10-704 Information Processing & Learning

Quiz 1 *March 5th 2015*

Name: : _____

Andrew Id: : _____

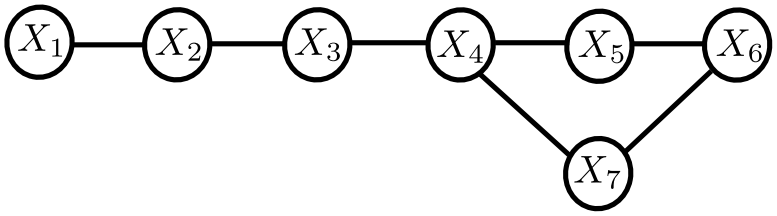
Department: : _____

Guidelines

1. **Please don't turn this page until instructed.**
2. Write your name, Andrew Id and department in the space provided above.
3. You have sixty **(60)** minutes for this exam.
4. This exam has **six (6)** pages on six sheets of paper including this page.
5. This exam has 2 Sections. The first section has 10 short questions and the second section has 3 long questions. The number of points allocated for each question is indicated next to the question. The total number of points is 100.
6. This exam is **open book**. You may use any material such as cheat sheets, class notes etc. No electronic devices are permitted.
7. The questions vary in difficulty. The points allocated per question **do not** entirely reflect the level of difficulty. Do not spend too much time on one question.
8. If any question is unclear, you may write your own interpretation and answer the question.
9. The questions appear only on one side of the paper. You may use the other side for rough work. If you still need extra paper, please ask one of the instructors.

Short Questions (50 Points)

Data Processing Inequality. Consider the Markov Random Field (Undirected Graphical Model) shown below.



Which of the statements are always true of the above graphical model?

1. **(5 Points)** $H(X_1|X_2) \leq H(X_1|X_3)$

2. **(5 Points)** $I(X_3; X_4) \geq I(X_3; X_5)$

3. **(5 Points)** $I(X_4; X_5) \geq I(X_4; X_6)$

Huffman Codes.

4. **(5 pts)** Construct a binary Huffman code for the following distribution on 5 symbols $p = (0.4, 0.25, 0.25, 0.05, 0.05)$. What is the expected length of this code ?

5. **(5 pts)** The above code does not meet the entropy lower bound $H(p) \approx 1.96$. Construct a distribution p' for which the above code meets the entropy lower bound $H(p')$.

Complexity Penalized ERM. Suppose you want to fit a Markov Chain (MC) distribution to your data. You don't know the order of the Markov chain to use and would like to set up a complexity penalized ERM approach using prefix codes to automatically do model selection for you. Let $\mathcal{F}_m$ be the class of Markov chain distributions of order m. In class, we saw how to encode any distribution in $\mathcal{F}_m$. You may assume that such a coding scheme is available – you *do not* need to derive it.

6. **(5 Points)** Propose a scheme to encode the *order* of the Markov Chain. You do not need to construct an optimal code for the order. (This is because the length of the code is dominated by the length to encode the Markov Chain given the order.)

7. **(5 Points)** Using your answer in part (a) state how you may construct a prefix code for a Markov chain distribution of any order – i.e. any distribution in the class $\mathcal{F} = \bigcup_{j=1}^{\infty} \mathcal{F}_j$.

8. **(5 Points)** Write down the corresponding complexity penalized ERM optimization problem in $\mathcal{F}$ using the minimum description length principle.

Redundancy and Mixture models. Suppose you would like to minimize redundancy with respect to a class of distributions $\{P_\theta\}_{\theta \in \Theta}$ where $P_\theta = \text{Bernoulli}(\theta)$ and $\Theta = \{0, 1\}$. You decide to use a mixture distribution $1/2P_0 + 1/2P_1$.

9. **(5 Points)** What is the worst-case redundancy $\sup_{P \in P_0, P_1} D(P \| 1/2P_0 + 1/2P_1)$?

10. **(5 Points)** Argue that coding using a mixture distribution $1/2P_0 + 1/2P_1$ is better than coding using either P_1 or P_0.

Long Questions (50 Points)

1. **(15 pts)** Let $X \sim p$ where p is a distribution on the set $\mathcal{X} = \{1, 2, \ldots, m\}$. We are given a subset $S \subset \mathcal{X}$. We ask whether $X \in S$ and receive the answer Y where $Y = 1$ if $X \in S$ and $Y = 0$ if $X \notin S$. Prove that $I(X;Y) = H(\text{Bern}(\mathbb{P}(X \in S)))$. Here $I(X;Y)$ is the mutual information between X and Y and $H(p)$ is the entropy of the distribution p.

2. Consider three dependent binary random variables (X_1, X_2, X_3) where $X_i \in \{0, 1\}$ $\forall i$. You have the following 4 data points from their joint distribution: $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 0)$ and $(1, 1, 1)$.

   (a) **(15 Points)** Obtain the plug-in estimates $\widehat{I}_{12}$, $\widehat{I}_{13}$ and $\widehat{I}_{23}$ for the Mutual Inforations $I(X_1; X_2)$, $I(X_1; X_3)$ and $I(X_2; X_3)$. Recall that $I(X, Y) = \sum_{x,y} p(x, y) \log \left(\frac{p(x,y)}{p(x)p(y)} \right)$.
   You do not need to simplify the expressions completely – you can leave fraction/ log terms.

   (b) **(5 Points)** Depict the Chow-Liu tree for (X_1, X_2, X_3) obtained from the above data.

3. **(15 Points)** We have samples $x_1, \ldots, x_n$ from a discrete distribution π on support $\mathcal{X}$. Let $D = |\mathcal{X}|$. We wish to estimate π using d features $\mathbf{f}$. Here $\mathbf{f} : \mathcal{X} \to \mathbb{R}^d$ and $\mathbf{f}(x) = [f_1(x), \ldots, f_d(x)]$. Since $D \gg n$, in order to improve generalization performance we wish to find the MaxEnt distribution subject to an ℓ_∞-ball constraint. I.e. we wish to solve,

$$\begin{aligned} \min_{p \in \Delta^{D-1}} \quad & D(p \| q_0) \\ \text{subject to} \quad & |\mathbb{E}_p[f_i(X)] - \mathbb{E}_n[f_i(X)]| \leq \beta \quad \forall i = 1, \ldots, d \end{aligned}$$

where $\Delta^{D-1} = \{x \in \mathbb{R}^D; \mathbf{1}^\top x = 1, x_i \geq 0\}$ and $\mathbb{E}_p$, $\mathbb{E}_n$ denote the expectation w.r.t p and the empirical expectations respectively.

Obtain the dual of this problem as a penalized Maximum Likelihood problem in the Exponential family $E_{\mathbf{f},q_0} = \{q(x) \propto q_0(x) \exp(\lambda^\top \mathbf{f}(x)) \,:\, \lambda \in \mathbb{R}^d\}$. *The dual should be in terms of empirically computable quantities.*
You may use any results we covered in class. You *do not* need to derive the dual from first principles.

Hint: The Fenchel conjugate of $U(u) = \mathbb{1}(\|r - u\|_\infty \leq \beta)$ is $U^*(\lambda) = \lambda^\top r + \beta \|\lambda\|_1$.