begin{align*}
\textbf{Fixed Points} \\
\text{Let us find the fixed points of the logistic map} \\
\text{Clear}[r, x]; \\
\text{Solve}[x == r x (1 - x), x] \\
\{ (x \to 0), \{x \to \frac{-1 - r}{r} \} \}
\end{align*}
\]

The stable equilibrium which we observed before is the fixed point. Suppose \( r = 2.5 \), then the fixed point is \[ 1 - \frac{1}{r} = 0.6 \]}
As we observed, an interesting behavior happened when \( r > 3 \). The map becomes unstable and shows cycles. These cycles correspond to the fixed points of the second order. We find these points:

\[
x_{n+2} = r x_{n+1}(1-x_{n+1}) \\
x_{n+1} = r x_n(1-x_n)
\]

We have found the first-order fixed points along with two second-order fixed points:

\[
\frac{1 + r \pm \sqrt{(r+1)(r-3)}}{2r}
\]

In this formula you should see the clue, why those cycle behavior happened when \( r > 3 \). The second-order fixed points are real when \( r > 3 \)!

Now if you want to see the second-order fixed points

\[
\{ 0.590164, 0.737705 \}
\]
\[ r = 3.05; \text{NestList}[r \# (1 - \#) \&, 0.9, 30]; \]\n\[ \text{ListPlot}[%; \text{PlotJoined} \to \text{True}; \text{PlotRange} \to \text{All}] \]
So, indeed those cycles are the fixed points of the second order.

Now we know that the 2-cycles start at \( r = 3 \). What about the 3-cycles? What are their values and where do they start?

\[
\begin{align*}
    x_{n+3} &= r x_{n+2}(1 - x_{n+2}) \\
    x_{n+2} &= r x_{n+1}(1 - x_{n+1}) \\
    x_{n+1} &= r x_n(1 - x_n)
\end{align*}
\]

Clear\([r]\);
Eliminate[
\{x[1 + n] == r (1 - x[n]) x[n], x[2 + n] == r (1 - x[1 + n]) x[1 + n],
    x[n] == r (1 - x[2 + n]) x[2 + n] \}, \{x[n + 1], x[n + 2] \}] / . x[n] \rightarrow x

\[
(1 - r^3) x + r^3 (1 + r + r^2) x^2 + r^4 (-2 - 2 r - 2 r^2) x^3 +
    r^4 (1 + r + 6 r^2 + r^3) x^4 + (-6 - 4 r) r^6 x^5 + r^6 (2 + 6 r) x^6 - 4 r^7 x^7 + r^7 x^8 = 0
\]

Factor[\%[[1]]];
\%[[3]];
pol = Collect[\%, x, Simplify]

\[
1 + r + r^2 - r (1 + 2 r + 2 r^2 + r^3) x + r^2 (1 + 3 r + 3 r^2 + 2 r^3) x^2 -
    r^3 (1 + 3 r + 5 r^2 + r^3) x^3 + r^4 (1 + 4 r + 3 r^2) x^4 - r^5 (1 + 3 r) x^5 + r^6 x^6
\]

Since we are looking for the 3-cycle, the above polynomial should have 3 double roots in the bifurcation point.

\[
\text{Sum}[a[i] x^i, \{i, 0, 3\}]^2
\]

\[
(a[0] + x a[1] + x^2 a[2] + x^3 a[3])^2
\]
CoefficientList[pol - Sum[a[i] x^i, {i, 0, 3}]^2, x]

\[(1 + r + r^2 - a[0]^2, -r (1 + 2 r + 2 r^2 + r^3) - 2 a[0] a[1],
\]
\[r^2 (1 + 3 r + 3 r^2 + 2 r^3) - a[1]^2 - 2 a[0] a[2],
\]
\[-r^3 (1 + 3 r + 5 r^2 + r^3) - 2 a[1] a[2] - 2 a[0] a[3],
\]

\# == 0 & /@ %;

Eliminate[% , a[0]]

\]
\[4 a[1]^2 == r^2 (1 + 3 r + 11 r^2 + 5 r^3) \&\& 10 r a[2] == (-12 - 17 r + r^2) a[3] \&\&
\]
\]
\]

The first polynomial is what we are looking for

Solve[-7 r^4 - 2 r^3 + r^6 == 0, r]

\[\{(r \rightarrow 0), (r \rightarrow 0), (r \rightarrow 0), (r \rightarrow 0), (r \rightarrow 1 - 2 \sqrt{2}), (r \rightarrow 1 + 2 \sqrt{2})\}\]

So 3-cycles start at \(1 + 2 \sqrt{2} \approx 3.82843\). Their values are

\[r = N[1 + 2 \text{Sqrt}[2]];\]

NSolve[pol == 0, x]

\[\{(x \rightarrow 0.159929), (x \rightarrow 0.159929), (x \rightarrow 0.514355),
\]
\[(x \rightarrow 0.514355), (x \rightarrow 0.956318), (x \rightarrow 0.956318)\}\]

Here is a picture, which demonstrates them.
\[ r = 3.85; \text{NestList}[r \# (1 - \#) &, 0.1, 30]; \]
\[ \text{ListPlot}[, \text{PlotJoined} \rightarrow \text{True}, \text{PlotRange} \rightarrow \text{All}] \]

- Graphics -

### The Bifurcation Diagram

*with credit to Robert Knapp and Mark Sofroniou:
http://library.wolfram.com/examples/iteration/

As the parameter \( r \) varies, the point at which the period of iterates doubles is known as a bifurcation point. A bifurcation diagram allows you to visualize what values the population take on as a function of the parameter. The idea works as follows. Start with some initial value and iterate the map many times. Then iterate the map several more times, and remove duplicate values amongst those iterates. If the parameter is in a range where there is a stable periodic cycle, then the number of values you have will be the period of the cycle. On the other hand, if you are in a range where there is chaos, then all of the values will be different and you will just see a range of points. To successfully make a bifurcation diagram, you need to be able to iterate the map many times for many different values of the parameter.
Clear[r];
BifurcationDiagram[f_,
    {r_, rmin_, rmax_, rstep_}, {x_, x0_}, start_, combine_] :=
    Module[{R, temp, MapFunction, i},
        R = Table[i, {i, rmin, rmax, rstep}];
        (* The range of values of the parameter *)
        MapFunction = MakeMapFunction[{r, x}, f];
        (* Construct the iterating function *)
        temp = Nest[MapFunction[R, #] & , x0 + 0. R, start + 1];
        (* Starting iterates *)
        temp = NestList[MapFunction[R, #] & , temp, combine - 1];
        (* Subsequent iterates *)
        temp = Map[Union, Transpose[temp]];  
        (* Remove duplicate values from cycles *)
        Flatten[MapThread[Thread[[#1, #2]] &, {R, temp}], 1];
    ];
MakeMapFunction[{r_, x_}, f_] := Function[{r, x}, f];
ListPlot[BifurcationDiagram[r x (1 - x), {r, 3, 3.9, 0.001},
    {x, .1}, 10000, 100], PlotStyle -> AbsolutePointSize[0.0001]]