Split Closure and Intersection Cuts *

Kent Andersen Gérard Cornuéjols Yanjun Li

Graduate School of Industrial Administration
Carnegie Mellon University, Pittsburgh, PA, USA
{kha,gc0v,yanjun}@andrew.cmu.edu

Abstract. In the seventies, Balas introduced intersection cuts for a Mixed Integer Linear Program (MILP), and showed that these cuts can be obtained by a closed form formula from a basis of the standard linear programming relaxation. In the early nineties, Cook, Kannan and Schrijver introduced the split closure of an MILP, and showed that the split closure is a polyhedron. In this paper, we show that the split closure can be obtained using only intersection cuts. We give two different proofs of this result, one geometric and one algebraic. Furthermore, the result is used to provide a new proof of the fact that the split closure is a polyhedron. Finally, we extend the result to more general two-term disjunctions.

1 Introduction

In the seventies, Balas showed how a cone and a disjunction can be used to derive a cut [1] for a Mixed Integer Linear Program (MILP). In that paper, the cone was obtained from an optimal basis to the standard linear programming relaxation. The cut was obtained by a closed form formula and was called the intersection cut.

Later in the seventies, Balas generalized the idea to polyhedra [2]. It was demonstrated that, given a polyhedron, and a valid but violated disjunction, a cut could be obtained by solving a linear program. The idea was further expanded in the early nineties, where Cook, Kannan and Schrijver [4] studied split cuts obtained from two-term disjunctions that are easily seen to be valid for an MILP. The intersection of all split cuts is called the split closure of an MILP. Cook, Kannan and Schrijver proved that the split closure of an MILP is a polyhedron.

Any basis of the constraint matrix describing the polyhedron can be used, together with a disjunction, to derive an intersection cut, i.e. the basis used does not have to be optimal or even feasible. A natural question is how intersection cuts relate to disjunctive cuts obtained from polyhedra. This question was answered by Balas and Perregaard [3] for the 0-1 disjunction for Mixed Binary Linear Programs. The conclusion was that any disjunctive cut obtained from a polyhedron and the 0-1 disjunction is identical to, or dominated by, intersection cuts obtained from the 0-1 disjunction and bases of the constraint matrix describing the polyhedron. We generalize this result from 0-1 disjunctions to more general two-term disjunctions. This is the main result of this paper. We provide two different proofs, one geometric and one algebraic. A consequence is a new proof of the fact that the split closure is a polyhedron.

We consider the Mixed Integer Linear Program (MILP):

$$\text{(MILP)} \qquad min\{c^T x : Ax \leq b, x_j \text{ integer}, j \in N_I\},$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $N_I \subseteq N := \{1, 2, \dots, n\}$ and A is an $m \times n$ matrix. r denotes the rank of A. LP is the Linear Programming problem obtained from MILP by dropping the integrality

* Supported by NSF grant DMI-0098427 and ONR grant N00014-97-1-0196.

conditions on x_j, $j \in N_I$. P_I and P denote the sets of feasible solutions to MILP and LP respectively. $M := \{1, 2, \ldots, m\}$ is used to index the rows of A. $a_{i.}$, for $i \in M$, denotes the i^{th} row of A. We assume $a_{i.} \neq 0_n$ for all $i \in M$. Given $S \subseteq M$, $r(S)$ denotes the rank of the sub-matrix of A induced by the rows in S ($r(M) = r$). Furthermore, for $S \subseteq M$, the relaxation of P obtained by dropping the constraints indexed by $M \setminus S$ from the description of P, is denoted $P(S)$, i.e. $P(S) := \{x \in \mathbb{R}^n : (a_{i.})^T x \leq b_i, \forall i \in S\}$ ($P(M) = P$). A *basis* of A is an n-subset B of M, such that the vectors $a_{i.}$, $i \in B$, are linearly independent. Observe that, if $r < n$, A does not have bases. $\mathcal{B}_k^*$, where k is a positive integer, denotes the set of k-subsets S of M, such that the vectors $a_{i.}$, $i \in S$, are linearly independent ($\mathcal{B}_n^*$ denotes the set of bases of A).

The most general two-term disjunction considered in this paper is an expression D of the form $D^1 x \leq d^1 \vee D^2 x \leq d^2$, where $D^1 : m_1 \times n$, $D^2 : m_2 \times n$, $d^1 : m_1 \times 1$ and $d^2 : m_2 \times 1$. The set of points in $\mathbb{R}^n$ satisfying D is denoted F_D. The set $conv(P \cap F_D)$ is called the *disjunctive set* defined by P and D in the remainder. In addition, given a subset S of the constraints, the set $conv(P(S) \cap F_D)$ is called the disjunctive set defined by S and D. Finally, given a set B in $\mathcal{B}_r^*$, the set $conv(P(B) \cap F_D)$ is called a *basic* disjunctive set.

An important two-term disjunction is the *split disjunction* $D(\pi, \pi_0)$ of the form $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $\pi_j = 0$ for all $j \notin N_I$. For the split disjunction, a complete description of the basic disjunctive sets is available. Given a set B in $\mathcal{B}_r^*$, the basic disjunctive set defined by B and $D(\pi, \pi_0)$ is the set of points in $P(B)$ that satisfies the intersection cut derived from B and $D(\pi, \pi_0)$ (Lemma 1). Let $\Pi^n(N_I) := \{(\pi, \pi_0) \in \mathbb{Z}^{n+1} : \pi_j = 0, j \notin N_I\}$. The *split closure* of MILP, denoted by SC, is defined to be the intersection of the disjunctive sets defined by P and $D(\pi, \pi_0)$ over all disjunctions (π, π_0) in $\Pi^n(N_I)$. Similarly, given $S \subseteq M$, SC(S), is defined to be the intersection of the disjunctive sets defined by $P(S)$ and $D(\pi, \pi_0)$ over all disjunctions (π, π_0) in $\Pi^n(N_I)$. A *split cut* is a valid inequality for SC.

The first contribution in this paper is a theorem (Theorem 1) stating that the split closure of MILP can be written as the intersection of the split closures of the sets $P(B)$ over all sets B in $\mathcal{B}_r^*$. (i.e. $SC = \cap_{B \in \mathcal{B}_r^*} SC(B)$). We prove this theorem by proving that the disjunctive set defined by P and $D(\pi, \pi_0)$, for a split disjunction $D(\pi, \pi_0)$, can be written as the intersection of the basic disjunctive sets (i.e. $conv(P \cap F_{D(\pi,\pi_0)}) = \cap_{B \in \mathcal{B}_r^*} conv(P(B) \cap F_{D(\pi,\pi_0)})$). We provide both a geometric and an algebraic proof of this result. The result implies that both the disjunctive set defined by P and $D(\pi, \pi_0)$ and the split closure of MILP can be obtained using only intersection cuts. This generalizes a result of Balas and Perregaard showing the same result for the disjunction $x_j \leq 0 \vee x_j \geq 1$ for Mixed Binary Linear Programs [3]. (In fact, in that paper, it was assumed that $r = n$, whereas the theorem presented here does not have this assumption). Furthermore, the result leads to a new proof of the fact that the split closure is a polyhedron (Theorem 3).

The second contribution in this paper is a theorem (Theorem 6) stating that the disjunctive set defined by P and D can be written as the intersection of disjunctive sets defined by D and $(r+1)$-subsets S of M (i.e. $conv(P \cap F_D) = \cap_{S \in \mathcal{C}_1^*} conv(P(S) \cap F_D)$, where $\mathcal{C}_1^*$ is some family of $(r+1)$-subsets of M to be defined later). The theorem implies that any valid inequality for the disjunctive set defined by P and D is identical to, or dominated by, inequalities derived from the disjunction D and $(r+1)$-subsets of the constraints describing the polyhedron. Furthermore, in the special case where $r = n$, we show that it is enough to consider a certain family $\mathcal{C}_2^*$ of n-subsets of the constraints describing the polyhedron.

The rest of this paper is organized as follows. In section 2, we consider basic disjunctive sets for split disjunctions. In section 3, the characterization of the split closure in terms of intersection cuts is presented, and a geometric argument for the validity of the result is presented. In section 4 we give a new proof of the fact that the split closure is a polyhedron. In section 5, we generalize the results of section 3 to more general two-term disjunctions. We also give an example showing that other extensions are incorrect. The arguments used in this section are mostly algebraic. In fact, section 3 and section 5 could be read independently. Finally, section 6 contains our conclusion.

2 A complete description of the basic disjunctive sets for split disjunctions

In this section, we describe the set $P(B)$ for B in B_r^* as the translate of a cone, and this cone is used together with a split disjunction to derive an intersection cut. The intersection cut is then used to characterize the basic disjunctive set obtained from the split disjunction.

Let $B \in \mathcal{B}_r^*$ be arbitrary. The set $P(B)$ was defined using half-spaces in the introduction. We now give an alternative description. Let $\bar{x}(B)$ satisfy $a_{i.}^T \bar{x}(B) = b_i$ for all $i \in B$. Furthermore, let $L(B) := \{x \in \mathbb{R}^n : a_{i.}^T x = 0, \forall i \in B\}$. Finally, let $r^i(B)$, where $i \in B$, be a solution to the system $(a_{k.})^T r^i(B) = 0$, for $k \in B \setminus \{i\}$, and $(a_{i.})^T r^i(B) = -1$. We have:

$$(1) \qquad P(B) = \bar{x}(B) + L(B) + \mathit{Cone}(\{r^i(B) : i \in B\}),$$

where $\mathit{Cone}(\{r^i(B) : i \in B\}) := \{x \in \mathbb{R}^n : x = \sum_{i \in B} \lambda_i r^i(B),\ \lambda_i \geq 0, i \in B\}$ denotes the cone generated by the vectors $r^i(B)$, $i \in B$. Observe that the vectors $r^i(B)$, $i \in B$, are linearly independent.

Let $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$. Assume that all points y in $\bar{x}(B) + L(B)$ violate the disjunction $D(\pi, \pi_0)$ (Lemma 1 below shows that this is the only interesting case). Observe that this implies that the linear function $\pi^T x$ is constant on $\bar{x}(B) + L(B)$. The intersection cut defined by B and $D(\pi, \pi_0)$ can now be described. Define $\epsilon(\pi, B) := \pi^T \bar{x}(B) - \pi_0$ to be the amount by which the points in $\bar{x}(B) + L(B)$ violate the first term in the disjunction. Also, for i in B, define:

$$(2) \qquad \alpha_i(\pi, B) := \begin{cases} -\epsilon(\pi, B)/(\pi^T r^i(B)) & \text{if } \pi^T r^i(B) < 0, \\ (1 - \epsilon(\pi, B))/(\pi^T r^i(B)) & \text{if } \pi^T r^i(B) > 0, \\ \infty & \text{otherwise.} \end{cases}$$

The interpretation of the numbers $\alpha_i(\pi, B)$, for $i \in B$, is the following. Let $x^i(\alpha, B) := \bar{x}(B) + \alpha r^i(B)$, where $\alpha \in \mathbb{R}_+$, denote the half-line starting in $\bar{x}(B)$ in the direction $r^i(B)$. The value $\alpha_i(\pi, B)$ is the smallest value of $\alpha \in \mathbb{R}_+$, such that $x^i(\alpha, B)$ satisfies the disjunction $D(\pi, \pi_0)$, i.e. $\alpha_i(\pi, B) = \inf\{\alpha \geq 0 : x^i(\alpha, B) \in F_{D(\pi,\pi_0)}\}$. Given the numbers $\alpha_i(\pi, B)$ for $i \in B$, the intersection cut associated with B and $D(\pi, \pi_0)$ is given by:

$$(3) \qquad \sum_{i \in B} (b_i - a_{i.}^T x)/\alpha_i(\pi, B) \geq 1.$$

The validity of this inequality for $\mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$ was proven by Balas [1]. In fact, we have:

Lemma 1 *Let $B \in \mathcal{B}_r^*$ and $D(\pi, \pi_0)$ be a split disjunction, where $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$.*

(i) If $\pi^T x \notin]\pi_0, \pi_0 + 1[$, for some $x \in \bar{x}(B) + L(B)$, then $\mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)}) = P(B)$.
(ii) If $\pi^T \bar{x} \in]\pi_0, \pi_0 + 1[$, for all $x \in \bar{x}(B) + L(B)$, then $\mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)}) = \{x \in P(B) : (3)\}$.

3 Split closure characterization

In this section, we give a geometric proof of the following theorem, which characterizes the split closure in terms of certain basic subsets of the constraints.

Theorem 1

$$SC = \bigcap_{B \in \mathcal{B}_r^*} SC(B). \tag{4}$$

We prove this result in the following lemmas and corollaries. Let $\alpha^T x \leq \beta$ and $\alpha^T x \geq \psi$ be inequalities, where $\alpha \in \mathbb{R}^n$ and $\beta < \psi$. In the case where $\alpha \neq 0_n$, $\alpha^T x \leq \beta$ and $\alpha^T x \geq \psi$ represent two non-intersecting half-spaces. We have the following key lemma:

Lemma 2 *Assume P is full-dimensional. Then:*

$$\begin{aligned} &Conv((P \cap \{x : \alpha^T x \leq \beta\}) \cup (P \cap \{x : \alpha^T x \geq \psi\})) = \\ \bigcap_{B \in \mathcal{B}_r^*} &Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\})). \end{aligned} \tag{5}$$

From Lemma 2 we immediately have the following:

Corollary 1 *Let $S \subset \mathbb{R}^{n+2}$ be a set of $(\alpha, \beta, \psi) \in \mathbb{R}^{n+2}$ such that $\alpha \in \mathbb{R}^n$ and $\beta < \psi$. When P is full-dimensional,*

$$\begin{aligned} &\bigcap_{(\alpha,\beta,\psi) \in S} Conv((P \cap \{x : \alpha^T x \leq \beta\}) \cup (P \cap \{x : \alpha^T x \geq \psi\})) = \\ \bigcap_{B \in \mathcal{B}_r^*} &\bigcap_{(\alpha,\beta,\psi) \in S} Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\})). \end{aligned} \tag{6}$$

By choosing S appropriately we get:

Corollary 2 *Equation* (4) *holds when P is full-dimensional.*

Now assume P is not full-dimensional. We first consider the case when P is empty:

Lemma 3 *Equation* (4) *holds when P is empty.*

In the remainder of this section we assume P is non-empty and not full-dimensional. Let $\mathit{Aff}(P)$ denote the affine hull of P. When P is not full-dimensional, it is full-dimensional in $\mathit{Aff}(P)$. $l := \dim(\mathit{Aff}(P)) < n$ denotes the dimension of P. Also, $M^= := \{i \in M : a_{i.}^T x = b_i, \ \forall x \in P\}$ denotes the constraints describing P satisfied with equality by all points in P. Since $\dim(P) = l$, there exists a set $B \in \mathcal{B}_{n-l}^*$, such that $\mathit{Aff}(P) = \{x \in \mathbb{R}^n : a_{i.}^T x = b_i, i \in B\}$. $\mathcal{S}^= := \{B \in \mathcal{B}_{n-l}^* : B \subseteq M^=\}$ denotes the set of all such sets B. The following properties are needed:

Lemma 4 *Assume P is non-empty and not full-dimensional. Then:*

(i) $\mathit{Aff}(P) = P(M^=)$.

(ii) *Let $i^* \in M^=$ be arbitrary. The linear program $min\{a_{i^*.}^T x : x \in P(M^= \backslash \{i^*\})\}$ is bounded, the optimal objective value is b_{i^*} and the set of optimal solutions is $Aff(P)$.*
(iii) *There exist $B' \in \mathcal{S}^=$ and $i' \in M^= \setminus B'$ such that $Aff(P) = P(B' \cup \{i'\})$.*
(iv) *Let i' and B' be as in (iii). There exists $i'' \in B'$ such that $B'' := (B' \backslash \{i''\}) \cup \{i'\} \in \mathcal{S}^=$ and $Aff(P) = P(B') \cap P(B'')$.*

Let B' and B'' be as in Lemma 4. The sets B' and B'' might not be of cardinality r, i.e. B' (B'') might not be a maximal subset of M such that the vectors $a_{i.}$, for $i \in B'$ ($i \in B''$), are linearly independent. Define $\gamma := r - (n - l)$. It follows that $0 \leq \gamma \leq l$. $\mathcal{B}_\gamma^A := \{B \in \mathcal{B}_\gamma^* : B \cup B' \in \mathcal{B}_r^*\}$ ($= \{B \in \mathcal{B}_\gamma^* : B \cup B'' \in \mathcal{B}_r^*\}$) denotes the family of γ-subsets B in $\mathcal{B}_\gamma^*$ such that $B' \cup B$ ($B'' \cup B$) is an r-subset in $\mathcal{B}_r^*$. Also, $\mathcal{B'}_r^* := \{B \in \mathcal{B}_r^* : B \supseteq B'\}$ and $\mathcal{B''}_r^* := \{B \in \mathcal{B}_r^* : B \supseteq B''\}$ denotes the families of r-subsets in $\mathcal{B}_r^*$ containing B' and B'' respectively. The following is immediate from the definitions of $\mathcal{B}_\gamma^A$, $\mathcal{B'}_r^*$ and $\mathcal{B''}_r^*$:

Lemma 5 *There is a one-to-one mapping from $\mathcal{B}_\gamma^A$ to $\mathcal{B'}_r^*$, and there is a one-to-one mapping from $\mathcal{B}_\gamma^A$ to $\mathcal{B''}_r^*$.*

We are now able to finish the proof of Theorem 1:

Lemma 6 *Equation* (4) *holds when P is non-empty and not full-dimensional in $\mathbb{R}^n$.*

Theorem 1 is implied by Corollary 2, Lemma 3 and Lemma 6. In fact the proofs allow us to extend Theorem 1 to arbitrary subsets of $\Pi^n(N_I)$:

Theorem 2 *Assume $S \subseteq \Pi^n(N_I)$. Then:*

$$\bigcap_{(\pi,\pi_0)\in S} Conv(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{B\in\mathcal{B}_r^*} \bigcap_{(\pi,\pi_0)\in S} Conv(P \cap F_{D(\pi,\pi_0)}). \tag{7}$$

4 Polyhedrality of split closure

In this section we assume $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^n$, i.e. that P is a rational polyhedron. Cook, Kannan and Schrijver proved the following result [4]

Theorem 3 *The split closure of MILP is a polyhedron.*

We give a new proof of this result using the characterization of the split closure obtained in the previous section. Let $\tilde{C} := \tilde{x} + Cone(\{\tilde{r}_i : i = 1, 2, \ldots, q\})$ be (the translate of) a cone with apex $\tilde{x} \in \mathbb{Q}^n$ and q linearly independent extreme ray vectors $\{\tilde{r}_i\}_{i=1}^q$, where $q \leq n$ and $\tilde{r}_i \in \mathbb{Z}^n$ for $1 \leq i \leq q$. The following lemma plays a key role in proving Theorem 3.

Lemma 7 *The split closure of $\tilde{C}$ is a polyhedron.*

Let $B \in \mathcal{B}_r^*$ be arbitrary. From section 2 we know that $P(B)$ can be written as $P(B) = \bar{x}(B) + L(B) + C(B)$, where $C(B) := Cone(\{r^i(B) : i \in B\})$ and $\{r^i(B) : i \in B\} \subseteq \mathbb{Z}^n$ are linearly independent (by scaling). The following lemmas are straightforward:

Lemma 8 *Let $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ and $B \in \mathcal{B}_r^*$ be arbitrary. If an inequality is valid for $Conv(P(B) \cap F_{D(\pi,\pi_0)})$ but not valid for $P(B)$, then $\pi^T = \sum_{i \in B} \alpha_i a_{i.}^T$, where $\alpha_i \in \mathbb{R}$ $(i \in B)$.*

Define $S_B := \{\pi \in \mathbb{Z}^n : \pi^T = \sum_{i \in B} \alpha_i a_{i.}^T,\ \alpha_i \in \mathbb{R},\ i \in B\}$. From Lemma 8 we have:

Lemma 9 *Let $B \in \mathcal{B}_r^*$, $(\pi, \pi_0) \in S_B \times \mathbb{Z}$.*

(i) Assume there exists a facet F of $Conv(P(B) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $P(B)$. Then F is unique and there exists a unique facet $\tilde{F}$ of $Conv((\bar{x}(B) + C(B)) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $\bar{x}(B) + C(B)$. Furthermore $F = L(B) + \tilde{F}$.

(ii) Assume there exists a facet $\tilde{F}$ of $Conv((\bar{x}(B) + C(B)) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $\bar{x}(B) + C(B)$. Then $\tilde{F}$ is unique and there exists a unique facet F of $Conv(P(B) \cap F_{D(\pi,\pi_0)})$, which is not a facet of $P(B)$. Furthermore $\tilde{F} = F \cap (\bar{x}(B) + C(B))$.

The following result is implied by Lemma 7 and Lemma 9:

Lemma 10 *$SC(B)$, where $B \in \mathcal{B}_r^*$, is a polyhedron.*

Furthermore, Theorem 3 follows the results of Theorem 1 and Lemma 10.

5 Disjunctive sets derived from polyhedra and two-term disjunctions

In this section, two decomposition results for the set $conv(P \cap F_D)$ are presented. The first decomposition result (Theorem 6) states that $conv(P \cap F_D)$ can be written as the intersection of sets $conv(P(T) \cap F_D)$ over sets $T \in C_1^*$, where C_1^* is some family of $(r+1)$-subsets of M. Furthermore, in the special case where $r = n$, we show that it is enough to consider r-subsets T of M. The second result strengthens the first result for split disjunctions $D(\pi, \pi_0)$, and states that the set $conv(P \cap F_{D(\pi,\pi_0)})$ can be written as the the intersection of the sets $conv(P(B) \cap F_{D(\pi,\pi_0)})$ for $B \in \mathcal{B}_r^*$. We start by proving the following:

Theorem 4 *Let $S \subseteq M$ be non-empty. If S satisfies $|S| \geq r(S) + 2$, then*

$$conv(P(S) \cap F_D) = \bigcap_{i \in S} conv(P(S \setminus \{i\}) \cap F_D). \tag{8}$$

Furthermore, (8) remains true if $r(S) = n$ and $|S| = n + 1$.

One direction is easy to prove:

Lemma 11 *Let $S \subseteq M$ be non-empty. Then,*

$$conv(P(S) \cap F_D) \subseteq \bigcap_{i \in S} conv(P(S \setminus \{i\}) \cap F_D). \tag{9}$$

The proof of the other direction involves the idea introduced by Balas [2] of lifting the set $Conv(P(S) \cap F_D)$ into a higher dimensional space. Specifically, $Conv(P(S) \cap F_D)$ can be described as the projection of the set, described by the following constraints, onto the space of x-variables:

$$
\begin{aligned}
&(10) \qquad & x = x^1 + x^2, & & \\
&(11) \qquad & a_{i.}^T x^1 \leq b_i \lambda^1, & \quad \forall i \in S, & \\
&(12) \qquad & a_{i.}^T x^2 \leq b_i \lambda^2, & \quad \forall i \in S, & \\
&(13) \qquad & \lambda^1 + \lambda^2 = 1, & & \\
&(14) \qquad & D^1 x^1 \leq d^1 \lambda^1, & & \\
&(15) \qquad & D^2 x^2 \leq d^2 \lambda^2, & & \\
&(16) \qquad & \lambda^1, \lambda^2 \geq 0. & &
\end{aligned}
$$

The description (10) – (16) can be projected into (x, x^1, λ^1)-space by using constraints (10) and (13). By doing this, we arrive at the following characterization of $Conv(P(S) \cap F_D)$. Later, the constraints below will be used in the formulation of an LP problem. Therefore, we have written the names of the corresponding dual variables next to the constraints:

$$
\begin{aligned}
&(17) \qquad & -\lambda^1 b_i + a_{i.}^T x^1 \leq 0, & \quad \forall i \in S, & \quad (u_i) \\
&(18) \qquad & -\lambda^1 b_i + a_{i.}^T x^1 \leq b_i - a_{i.}^T x, & \quad \forall i \in S, & \quad (v_i) \\
&(19) \qquad & \lambda^1 \leq 1, & & \quad (w_0) \\
&(20) \qquad & -\lambda^1 d^1 + D^1 x^1 \leq 0_{m_1}, & & \quad (u^0) \\
&(21) \qquad & \lambda^1 d^2 - D^2 x^1 \leq d^2 - D^2 x, & & \quad (v^0) \\
&(22) \qquad & \lambda^1 \geq 0. & & \quad (t_0)
\end{aligned}
$$

Consider now relaxing constraints (20) and (21), i.e. replacing (20) and (21) by the following constraints:

$$
\begin{aligned}
&(23) \qquad & -\lambda^1 d^1 + D^1 x^1 - s 1_{m_1} \leq 0_{m_1}, & \quad (u^0) \\
&(24) \qquad & \lambda^1 d^2 - D^2 x^1 - s 1_{m_2} \leq d^2 - D^2 x, & \quad (v^0) \\
&(25) \qquad & s \geq 0. & \quad (t_1)
\end{aligned}
$$

Now, the problem of deciding whether or not a given vector $x \in \mathbb{R}^n$ belongs to $Conv(P(S) \cap F_D)$ can be decided by solving the following linear program, which will be called $P_{LP}(x, S)$ in the following:

$$
\begin{aligned}
&\max \ -s \\
&\text{s.t. } (17) - (19), (22) \text{ and } (23) - (25). \qquad\qquad (P_{LP}(x,S))
\end{aligned}
$$

Observe that $P_{LP}(x, S)$ is feasible if and only if $x \in P(S)$, and that $P_{LP}(x, S)$ is always bounded above by zero. Finally, note that $x \in Conv(P(S) \cap F_D)$ if and only if $P_{LP}(x, S)$ is feasible and bounded, and there exists an optimal solution in which the variable s has the value zero.

The other direction is proved with the aid of the problem $P_{LP}(x, S)$ and its dual $D_{LP}(x, S)$. Suppose $S \subseteq M$, satisfies $S \neq \emptyset$, $|S| \geq r(S) + 1$ and that $\bar{x} \in \cap_{i \in S} Conv(P(S \setminus \{\bar{i}\}) \cap F_D)$. Then

$\bar{x} \in \cap_{i\in S} P(S\setminus\{i\})$, and since $|S| \geq 2$, we have $\bar{x} \in P(S)$. Hence $P_{LP}(\bar{x}, S)$ is feasible and bounded if S satisfies $|S| \geq r(S)+1$ and $\bar{x} \in \cap_{i\in S} Conv(P(S\setminus\{\bar{i}\})\cap F_D)$. In the case where $P_{LP}(\bar{x}, S)$ is feasible and bounded, $(\bar{x}^1, \bar{\lambda}^1, \bar{s})$ denotes an optimal basic feasible solution to $P_{LP}(\bar{x}, S)$ and $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ denotes a corresponding optimal basic feasible solution to $D_{LP}(\bar{x}, S)$.

For $u^0 \geq 0_{m_1}$, $u_i \geq 0$ for $i \in S$ and $j \in N$, define the quantities $\alpha_j^1(S, u, u^0) := \sum_{i\in S} u_i a_{i,j} + (u^0)^T D^1_{.j}$ and $\beta^1(S, u, u^0) := \sum_{i\in S} u_i b_i + (u^0)^T d^1$, where $D^1_{.j}$ denotes the j^{th} column of D^1. The inequality $(\alpha^1(S, u, u^0))^T x \leq \beta^1(S, u, u^0)$ is valid for $\{x \in P(S) : D^1 x \leq d^1\}$. Similarly, for $v^0 \geq 0_{m_2}$, $v_i \geq 0$ for $i \in S$ and $j \in N$, defining the quantities $\alpha_j^2(S, v, v^0) := \sum_{i\in S} v_i a_{i,j} + (v^0)^T D^2_{.j}$ and $\beta^2(S, v, v^0) := \sum_{i\in S} v_i b_i + (v^0)^T d^2$, gives the inequality $(\alpha^2(S, v, v^0))^T x \leq \beta^2(S, v, v^0)$, which is valid for $\{x \in P(S) : D^2 x \leq d^2\}$. With these quantities, the dual $D_{LP}(\bar{x}, S)$ of $P_{LP}(\bar{x}, S)$ can be formulated as follows:

$$
\begin{array}{llrll}
 & \min & \beta^2(S, v, v^0) - (\alpha^2(S, v, v^0))^T \bar{x} + w_0 & & \\
 & \text{s.t.} & & & \\
(26) & & \alpha_j^1(S, u, u^0) - \alpha_j^2(S, v, v^0) = 0, & \forall j \in N, & (x_j^1) \\
(27) & & \beta^2(S, v, v^0) - \beta^1(S, u, u^0) + w_0 - t_0 = 0, & & (\lambda^1) \\
(28) & & 1_{m_1}^T u^0 + 1_{m_2}^T v^0 + t_1 = 1, & & (\text{s}) \\
(29) & & u^0 \geq 0_{m_1}, & & \\
(30) & & v^0 \geq 0_{m_2}, & & \\
(31) & & w_0, t_0, t_1 \geq 0, & & \\
(32) & & u_i, v_i \geq 0, & \forall i \in S. &
\end{array}
$$

We have the following:

Lemma 12 *Let $S \subseteq M$ be arbitrary. Suppose $\bar{x} \in P(S) \setminus Conv(P(S) \cap F_D)$. Then $\bar{u}^0 \neq 0_{m_1}$ and $\bar{v}^0 \neq 0_{m_2}$.*

The next lemma is essential for the proof of the converse direction of Theorem 4:

Lemma 13 *Let $S \subseteq M$, and let $T \subseteq S$ satisfy $|T| \geq 2$. Also, suppose $\bar{x} \in \cap_{i\in T} Conv(P(S \setminus \{i\}) \cap F_D)$ and $\bar{x} \notin Conv(P(S) \cap F_D)$. Then $D_{LP}(\bar{x}, S)$ is feasible and bounded. Furthermore, $\bar{u}_i > 0$ or $\bar{v}_i > 0$ for all $i \in T$.*

With the above lemmas, we are ready to prove the converse of Theorem 4:

Lemma 14 *Let $S \subseteq M$, and suppose that either $|S| \geq r(S)+2$ or $r(S) = n$ and $|S| = n+1$. Then*

$$
Conv(P(S) \cap F_D) \supseteq \bigcap_{i\in S} Conv(P(S \setminus \{i\}) \cap F_D). \tag{33}
$$

Now, we strengthen Theorem 4 for the case where $|S| \geq r(S) + 2$. Let $\bar{I}(S)$ be the set of constraints $i \in S$ for which $r(S \setminus \{i\}) = r(S)$, i.e. $\bar{I}(S) := \{i \in S : r(S) = r(S \setminus \{i\})\}$. We have:

Theorem 5 *Let $S \subseteq M$ satisfy $|S| \geq r(S)+2$. Then*

$$Conv(P(S) \cap F_D) = \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D). \tag{34}$$

Like in Theorem 4, and with the same proof, one direction of Theorem 5 is easy. For the converse, observe that $|S| \geq r(S)+2$ implies $|\bar{I}(S)| \geq 2$. It also implies $\cap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D) \subseteq P(S)$. From Lemma 13, we have:

Corollary 3 *Suppose $|S| \geq r(S)+2$, $\bar{x} \in \cap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D)$ and $\bar{x} \notin Conv(P(S) \cap F_D)$. Then $\bar{u}_i > 0$ or $\bar{v}_i > 0$ for all $i \in \bar{I}(S)$.*

We can now prove the converse of Theorem 5:

Lemma 15 *Let $S \subseteq M$ satisfy $|S| \geq r(S)+2$. Then:*

$$Conv(P(S) \cap F_D) \supseteq \bigcap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D). \tag{35}$$

Applying Theorem 5 iteratively, and Theorem 4 for sets of size $n+1$, we get the following:

Theorem 6 *Suppose $|M| = m \geq r+1$. Also, define $\mathcal{C}_1^* := \{S \subseteq M : |S| = r+1$ and $r(S) = r\}$ and $\mathcal{C}_2^* := \{S \subseteq M : |S| = r$ and $(r(S) = r \vee r(S) = r-1)\}$. We have,*

$$Conv(P \cap F_D) = \bigcap_{T \in C_1^*} Conv(P(T) \cap F_D). \tag{36}$$

Furthermore, if $r = n$,

$$Conv(P \cap F_D) = \bigcap_{T \in C_2^*} Conv(P(T) \cap F_D). \tag{37}$$

The example in figure 1 in appendix 2 demonstrates that the assumption $|S| \geq r(S)+2$ is necessary for (35) to hold. The example does not exclude, however, that (35) is true for sets S satisfying $|S| = r(S)+1$ for the special case of a split disjunction. In fact, the example suggests that it is also necessary for the disjunction to be a split disjunction. In the following, we will prove that (35) remains valid for $|S| = r(S)+1$ for the special case of a split disjunction.

Let $D(\pi, \pi_0)$ be a split disjunction. The problem $P_{LP}(\bar{x}, S)$, for a split disjunction $D(\pi, \pi_0)$, which will be called $P^S_{LP}(\bar{x}, S)$ in the following, is obtained from the problem $P_{LP}(\bar{x}, S)$ by replacing (23) and (24) with:

$$-\lambda^1 \pi_0 + \pi^T x^1 - s \leq 0, \qquad (u^0) \tag{38}$$

$$-\lambda^1(\pi_0+1) + \pi^T x^1 - s \leq -(\pi_0+1) + \pi^T \bar{x}. \qquad (v^0) \tag{39}$$

The dual of $P_{LP}^S(\bar{x}, S)$ is the problem $D_{LP}^S(\bar{x}, S)$ defined as:

$$
\begin{array}{lll}
\min & \sum_{i \in S} v_i(b_i - a_{i.}^T \bar{x}) + w_0 + v^0(\pi^T \bar{x} - (\pi_0 + 1)) & \\
\text{s.t.} & & \\
(40) & \sum_{i \in S} a_{i.}(u_i - v_i) + \pi(u_0 + v_0) = 0_n, & (x^1) \\
(41) & \sum_{i \in S} b_i(v_i - u_i) - \pi_0(u_0 + v_0) - v^0 + w_0 - t_0 = 0, & (\lambda^1) \\
(42) & u^0 + v^0 + t_1 = 1, & (s) \\
(43) & u^0, v^0, w_0, t_0, t_1 \geq 0, & \\
(44) & u_i, v_i \geq 0, \qquad \forall i \in S. &
\end{array}
$$

The solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0,\ \bar{t}_0, \bar{t}_1)$ to $D_{LP}^S(\bar{x}, S)$, for the case where $|S| = r(S) + 1$, can be characterized as follows (see also Lemma 2 in [3]):

Lemma 16 *Suppose $|S| = r(S)+1$, $\bar{x} \in \cap_{i \in \bar{I}(S)}$ Conv$(P(S \setminus \{i\}) \cap F_{D(\pi,\pi_0)})$ and $\bar{x} \notin$ Conv$(P(S) \cap F_{D(\pi,\pi_0)})$. Let $B_u := \{i \in S : u_i$ basic $\}$ and $B_v := \{i \in S : v_i$ basic $\}$ be the set of basic u's and v's respectively in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0,\ \bar{t}_0, \bar{t}_1)$ to $D_{LP}^S(\bar{x}, S)$. Then $B_u \cap B_v = \emptyset$, $r(S) = n$, $|B_u \cup B_v| = n$, and the vectors $a_{i.}$, $i \in B_u \cup B_v$, are linearly independent.*

From the above lemma, we immediately have the desired extension of Theorem 5 for the split disjunction:

Lemma 17 *Suppose $S \subseteq M$ satisfies $|S| = r(S) + 1$. Then:*

$$
(45) \qquad Conv(P(S) \cap F_{D(\pi,\pi_0)}) = \cap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_{D(\pi,\pi_0)}).
$$

From Theorem 5 and Lemma 17, we get the following:

Theorem 7

$$
(46) \qquad Conv(P \cap F_{D(\pi,\pi_0)}) = \cap_{T \in \mathcal{B}_r^*} Conv(P(T) \cap F_{D(\pi,\pi_0)}).
$$

By intersection over all possible split disjunctions, and interchanging intersections, we get Theorem 1 of section 3.

6 Conclusion

We have generalized a result of Balas and Perregaard in [3] to more general two-term disjunctions. We have showed that, given any two-term disjunction and a polyhedron in $\mathbb{R}^n$, every valid inequality for the set of points in the polyhedron, that satisfies the disjunction, is identical to, or dominated by, valid inequalities derived from the disjunction and $(r+1)$-subsets of the constraints describing the polyhedron, where r denotes the rank of the constraint matrix. In the special case where $r = n$, we show that it is enough to consider n subsets of the constraints. Furthermore, for the special case of a split disjunction, we demonstrated that it is sufficient to consider those r-subsets of the constraints that define a basis. It would be interesting to investigate whether or not a similar decomposition result holds for disjunctions with more than two terms. Another direction for future work is computational, based on the fact that the split closure can be obtained using only intersection cuts and the fact that intersection cuts have a closed form formula. In this vein, we are currently investigating how to obtain "strong" split cuts using this equivalence.

References

1. E. Balas. Intersection cuts - a new type of cutting planes for integer programming. *Operations Research*, 19:19–39, 1971.
2. E. Balas. Disjunctive programming. *Annals of Discrete Mathematics*, 5:3–51, 1979.
3. E. Balas and M. Perregaard. A precise correspondence between lift-and-project cuts, simple disjunctive cuts, and mixed integer gomory cuts for 0-1 programming. Technical Report MSRR-631, GSIA, Carnegie Mellon University, Pittsburgh, PA, USA, 2000. working paper.
4. W. Cook, R. Kannan, and A. Schrijver. Chvátal closures for mixed integer programs. *Mathematical Programming*, 47:155–174, 1990.

A Proofs

Proof of Lemma 1:

(i) Let $x \in \bar{x}(B)+L(B)$ satisfy $\pi^T x \notin]\pi_0, \pi_0+1[$. First, we prove that this implies $\bar{x}(B)+L(B) \subseteq conv(P(B) \cap F_{D(\pi,\pi_0)})$. Suppose $\tilde{x} \in (\bar{x}(B) + L(B)) \setminus conv(P(B) \cap F_{D(\pi,\pi_0)})$. The line through x and $\tilde{x}$ contains two points x^1 and x^2 in $F_{D(\pi,\pi_0)}$, such that $\tilde{x}$ is a convex combination of x^1 and x^2, which is a contradiction.

Now we prove $conv(P(B) \cap F_{D(\pi,\pi_0)}) = P(B)$. Clearly $conv(P(B) \cap F_{D(\pi,\pi_0)}) \subseteq P(B)$, so we only show $conv(P(B) \cap F_{D(\pi,\pi_0)}) \supseteq P(B)$. Therefore, let $\bar{y} \in P(B)$ be arbitrary. Since $\bar{y} \in P(B)$, we have $\bar{y} = x^0 + \sum_{i \in B} \bar{s}_i r^i(B)$, where $\bar{s}_i \geq 0$, $i \in B$, and $x^0 \in \bar{x}(B)+L(B)$. If $\bar{y}$ satisfies the disjunction $D(\pi, \pi_0)$, we are done, so assume $\pi^T \bar{y} \in]\pi_0, \pi_0 + 1[$. Furthermore, we can assume $x^0 \in F_{D(\pi,\pi_0)}$.

Wlog, assume $\pi^T x^0 \leq \pi_0$ (the proof for the case $\pi^T x^0 \geq \pi_0 + 1$ is symmetric to the proof we are about to present). Define $B^+ := \{i \in B : \pi^T r^i(B) > 0 \text{ and } \bar{s}_i > 0\}$. Since $\pi^T \bar{y} \in]\pi_0, \pi_0 + 1[$, we must have $B^+ \neq \emptyset$. Let $z^+ := \sum_{i \in B^+} \bar{s}_i r^i(B) \neq 0_n$ and $z^- := \sum_{i \in B \setminus B^+} \bar{s}_i r^i(B)$. By definition of the vectors $r^i(B)$, for $i \in B$, we have $a_{i.}^T z^+ \leq 0$ and $a_{i.}^T z^- \leq 0$ for all $i \in B$. Choose $\bar{\lambda} \in]0, 1]$ to satisfy $\pi^T(x^0 + z^- + \frac{1}{\bar{\lambda}} z^+) \geq \pi_0 + 1$ and let $y^1 := x^0 + z^- + \frac{1}{\bar{\lambda}} z^+$ and $y^2 := x^0 + z^-$. Then $\bar{y} = x^0 + z^- + z^+ = \bar{\lambda} y^1 + (1 - \bar{\lambda}) y^2$. By the choice of $\bar{\lambda}$, y^1 satisfies the disjunction $D(\pi, \pi_0)$. Furthermore, since $\pi^T(x^0 + z^-) \leq \pi_0$, y^2 satisfies the disjunction $D(\pi, \pi_0)$. Finally, $a_{i.}^T y^1 \leq b_i$ and $a_{i.}^T y^2 \leq b_i$ for all $i \in B$. Hence $\bar{y} \in conv(P(B) \cap F_{D(\pi,\pi_0)})$.

(ii) Since (3) is valid for $conv(P(B) \cap F_{D(\pi,\pi_0)})$, $conv(P(B) \cap F_{D(\pi,\pi_0)}) \subseteq \{x \in P(B) : (3)\}$. Therefore, let $\bar{y} \in \{x \in P(B) : (3)\}$ be arbitrary. We have $\bar{y} \in P(B)$. If $\bar{y} \in F_{D(\pi,\pi_0)}$, we are done, so suppose $\pi^T \bar{y} \in]\pi_0, \pi_0+1[$. Again, we may write $\bar{y} = x^0 + \sum_{i \in B} \bar{s}_i r^i(B)$, where $\bar{s}_i \geq 0$, $i \in B$, and $x^0 \in \bar{x}(B) + L(B)$. Observe that, by definition of the vectors $r^i(B)$ for $i \in B$, we have $b_i - a_{i.}^T \bar{y} = \bar{s}_i$ for all $i \in B$.

First assume $\sum_{i \in B}(b_i - a_{i.}^T \bar{y})/\alpha_i(\pi, B) = 1$, and let $\lambda_i := \bar{s}_i/\alpha_i(\pi, B)$ for $i \in B$. Then we may write $\bar{y} = x^0 + \sum_{i \in B} \bar{s}_i r^i(B) = \sum_{i \in B : \pi^T r^i(B) \neq 0} \lambda_i (x^0 + \alpha_i(\pi, B) r^i(B)) = \sum_{i \in B : \pi^T r^i(B) \neq 0} \lambda_i \tilde{x}^i(\alpha_i(\pi, B), B)$, where $\tilde{x}^i(\alpha, B) := x^0 + \alpha r^i(B)$. Since the vectors $\tilde{x}^i(\alpha_i(\pi, B), B)$, for i in B satisfying $\pi^T r^i(B) \neq 0$, satisfy the disjunction $D(\pi, \pi_0)$, this shows $\bar{y} \in conv(P(B) \cap F_{D(\pi,\pi_0)})$.

Now suppose $\sum_{i \in B}(b_i - a_{i.}^T \bar{y})/\alpha_i(\pi, B) > 1$. Let z be the point on the line between $\bar{y}$ and x^0 that satisfies (3) with equality. We have $\bar{y} = x^0 + \bar{\mu}(z - x^0)$, for some $\bar{\mu} > 1$, $z \in P(B)$ and z satisfies (3) with equality. By what we have just shown, we may write z as $z = \lambda^1 z^1 + \lambda^2 z^2$, where $z^1, z^2 \in P(B) \cap F_{D(\pi,\pi_0)}$, $\lambda^1, \lambda^2 \geq 0$ and $\lambda^1 + \lambda^2 = 1$. But then $\bar{y} = \lambda^1(x^0 + \bar{\mu}(z^1 - x^0)) + \lambda^2(x^0 + \bar{\mu}(z^2 - x^0))$, so that $\bar{y} \in conv(P(B) \cap F_{D(\pi,\pi_0)})$. □

Proof of Lemma 2:

The following notation will be convenient. Define $P_1 := P \cap \{x : \alpha^T x \leq \beta\}$ and $P_2 := P \cap \{x : \alpha^T x \geq \psi\}$. Furthermore, given a set $B \in \mathcal{B}_r^*$, let $P_1(B) := P(B) \cap \{x : \alpha^T x \leq \beta\}$ and $P_2(B) := P(B) \cap \{x : \alpha^T x \geq \psi\}$.

When $\alpha = 0_n$, at least one of P_1 and P_2 is empty. If both are empty, the lemma holds trivially with $\emptyset = \emptyset$. If one is not empty, $Conv(P_1 \cup P_2) = P$ and, similarly, $\bigcap_{B \in \mathcal{B}_r^*} Conv((P_1(B) \cup P_2(B)) = \bigcap_{B \in \mathcal{B}_r^*} P(B) = P$. The last equality is due to the assumption $a_{i.} \neq 0_n$ for $i \in M$. Therefore, we assume $\alpha \neq 0_n$ in the rest of the proof.

Because $P \subseteq P(B)$ for any $B \in \mathcal{B}_r^*$, it is clear that $Conv(P_1 \cup P_2) \subseteq \bigcap_{B \in \mathcal{B}_r^*} Conv((P_1(B) \cup P_2(B))$. Therefore, we only need to show the other direction of the inclusion.

Observe that it suffices to show that any valid inequality for $Conv(P_1 \cup P_2)$ is valid for $Conv(P_1(B) \cup P_2(B))$ for at least one $B \in \mathcal{B}_r^*$. Therefore, let $\delta^T x \leq \delta_0$ be a valid inequality for $Conv(P_1 \cup P_2)$. Clearly, we can assume $\delta^T x \leq \delta_0$ is facet defining for $Conv(P_1 \cup P_2)$. This is clearly true for the valid inequalities of P, since we can always choose a $B \in \mathcal{B}_r^*$, by applying the techniques of linear programming, such that the valid inequality of P is valid for $P(B)$. So we may assume that the inequality is valid for $Conv(P_1 \cup P_2)$ but not valid for P.

Case 1. $P_2 = \emptyset$.

Since $P_2 = \emptyset$, $Conv(P_1 \cup P_2) = P_1 = P \cap \{x : \alpha^T x \leq \beta\}$. Hence $\alpha^T x \leq \beta$ is a valid inequality for $Conv(P_1 \cup P_2)$. We just want to show that it is also valid for $Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$ for some $B \in \mathcal{B}_r^*$. Because $P_2 = \emptyset$ and P is full-dimensional, applying the techniques of linear programming shows that the value $\tilde{\gamma} = \max\{\gamma : P \cap \{x : \alpha^T x = \gamma\} \neq \emptyset\}$ specifies $B \in \mathcal{B}_r^*$ such that $P(B) \cap \{x : \alpha^T x \geq \psi\} = \emptyset$ and $\tilde{\gamma} = \max\{\gamma : P(B) \cap \{x : \alpha^T x = \gamma\} \neq \emptyset\}$, where $\tilde{\gamma} < \psi$. And then we see $Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\})) = P(B) \cap \{x : \alpha^T x \leq \beta\}$ for this particular B. Therefore, $\alpha^T x \leq \beta$ is valid for $Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$.

Case 2. P_1 and P_2 are full-dimensional.

Consider an arbitrary facet F of $Conv(P_1 \cup P_2)$ which does not induce a valid inequality for P. We also consider the corresponding F-defining inequality (half-space) and F-defining equality (hyperplane). We will show that the F-defining inequality is valid for $Conv\ (P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$ for some $B \in \mathcal{B}_r^*$.

Let $F_1 := F \cap P_1$ and $F_2 := F \cap P_2$. Since the F-defining inequality is valid for P_1 and P_2 but not valid for P, we can deduce $F_1 \subseteq \{x \in \mathbb{R}^n | \alpha^T x = \beta\}$ and $F_2 \subseteq \{x \in \mathbb{R}^n : \alpha^T x = \psi\}$. F is the convex combination of F_1 and F_2, where F_1 is a k-dimensional face of P_1 and F_2 is an m-dimensional face of P_2. Since F has dimension $n-1$, we know $0 \leq k \leq n-2$ and $n-k-2 \leq m \leq n-2$.

Because the F-defining hyperplane intersection $\alpha^T x = \beta$ (or $\alpha^T x = \psi$) defines a $(n-2)$-dimensional affine subspace and F, as a $(n-1)$-dimensional facet, is a convex combination of F_1 and F_2, $Aff(F_2)$ contains two affine subspaces S_1 and S_2 of dimensions $n-k-2$ and $m-(n-k-2)$ respectively, where S_1 is orthogonal $Aff(F_1)$ and S_2 is parallel to $Aff(F_1)$. In other words, $Aff(F_2)$ can be orthogonally decomposed into two affine subspaces S_1 and S_2 such that $S_1 \cap S_2$ has an unique point x_0, $(x_1 - x_0)^T(x_2 - x_0) = 0$ for any $x_1 \in S_1$ and any $x_2 \in S_2$, and for some $x_3 \in Aff(F_1)$ we have $\{x_i - x_0 : x_i \in S_2\} \subseteq \{x_i - x_3 : x_i \in Aff(F_1)\}$ and $(x_1 - x_0)^T(x_4 - x_3) = 0$ for any $x_1 \in S_1$ and any $x_4 \in Aff(F_1)$.

There exist $n-k-1$ constraints of P such that the corresponding hyperplanes, together with $\alpha^T x = \beta$, define $Aff(F_1)$. Let these hyperplanes be $A_{(n-k-1)\times n} x = b_1$. Similarly, there are $n-m-1$ constraints of P such that the corresponding hyperplanes, together with $\alpha^T x = \psi$, define $Aff(F_2)$. Let them be $A_{(n-m-1)\times n} x = b_2$. From the discussion in the previous paragraphs, one can easily see that the equations $A_{(n-k-1)\times n} x = 0$, $A_{(n-m-1)\times n} x = 0$ have solution space $\{x_i - x_0 : x_i \in S_2\}$ with dimension $m-(n-k-2)$. Since $m-(n-k-2) \equiv n - [(n-m-1)+(n-k-1)]$, the matrix $\binom{A_{(n-k-1)\times n}}{A_{(n-m-1)\times n}}$ is full row-rank. Because the rank of A is r, $(n-k-1)+(n-m-1) \leq r$. This allows us to choose another $r-(n-k-1)-(n-m-1)$ constraints $A_{[r-(n-k-1)-(n-m-1)]\times n} x \leq b_3$ from $Ax \leq b$, together with $A_{(n-k-1)\times n} x \leq b_1$ and $A_{(n-m-1)\times n} x \leq b_2$, to construct $B \in \mathcal{B}_r^*$, and the F-defining inequality is valid for $Conv(P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))$.

Case 3. $P_1 \neq \emptyset$, $P_2 \neq \emptyset$, and some of P_1 and P_2 is not full-dimensional.

Instead of considering two inequalities $\alpha^T x \leq \beta$ and $\alpha^T x \geq \psi$, we construct two inequalities $\alpha^T x \leq \beta+\epsilon$ and $\alpha^T x \geq \psi-\epsilon$, where ϵ is an arbitrarily small positive number satisfying $\beta+\epsilon < \psi-\epsilon$. Let $P_1^\epsilon := P \cap \{x : \alpha^T x \leq \beta + \epsilon\}$ and $P_2^\epsilon := P \cap \{x : \alpha^T x \geq \psi - \epsilon\}$. Since $P_1 \neq \emptyset$ and $P_2 \neq \emptyset$, P_1^ϵ and P_2^ϵ are full-dimensional polyhedra.

Because $|\mathcal{B}_r^*|$ is finite and B and P are closed sets in $\mathbb{R}^n$, by the definition of *Conv* operation and the proved result in Case 2, we have

$$\begin{aligned}
&\bigcap_{B\in\mathcal{B}_r^*} Conv((P(B) \cap \{x : \alpha^T x \leq \beta\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi\}))\\
&= \bigcap_{B\in\mathcal{B}_r^*} lim_{\epsilon\to 0^+} Conv((P(B) \cap \{x : \alpha^T x \leq \beta+\epsilon\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi-\epsilon\}))\\
&= lim_{\epsilon\to 0^+} \bigcap_{B\in\mathcal{B}_r^*} Conv((P(B) \cap \{x : \alpha^T x \leq \beta+\epsilon\}) \cup (P(B) \cap \{x : \alpha^T x \geq \psi-\epsilon\}))\\
&= lim_{\epsilon\to 0^+} Conv((P \cap \{x : \alpha^T x \leq \beta+\epsilon\}) \cup (P \cap \{x : \alpha^T x \geq \psi-\epsilon\}))\\
&= Conv((P \cap \{x : \alpha^T x \leq \beta\}) \cup (P \cap \{x : \alpha^T x \geq \psi\})).
\end{aligned}$$

□

Proof of Lemma 3:

There always exist $\bar{S} \subseteq M$ and $\bar{i} \in M$, where $\bar{i} \notin \bar{S}$, such that $\bar{S}$ contains $\bar{B} \in \mathcal{B}_r^*$ and $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{S}\} \neq \emptyset$ and $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{S} \cup \{\bar{i}\}\} = \emptyset$. Actually, $\bar{S}$ and $\bar{i}$ can be chosen by an iterative procedure. In iteration k ($k \geq 0$), an $i_k \in M_k \subseteq M$ is chosen to ensure $M_k \backslash \{i_k\}$ contains $B \in \mathcal{B}_r^*$, where $M_0 := M$. If $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in M_k \backslash \{i_k\}\} \neq \emptyset$, then $\bar{S} := M_k \backslash \{i_k\}$ and $\bar{i} := i_k$. Otherwise, let $M_{k+1} := M_k \backslash \{i_k\}$ and proceed until finally we obtain $\bar{S}$ and $\bar{i}$. The fact of $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in B\} \neq \emptyset$ for any $B \in \mathcal{B}_r^*$ ensures the availability of $\bar{S}$ and $\bar{i}$.

By applying the techniques of linear programming, we see that $\bar{S}$ contains $\bar{B}^* \in \mathcal{B}_r^*$ such that $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^* \cup \{\bar{i}\}\} = \emptyset$. It is possible to choose $\tilde{i} \in \bar{B}^*$ such that $\tilde{B}^* := (\bar{B}^* \backslash \{\tilde{i}\} \cup \{\bar{i}\} \in \mathcal{B}_r^*$. Then $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^*\} \cap \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \tilde{B}^*\} = \emptyset$.

Because $P = \emptyset$, $\mathrm{SC} = \emptyset$. $\mathrm{SC}(\bar{B}^*) \cap \mathrm{SC}(\tilde{B}^*) = \emptyset$ follows $\mathrm{SC}(\bar{B}^*) \subseteq \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^*\}$, $\mathrm{SC}(\tilde{B}^*) \subseteq \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \tilde{B}^*\}$ and $\{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \bar{B}^*\} \cap \{x \in \mathbb{R}^n : a_{i.}^T x \leq b_i,\ i \in \tilde{B}^*\} = \emptyset$. Therefore, $\mathrm{SC} = \emptyset = \mathrm{SC}(\bar{B}^*) \cap \mathrm{SC}(\tilde{B}^*) = \bigcap_{B\in\mathcal{B}_r^*} \mathrm{SC}(B)$. □

Proof of Lemma 4:

The proof of (i), (ii) and (iv) is trivial. Next we prove (iii).

Because of (i), we have $|M^=| \geq n-l+1$. Choose $i' \in M^=$ such that $(M^= \backslash \{i'\}) \cap \mathcal{B}_{n-l}^* \neq \emptyset$. We know that (ii) is true for $i^* = i'$. Since $(M^= \backslash \{i'\}) \cap \mathcal{B}_{n-l}^* \neq \emptyset$, the optimal dual solution of $\min\{a_{i'}^T x : a_{i.}^T \leq b_i,\ i \in M^= \backslash \{i'\}\}$ specifies a $B' \in (M^= \backslash \{i'\}) \cap \mathcal{B}_{n-l}^*$ such that $\min\{a_{i'}^T x : a_{i.}^T \leq b_i,\ i \in B'\} = b_{i'}$ with optimal solution set $\mathit{Aff}(P)$. Therefore, $\mathit{Aff}(P) = \{x \in \mathbb{R}^n : a_{i.}^T \leq b_i,\ i \in B' \cup \{i'\}\}$. □

Proof of Lemma 6:

P is full-dimensional in $\mathit{Aff}(P)$. If there exists $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $\mathit{Aff}(P)$ is between the hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$ and $\mathit{Aff}(P)$ does not intersect them, then Lemma 6 is trivially true with $\emptyset = \emptyset$. Otherwise, we only need to consider the $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ satisfying that both hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$ intersect $\mathit{Aff}(P)$ and neither of them contains $\mathit{Aff}(P)$. Denote the set of these (π, π_0) by S_A.

Now we have $\bigcap_{(\pi,\pi_0)\in\mathbb{Z}^{n+1}} Conv(P \cap F_{D(\pi,\pi_0)}) = \bigcap_{(\pi,\pi_0)\in S_A} Conv((P \cap \{x \in \mathit{Aff}(P) : \pi^T x \leq \pi_0\}) \cup (P \cap \{x \in \mathit{Aff}(P) : \pi^T x \geq \pi_0 + 1\}))$. Applying Corollary 1 to the affine subspace $\mathit{Aff}(P)$, we see that the latter is equal to $\bigcap_{B\in\mathcal{B}_\gamma^A} \bigcap_{(\pi,\pi_0)\in S_A} Conv(((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) :$

$\pi^T x \leq \pi_0\}) \cup ((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) : \pi^T x \geq \pi_0 + 1\}))$. By Lemma 4(iv) and Lemma 5, for any $P(B)$, where $B \in \mathcal{B}_\gamma^A$, there always exist $P(\tilde{B}')$ and $P(\tilde{B}'')$, where $\tilde{B}' \in \mathcal{B'}_r^*$ and $\tilde{B}'' \in \mathcal{B''}_r^*$, such that $\tilde{B}' = B' \cup B$, $\tilde{B}'' = B'' \cup B$ and $P(B) \cap \mathit{Aff}(P) = P(\tilde{B}') \cap P(\tilde{B}'')$. Therefore, $\bigcap_{B \in \mathcal{B}_\gamma^A} \bigcap_{(\pi,\pi_0) \in S_A} \mathit{Conv}(((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) : \pi^T x \leq \pi_0\}) \cup ((P(B) \cap \mathit{Aff}(P)) \cap \{x \in \mathit{Aff}(P) : \pi^T x \geq \pi_0 + 1\})) \supseteq \bigcap_{B \in \mathcal{B}_r^*} \bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{n+1}} \mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$, which implies $\bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{n+1}} \mathit{Conv}(P \cap F_{D(\pi,\pi_0)}) \supseteq \bigcap_{B \in \mathcal{B}_r^*} \bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{n+1}} \mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$. Because $P \subseteq P(B)$ for any $B \in \mathcal{B}_r^*$, it is easy to obtain $\bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{n+1}} \mathit{Conv}(P \cap F_{D(\pi,\pi_0)}) \subseteq \bigcap_{B \in \mathcal{B}_r^*} \bigcap_{(\pi,\pi_0) \in \mathbb{Z}^{n+1}} \mathit{Conv}(P(B) \cap F_{D(\pi,\pi_0)})$. The lemma is proved. □

Proof of Lemma 7:

Suppose that the disjunction $\pi^T x \leq \pi_0$ and $\pi^T x \geq \pi_0 + 1$ induces a split cut that is not valid for some part of $\tilde{C}$. Then $\tilde{x}$ violates this split cut as well. So we know $\pi_0 < \pi^T \tilde{x} < \pi_0 + 1$, i.e. the point $\tilde{x}$ is between two hyperplanes $\pi^T x = \pi_0$ and $\pi^T x = \pi_0 + 1$.

Choose an extreme ray generated by vector $\tilde{r}_i$ and assume that the hyperplane $\pi^T x = \pi_0$ intersects the extreme ray at $\tilde{x} := \tilde{x} + \tilde{\alpha}_i \tilde{r}_i$, where $\tilde{\alpha}_i > 0$ ($\tilde{\alpha}_i = +\infty$ is allowed). Then $\tilde{\alpha}_i = \frac{\epsilon}{-\pi^T \tilde{r}_i}$ can be easily calculated, where $\pi^T \tilde{r}_i \leq 0$, $\epsilon := \pi^T \tilde{x} - \pi_0$ and $0 < \epsilon < 1$.

We claim that $\tilde{\alpha}_i$ is either $+\infty$ or bounded above by 1. When $\pi^T \tilde{r}_i = 0$, $\tilde{\alpha}_i = +\infty$, in which case the hyperplane $\pi^T x = \pi_0$ is parallel to the vector $\tilde{r}_i$. When $\pi^T \tilde{r}_i < 0$, $0 < \tilde{\alpha}_i < +\infty$, which means that the hyperplane intersects the ray at some finite point. In this case, because π and $\tilde{r}_i$ are in $\mathbb{Z}^n$, we know $-\pi \tilde{r}_i \geq 1$. Hence, $\tilde{\alpha}_i = \frac{\epsilon}{-\pi^T \tilde{r}_i} \leq \epsilon < 1$.

Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^T \in Q^n$. Let g be the least common multiple of all the denominators of $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n$. Then, using the fact that $\tilde{\alpha}_i = \frac{\epsilon}{-\pi^T \tilde{r}_i}$, it follows that $\tilde{\alpha}_i$ can be expressed as $\frac{p}{wg}$, where $p, w \in \mathbb{Z}_+$ and $0 < p < g$.

By the following claim, we actually prove the lemma.

Claim. There are only finite undominated split cuts for cone $\tilde{C}$.

Proof of claim. By induction on q, the number of extreme rays of $\tilde{C}$.

When $q = 1$, $\tilde{C} = \{x \in \mathbb{R}^n : x = \tilde{x} + \alpha_1 \tilde{r}_1,\ \alpha_1 \geq 0\}$. Because $\tilde{\alpha}_1$ is either $+\infty$ or bounded above by 1, if $\tilde{\alpha}_1$ reaches maximum by intersecting some hyperplane $\pi^* x = \pi_0^*$, then the split cut $\pi^* x \leq \pi_0^*$ would be an undominated split cut of $\tilde{C}$. Note that the maximum value of $\tilde{\alpha}_1$ is reachable, because $\tilde{\alpha}_1$ has the form of $\frac{p}{wg}$ as mentioned above.

Assume that the claim is true for $q = k < n$. Let us consider the case of $q = k + 1$.

Let $\tilde{C}_i := \{x \in \mathbb{R}^n : x = \tilde{x} + \sum_{j \neq i} \alpha_j \tilde{r}_j,\ \alpha_j \geq 0,\ 1 \leq j \leq k+1,\ j \neq i\}$, where $1 \leq i \leq k+1$. Each $\tilde{C}_i$ is a polyhedral cone with apex $\tilde{x}$ and k linearly independent extreme rays. By induction hypothesis, there are only finite undominated split cuts for each cone $\tilde{C}_i$. Among those points obtained by intersecting the undominated split cuts for $\tilde{C}_i$ with the extreme ray generated by $\tilde{r}_i$, let z_i be the closest point to $\tilde{x}$.

Now we claim that any undominated split cut of $\tilde{C}$ cannot intersect the extreme ray generated by $\tilde{r}_i$ $(1 \leq i \leq k+1)$ at a point which is closer to $\tilde{x}$ than z_i. Otherwise, let us assume that there is an undominated split cut $\mathcal{H}$ of $\tilde{C}$ which intersects the extreme ray generated by $\tilde{r}_i$ at a point $\tilde{z}_i$ that is between $\tilde{x}$ and z_i. By the definition of z_i, the cut (when restricted to $\tilde{C}_i$) must be dominated by a cut of $\tilde{C}_i$, say $\mathcal{H}'$. Wlog, assume that the cut $\mathcal{H}'$ is an undominated cut for $\tilde{C}_i$. So $\mathcal{H}'$ must intersect the extreme ray generated by $\tilde{r}_i$ at z_i' that is not between $\tilde{x}$ and z_i. But now $\mathcal{H}'$ dominates $\mathcal{H}$, a contradiction to the choice of $\mathcal{H}$.

We know that the intersection point of any undominated split cut with the extreme ray of $\tilde{r}_i$ ($1 \leq i \leq k+1$) is either infinity or between z_i and $\tilde{x} + \tilde{r}_i$. Since $\tilde{\alpha}_i = \frac{p}{wg}$, there are only finite points between z_i and $\tilde{x} + \tilde{r}_i$ that could be the intersections of the split cuts with the extreme ray. Therefore, we see that the claim is true when $q = k+1$. □

Proof of Lemma 11:
Clearly $P(S) \subseteq P(S \setminus \{i\})$ for all $i \in S$. Intersecting with F_D on both sides gives $P(S) \cap F_D \subseteq P(S \setminus \{i\}) \cap F_D$ for all $i \in S$. Convexifying both sides results in $Conv(P(S) \cap F_D) \subseteq Conv(P(S \setminus \{i\}) \cap F_D)$ for all $i \in S$, and finally, since this holds for all $i \in S$, the result follows. □

Proof of Lemma 12:
Let $\bar{x}$ as stated, and suppose firstly that $\bar{v}^0 = 0_{m_2}$. The inequality $(\alpha^2(S, \bar{v}, 0_{m_2}))^T x \leq \beta^2(S, \bar{v}, 0_{m_2})$ is valid for $P(S)$. However, since the optimal objective value to $D_{LP}(\bar{x}, S)$ is negative, and $\bar{w}_0 \geq 0$, we have $\beta^2(S, \bar{v}, 0_{m_2}) -\ (\alpha^2(S, \bar{v}, 0_{m_2}))^T \bar{x} < 0$, which contradicts the assumption that $\bar{x} \in P(S)$.

Now suppose $\bar{u}_0 = 0_{m_1}$. The inequality $(\alpha^1(S, \bar{u}, 0_{m_1}))^T x \leq \beta^1(S, \bar{u}, 0_{m_1})$ is valid for $P(S)$, but $\beta^1(S, \bar{u}, 0_{m_1}) -\ (\alpha^1(S, \bar{u}, 0_{m_1}))^T \bar{x} \leq \beta^2(S, \bar{v}, \bar{v}^0) + w_0 -\ (\alpha^2(S, \bar{v}, \bar{v}^0))^T \bar{x} < 0$, where we have used (20) and (21). This contradicts the assumption that $\bar{x} \in P(S)$. □

Proof of Lemma 13: Let S, T and $\bar{x}$ be as stated. The fact that $D_{LP}(\bar{x}, S)$ is feasible and bounded follows from the facts $\bar{x} \in \cap_{i \in T} Conv(P(S \setminus \{i\}) \subseteq P(S)$ and $|T| \geq 2$.

Now, suppose $\bar{u}_{i'} = 0$ and $\bar{v}_{i'} = 0$ for some $i' \in T$. Then the problem $(D'_{LP}(\bar{x}, S \setminus \{i'\}))$, obtained from $D_{LP}(\bar{x}, S)$ by eliminating $u_{i'}$, $v_{i'}$ and the normalization constraint:

$$
\begin{array}{lrll}
\min & \beta^2(S \setminus \{i'\}, v, v^0) - (\alpha^2(S \setminus \{i'\}, v, v^0))^T \bar{x} + w_0 & & \\
\text{s.t.} & & & \\
& \alpha_j^1(S \setminus \{i'\}, u, u^0) - \alpha_j^2(S \setminus \{i'\}, v, v^0) = 0, & \forall j \in N, & (x_j^1) \\
& \beta^2(S \setminus \{i'\}, v, v^0) - \beta^1(S \setminus \{i'\}, u, u^0) + w_0 - t_0 = 0, & & (\lambda^1) \\
& u^0 \geq 0_{m_1}, & & \\
& v^0 \geq 0_{m_2}, & & \\
& w_0, t_0 \geq 0, & & \\
& u_i, v_i \geq 0, & \forall i \in S \setminus \{i'\}, &
\end{array}
$$

is unbounded (since $\bar{x} \notin Conv(P(S) \cap F_D)$). This means that the dual of $D'_{LP}(\bar{x}, S \setminus \{i'\})$, the problem $P'_{LP}(\bar{x}, S \setminus \{i'\})$, is infeasible:

$$
\begin{array}{lll}
\max & 0_n^T x^1 + 0\lambda^1 & \\
\text{s.t.} & & \\
& -\lambda^1 b_i + a_{i.}^T x^1 \leq 0, \qquad \forall i \in S \setminus \{i'\}, & (u_i) \\
& \lambda^1 b_i - a_{i.}^T x^1 \leq b_i - a_{i.}^T \bar{x}, \qquad \forall i \in S \setminus \{i'\}, & (v_i) \\
& \lambda^1 \leq 1, & (w_0) \\
& -\lambda^1 d^1 + D^1 x^1 \leq 0_{m_1}, & (u^0) \\
& \lambda^1 d^2 - D^2 x^1 \leq d^2 - D^2 \bar{x}, & (v^0) \\
& \lambda^1 \geq 0. & (t_0)
\end{array}
$$

However, these constraints are the conditions that must be satisfied for $\bar{x}$ to be in $Conv(P(S \setminus \{i'\}) \cap F_D)$, which is a contradiction. $\square$

Proof of Lemma 14:
Let $\bar{x} \in \cap_{i \in S} Conv(P(S \setminus \{i\}) \cap F_D)$, and suppose $\bar{x} \notin Conv(P(S) \cap F_D)$. Define $B_u := \{i \in S : u_i \text{ basic }\}$ and $B_v := \{i \in S : v_i \text{ basic }\}$ to be the set of basic u's and v's respectively in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}(\bar{x}, S)$. From Lemma 13 and the fact that a variable with positive value, that does not have an upper bound, is basic, it follows that $(B_u \cup B_v) = S$.

The feasible set for the problem $D_{LP}(\bar{x}, S)$ is of the form $\{y \in \mathbb{R}^{n'} : Wy = w_0, y \geq 0_{n'}\}$, where W and w_0 are of suitable dimensions. The column of W corresponding to the variable u_i, $i \in S$, is given by $[a_{i.}^T, -b_i, 0]^T$. Similarly, the column of W corresponding to the variable v_i, $i \in S$, is given by $[-a_{i.}^T, b_i, 0]^T$. Since for all $i \in S$, either u_i or v_i is basic, the vectors $[a_{i.}^T, -b_i]^T$, $i \in S$, are linearly independent. Clearly, there can be at most $r(S) + 1$ of these. Hence $|S| = r(S) + 1$. This excludes the case $|S| \geq r(S) + 2$, so we must have $r(S) = n$ and $|S| = n + 1$.

The number of basic variables in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}(\bar{x}, S)$ is at most $n + 2$, since the number of basic variables is bounded by the number of equality constraints in $D_{LP}(\bar{x}, S)$. The number of basic variables among the variables u_i and v_i, $i \in S$, is $|S| = n + 1$. However, according to Lemma 12, at least two of the variables in $(\bar{u}^0, \bar{v}^0)$ are basic, which gives a total of $n + 3$ basic variables - contradiction. $\square$

Proof of Lemma 15:
Let $\bar{x} \in \cap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_D)$, and suppose $\bar{x} \notin Conv(P(S) \cap F_D)$. Observe that it suffices to prove that the vectors $[a_{i.}^T, -b_i]$, $i \in S$, are linearly independent, since that would contradict $|S| \geq r(S) + 2$. Suppose they are linearly dependent, and let $\bar{i} \in S$ satisfy $[a_{\bar{i}.}^T, -b_{\bar{i}}] = \sum_{i \in S \setminus \{i\}} \mu_i [a_{i.}^T, -b_i]$, where μ_i, for $i \in S \setminus \{i\}$, are scalars, not all zero. Suppose firstly that $\bar{i} \in S \setminus \bar{I}(S)$. Then we have $a_{\bar{i}.} \in span(\{a_{i.} : i \in S \setminus \{\bar{i}\})$. However, that implies $r(S) = r(S \setminus \{\bar{i}\})$, which is a contradiction. Hence, we must have $\bar{i} \in \bar{I}(S)$ and $\mu_i = 0$ for all $i \in S \setminus \bar{I}(S)$. However, according to Corollary 3, the vectors $[a_{i.}^T, -b_i]$, $i \in \bar{I}(S)$, are linearly independent. $\square$

Proof of Lemma 16:
As mentioned earlier, the feasible set for $D_{LP}^S(\bar{x}, S)$ can be written as $\{y \in \mathbb{R}^{n'} : Wy = z_0, y \geq 0_{n'}\}$, where W and z_0 are of suitable dimensions.

We first argue that the variables w_0, t_0 and t_1 are non-basic in the solution $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ to $D_{LP}^S(\bar{x}, S)$. t_1 is clearly non-basic, since s is basic in $P_{LP}^S(\bar{x}, S)$. From Lemma 12 it follows that both u^0 and v^0 are basic. The column corresponding to u^0 is $[\pi, -\pi_0, 1]$ and the column corresponding to v^0 is $[\pi, -(\pi_0+1), 1]$. Deducting the column corresponding to v_0 from the column corresponding to u_0 gives e_{n+1}, i.e. the $(n+1)^{th}$ unit vector in $\mathbb{R}^{n+2}$. Since this is exactly the column corresponding to w_0 and $-t_0$, and since basic columns must be linearly independent, it follows that w_0 and t_0 are both non-basic.

As argued earlier, not both v_i and u_i, $i \in S$, can be in the basis, since their corresponding columns in W are multiples of each other. Hence $B_u \cap B_v = \emptyset$. Now, since $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ is a basic solution to $D^S_{LP}(\bar{x}, S)$, the solution to the system:

$$
\begin{aligned}
&(47) \qquad && \textstyle\sum_{i \in B_u} a_{i.} u_i - \sum_{i \in B_v} a_{i.} v_i + \pi &&= 0_n \\
&(48) \qquad && \textstyle\sum_{i \in B_v} b_i v_i - \sum_{i \in B_u} b_i u_i - \pi_0 - v^0 &&= 0 \\
&(49) \qquad && u_0 + v_0 &&= 1
\end{aligned}
$$

is unique. The system (47)-(49) is of the form $W^B y = z_0^B$. The number of rows of W^B (and z_0^B) is $n+2$, and the number of columns is $|B_u \cup B_v| + 2$. All columns of W^B are linearly independent. If $|B_u \cup B_v| + 2 < n+2$, multiple solutions would exist. Hence, we must have $|B_u \cup B_v| = n$.

Now suppose the vectors $a_{i.}$, $i \in B_u \cup B_v$, are linearly dependent. Then there exists a non-zero solution (u^*, v^*) to the system $\sum_{i \in B_u} a_{i.} u_i^* - \sum_{i \in B_v} a_{i.} v_i^* = 0_n$. Define the scalars $u_i(\delta) := \bar{u}_i + \delta u_i^*$ for $i \in B_u$ and $v_i(\delta) := \bar{v}_i + \delta v_i^*$ for $i \in B_v$, where $\delta \in \mathbb{R}$. We have that $(u(\delta), v(\delta), u_0, v_0)$ satisfies (47)-(49) if and only if $u_0 + v_0 = 1$ and $\bar{v}_0 + \delta(\sum_{i \in B_v} b_i v_i^* - \sum_{i \in B_u} b_i u_i^*) - v_0 = 0$. Defining $v_0(\delta) := \bar{v}_0 + \delta(\sum_{i \in B_v} b_i v_i^* - \sum_{i \in B_u} b_i u_i^*)$ and $u_0(\delta) := 1 - v_0(\delta)$, we have that $(u(\delta), v(\delta), u_0(\delta), v_0(\delta))$ satisfies (47)-(49). Since u_i^* for $i \in B_u$ and v_i^* for $i \in B_v$ are not all zero, and none of the vectors $a_{i.}$, $i \in M$, are zero vectors, there must exist $\delta^* \in \mathbb{R}$ such that $(u(\delta^*), v(\delta^*), u_0(\delta^*), v_0(\delta^*))$ is a different solution to (47)-(49) than $(\bar{u}, \bar{v}, \bar{u}_0, \bar{v}_0)$ - contradiction. $\square$

Proof of Lemma 17:
Suppose $\bar{x} \in \cap_{i \in \bar{I}(S)} Conv(P(S \setminus \{i\}) \cap F_{D(\pi,\pi_0)})$ and $\bar{x} \notin Conv(P(S) \cap F_{D(\pi,\pi_0)})$. Let B_u and B_v and $(\bar{u}, \bar{v}, \bar{u}^0, \bar{v}^0, \bar{w}_0, \bar{t}_0, \bar{t}_1)$ be as in Lemma 6. We have $B_u \cap B_v = \emptyset$, $r(S) = n$, $|B_u \cup B_v| = n$, and the vectors $a_{i.}$, $i \in B_u \cup B_v$, are linearly independent. Let $\{\bar{i}\} = S \setminus (B_u \cup B_v)$. We can not have $\bar{i} \in \bar{I}(S)$, since by Corollary 1, that would imply $\bar{u}_{\bar{i}} > 0$ or $\bar{v}_{\bar{i}} > 0$, which contradicts $\bar{i} \notin B_u \cup B_v$. Hence, we must have $\bar{i} \in S \setminus \bar{I}(S)$. But that means that $\bar{i}$ is in every basis of S, which contradicts $\bar{i} \notin B_u \cup B_v$. $\square$

B Example of two-term disjunction

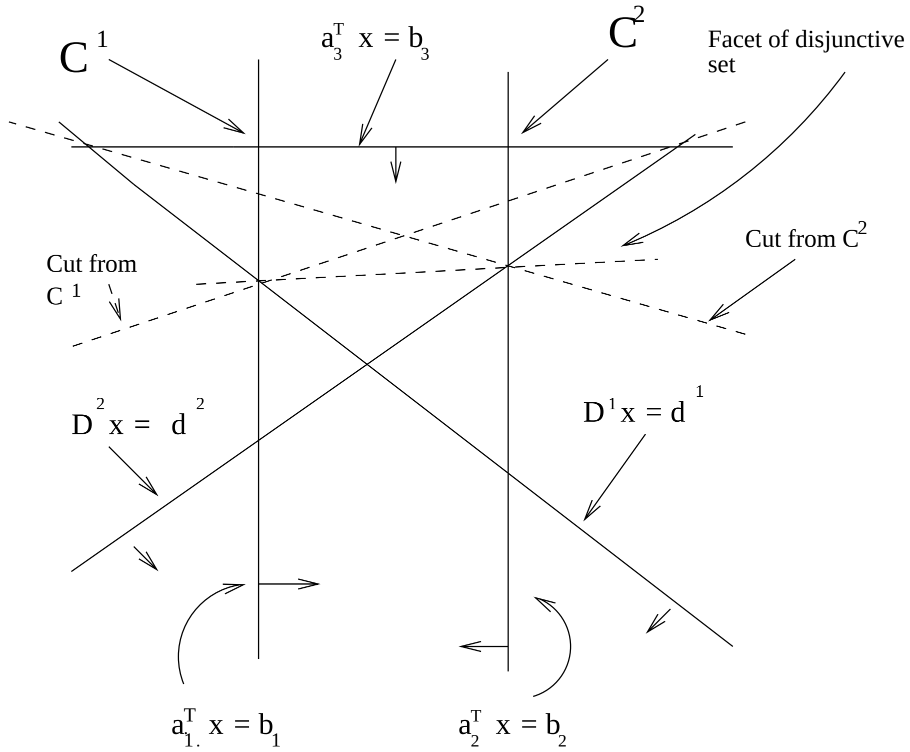

Fig. 1. Example of two-term disjunction

In this example, P has 3 constraints $a_{i.}^T x \leq b_i$, $i = 1,2,3$, and D is a two-term disjunction involving the 2 constraints $D^1 x \leq d^1$ and $D^2 x \leq d^2$. C^1 denotes the cone defined by $a_{1.}^T x \leq b_1$ and $a_{3.}^T x \leq b_3$, and C^2 denotes the cone defined by $a_{2.}^T x \leq b_2$ and $a_{3.}^T x \leq b_3$. In this example, $Conv(P(M) \cap F_D) \neq \cap_{i \in \bar{I}(M)} Conv(P(M \setminus \{i\}) \cap F_D)$. Observe, however, that $Conv(P(S) \cap F_D)$ can be generated by considering the n-subsets of the constraints (n=2). In fact, only one 2-subset need to considered (the set containing 1 and 2).