10-704 Homework 2
Due: Thursday 2/26/2015

Instructions: Turn in your homework in class on Thursday 2/26/2015

1. **Maximum Entropy**

   (a) Suppose we want to maximize the entropy of a distribution supported on the non-negative integers ($\mathbb{N} \cup \{0\}$) subject to a mean constraint:

   $$p^\star = \max_p - \sum_{i \in \mathbb{N} \cup \{0\}} p_i \log p_i \qquad \text{s.t.} \sum_{i \in \mathbb{N} \cup \{0\}} p_i = 1 \sum_{i \in \mathbb{N} \cup \{0\}} i p_i = \alpha$$

   Verify that the solution to this program (the MaxEnt distribution) is a Geometric distribution (i.e. $p_k = (1-\lambda)^k \lambda$ for some parameter $\lambda > 0$).

   (b) Consider the following MaxEnt of a joint distribution $p(x_1, \dots x_d) = p(\mathbf{x})$:

   $$p^\star = \max_p - \int p(\mathbf{x}) \log p(\mathbf{x}) d\mathbf{x} \qquad \text{s.t.} \int p(\mathbf{x}) d\mathbf{x} = 1, \forall i \in [d] p_i(\cdot) = \int p(\mathbf{x}) d\mathbf{x}_{-i} = f_i(\cdot)$$

   Specifically, given marginal functions $f_i : \mathcal{X}_i \to \mathbb{R}$ (assume $\int f_i(x_i) dx_i = 1$ and $f_i(x_i) \geq 0$), we want the actual marginals of p to match. The notation $d\mathbf{x}_{-i}$ denotes integration over all but the ith variable, which is an argument to $p_i(\cdot)$.

2. **Relative Entropy**

   (a) Show that relative entropy $D(p\|q)$ is convex in p.

   (b) Derive the conjugate function of D w.r.t. p

3. **Source Coding**

   (a) A set of symbols have a distribution p. You encode the symbols so that the length ℓ of a symbol x is $\ell(x) = \lceil \log \frac{1}{q(x)} \rceil$ for some other distribution q. Show that:

   $$H(p) + D(p\|q) \leq \mathbb{E}\ell(x) < H(p) + D(p\|q) + 1.$$

   (b) Consider the following method for generating a code for a random variable X on p symbols $\{1, 2, \dots, m\}$ with probabilities $p_1 \geq p_2 \geq \dots p_m$. Define

   $$F_i = \sum_{k=1}^{i-1} p_k$$

   The codeword for i is the number $F_i \in [0, 1]$ rounded off to ℓ_i bits where $\ell_i = \lceil \log \frac{1}{p_i} \rceil$. (E.g. for the symbols $\{a, b, c, d\}$ with probabilities $\{0.5, 0.25, 0.125, 0.125\}$ the codeword assignment would be $\{0, 10, 110, 111\}$.) Show that

   i. That this code is a prefix code

   ii. The code satisfies $H(X) \leq \mathbb{E}\ell_i \leq H(X) + 1$