Boosting

Can we make dumb learners smart?

Aarti Singh

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Slides Courtesy: Carlos Guestrin, Freund & Schapire
Why boost weak learners?

**Goal:** Automatically categorize type of call requested (Collect, Calling card, Person-to-person, etc.)

- yes I’d like to place a collect call long distance please (Collect)
- operator I need to make a call but I need to bill it to my office (ThirdNumber)
- yes I’d like to place a call on my master card please (CallingCard)

- **Easy to find “rules of thumb” that are “often” correct.**
  
  E.g. If ‘card’ occurs in utterance, then predict ‘calling card’

- **Hard to find single highly accurate prediction rule.**
Fighting the bias-variance tradeoff

- **Simple (a.k.a. weak) learners** e.g., naïve Bayes, logistic regression, decision stumps (or shallow decision trees)

  - Are good 😊 - Low variance, don’t usually overfit
  - Are bad 😞 - High bias, can’t solve hard learning problems

- Can we make weak learners always good???
  - No!!! But often yes…
Voting (Ensemble Methods)

- Instead of learning a single (weak) classifier, learn **many weak classifiers** that are **good at different parts of the input space**

- **Output class:** (Weighted) vote of each classifier
  - Classifiers that are most “sure” will vote with more conviction
  - Classifiers will be most “sure” about a particular part of the space
  - On average, do better than single classifier!

\[ H: X \rightarrow Y \ (-1,1) \]
\[ H(X) = \text{sign} \left( \sum_{t} \alpha_{t} h_{t}(X) \right) \]
\[ H(X) = h_{1}(X) + h_{2}(X) \]
Voting (Ensemble Methods)

• Instead of learning a single (weak) classifier, learn many weak classifiers that are good at different parts of the input space

• Output class: (Weighted) vote of each classifier
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  – On average, do better than single classifier!

• But how do you ???
  – force classifiers $h_t$ to learn about different parts of the input space?
  – weigh the votes of different classifiers? $\alpha_t$
Boosting [Schapire’89]

• **Idea:** given a weak learner, run it multiple times on (rewighted) training data, then let learned classifiers vote

• On each iteration $t$:
  – weight each training example by how incorrectly it was classified
  – Learn a weak hypothesis – $h_t$
  – A strength for this hypothesis – $\alpha_t$

• Final classifier: \[ H(X) = \text{sign}(\sum \alpha_t h_t(X)) \]

• Practically useful
• Theoretically interesting
Learning from weighted data

• Consider a weighted dataset
  – \( D(i) \) – weight of \( i \) th training example \((x^i, y^i)\)
  – Interpretations:
    • \( i \) th training example counts as \( D(i) \) examples
    • If I were to “resample” data, I would get more samples of “heavier” data points

• Now, in all calculations, whenever used, \( i \) th training example counts as \( D(i) \) “examples”
  – e.g., in MLE redefine \( \text{Count}(Y=y) \) to be weighted count

Unweighted data

\[
\text{Count}(Y=y) = \sum_{i=1}^{m} 1(Y^i=y)
\]

Weights \( D(i) \)

\[
\text{Count}(Y=y) = \sum_{i=1}^{m} D(i)1(Y^i=y)
\]
AdaBoost [Freund & Schapire’95]

Given: \((x_1, y_1), \ldots, (x_m, y_m)\) where \(x_i \in X, y_i \in Y = \{-1, +1\}\)

Initialize \(D_1(i) = 1/m\). Initially equal weights

For \(t = 1, \ldots, T\):

- Train weak learner using distribution \(D_t\). Naïve bayes, decision stump
- Get weak classifier \(h_t : X \rightarrow \mathbb{R}\).
- Choose \(\alpha_t \in \mathbb{R}\). Magic (+ve)
- Update:

\[
D_{t+1}(i) = \frac{D_t(i)}{Z_t} \begin{cases} 
  e^{-\alpha_t} & \text{if } y_i = h_t(x_i) \\
  e^{\alpha_t} & \text{if } y_i \neq h_t(x_i)
\end{cases}
\]

\[
= \frac{D_t(i) \exp(-\alpha_i y_i h_t(x_i))}{Z_t}
\]

where \(Z_t\) is a normalization factor

Increase weight if wrong on pt \(i\)
\(y_i h_t(x_i) = -1 < 0\)
**AdaBoost [Freund & Schapire’95]**

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\[
D_{t+1}(i) = \frac{D_t(i) \exp(-\alpha_t y_i h_t(x_i))}{Z_t}
\]

where \(Z_t\) is a normalization factor

\[
Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))
\]

Weights for all pts must sum to 1

\[
\sum_{t} D_{t+1}(i) = 1
\]
**AdaBoost** [Freund & Schapire’95]

Given: \((x_1, y_1), \ldots, (x_m, y_m)\) where \(x_i \in X, y_i \in Y = \{-1, +1\}\)

Initialize \(D_1(i) = 1/m.\) \hspace{1cm} \textit{Initially equal weights}

For \(t = 1, \ldots, T:\)

- Train weak learner using distribution \(D_t.\) \hspace{1cm} \textit{Naïve bayes, decision stump}
- Get weak classifier \(h_t : X \rightarrow \mathbb{R}.\)
- Choose \(\alpha_t \in \mathbb{R}.\) \hspace{1cm} \textit{Magic (+ve)}
- Update:

\[
D_{t+1}(i) = \frac{D_t(i) \exp(-\alpha_t y_i h_t(x_i))}{Z_t}
\]

where \(Z_t\) is a normalization factor

Output the final classifier:

\[
H(x) = \text{sign} \left( \sum_{t=1}^{T} \alpha_t h_t(x) \right).
\]
What $\alpha_t$ to choose for hypothesis $h_t$?

Weight Update Rule:

$$D_{t+1}(i) = \frac{D_t(i) \exp(-\alpha_t y_i h_t(x_i))}{Z_t}$$

$$\alpha_t = \frac{1}{2} \ln \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)$$

[Freund & Schapire’95]

Weighted training error

$$\epsilon_t = P_{i \sim D_t(i)}[h_t(x^i) \neq y^i] = \sum_{i=1}^{m} D_t(i) \delta(h_t(x_i) \neq y_i)$$

Does $h_t$ get $i^{th}$ point wrong

$\epsilon_t = 0$ if $h_t$ perfectly classifies all weighted data pts
$\epsilon_t = 1$ if $h_t$ perfectly wrong => $-h_t$ perfectly right
$\epsilon_t = 0.5$

$\alpha_t = \infty$
$\alpha_t = -\infty$
$\alpha_t = 0$
Boosting Example (Decision Stumps)

\[ D_1 \]

\[ D_2 \]

\[ D_3 \]

\[ h_1 \]

\[ h_2 \]

\[ h_3 \]

\[ \varepsilon_1 = 0.30 \]
\[ \alpha_1 = 0.42 \]

\[ \varepsilon_2 = 0.21 \]
\[ \alpha_2 = 0.65 \]

\[ \varepsilon_3 = 0.14 \]
\[ \alpha_3 = 0.92 \]
Boosting Example (Decision Stumps)

\[ H_{\text{final}} = \text{sign} \left( 0.42 + 0.65 + 0.92 \right) \]
Analysis reveals:

- What $\alpha_t$ to choose for hypothesis $h_t$?
  $\alpha_t = \frac{1}{2} \ln \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)$

- If each weak learner $h_t$ is slightly better than random guessing ($\epsilon_t < 0.5$), then training error of AdaBoost decays exponentially fast in number of rounds $T$.

$$\frac{1}{m} \sum_{i=1}^{m} \delta(H(x_i) \neq y_i) \leq \exp \left( -2 \sum_{t=1}^{T} (1/2 - \epsilon_t)^2 \right)$$

Training Error
Analyzing training error

Training error of final classifier is bounded by:

\[
\frac{1}{m} \sum_{i=1}^{m} \delta(H(x_i) \neq y_i) \leq \frac{1}{m} \sum_{i=1}^{m} \exp(-y_i f(x_i))
\]

Convex upper bound

Where \( f(x) = \sum_{t} \alpha_t h_t(x); H(x) = \text{sign}(f(x)) \)

If boosting can make upper bound \( \rightarrow 0 \), then training error \( \rightarrow 0 \)
Training error of final classifier is bounded by:

\[
\frac{1}{m} \sum_{i=1}^{m} \delta(H(x_i) \neq y_i) \leq \frac{1}{m} \sum_{i=1}^{m} \exp(-y_i f(x_i)) = \prod_{t} Z_t
\]

Where \( f(x) = \sum_{t} \alpha_t h_t(x) \); \( H(x) = \text{sign}(f(x)) \)

**Proof:** Using Weight Update Rule

\[
D_{t+1}(i) = \frac{D_t(i) \exp(-\alpha_t y_i h_t(x_i))}{Z_t}
\]

\[
D_1(i) = \frac{1}{m}
\]

\[
D_2(i) = \frac{1}{m} \frac{e^{-\alpha_1 y_i h_1(x_i)}}{Z_1}
\]

\[
D_3(i) = \frac{1}{m} \frac{e^{-\alpha_1 y_i h_1(x_i)} e^{-\alpha_2 y_i h_2(x_i)}}{Z_1 Z_2}
\]

\[
\vdots
\]

\[
D_{T+1}(i) = \frac{1}{m} \frac{\exp(-y_i f(x_i))}{\prod_{t} Z_t}
\]

Wts of all pts add to 1

\[
\sum_{i=1}^{m} D_{T+1}(i) = 1
\]
Analyzing training error

Training error of final classifier is bounded by:

\[
\frac{1}{m} \sum_{i=1}^{m} \delta(H(x_i) \neq y_i) \leq \frac{1}{m} \sum_{i=1}^{m} \exp(-y_i f(x_i)) = \prod_{t} Z_t
\]

Where \( f(x) = \sum_{t} \alpha_t h_t(x) \); \( H(x) = \text{sign}(f(x)) \)

If \( Z_t < 1 \), training error decreases exponentially (even though weak learners may not be good \( \varepsilon_t \sim 0.5 \))
What $\alpha_t$ to choose for hypothesis $h_t$?

Training error of final classifier is bounded by:

$$\frac{1}{m} \sum_{i=1}^{m} \delta(H(x_i) \neq y_i) \leq \frac{1}{m} \sum_{i=1}^{m} \exp(-y_if(x_i)) = \prod_t Z_t$$

Where \( f(x) = \sum_t \alpha_t h_t(x); H(x) = \text{sign}(f(x)) \)

If we minimize $\prod_t Z_t$, we minimize our training error

We can tighten this bound greedily, by choosing $\alpha_t$ and $h_t$ on each iteration to minimize $Z_t$.

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha ty_i h_t(x_i))$$
What $\alpha_t$ to choose for hypothesis $h_t$?

We can minimize this bound by choosing $\alpha_t$ on each iteration to minimize $Z_t$:

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))$$

For boolean target function, this is accomplished by [Freund & Schapire ’97]:

$$\alpha_t = \frac{1}{2} \ln \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)$$

Proof:

$$Z_t = \sum_{i:y_i \neq h_t(x_i)} D_t(i) e^{\alpha_t} + \sum_{i:y_i = h_t(x_i)} D_t(i) e^{-\alpha_t}$$

$$= \epsilon_t e^{\alpha_t} + (1 - \epsilon_t) e^{-\alpha_t}$$

$$\frac{\partial Z_t}{\alpha_t} = \epsilon_t e^{\alpha_t} - (1 - \epsilon_t) e^{-\alpha_t} = 0$$

$$\Rightarrow e^{2\alpha_t} = \frac{1 - \epsilon_t}{\epsilon_t}$$
What $\alpha_t$ to choose for hypothesis $h_t$?

We can minimize this bound by choosing $\alpha_t$ on each iteration to minimize $Z_t$.

$$ Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i)) $$

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$$ \alpha_t = \frac{1}{2} \ln \left( \frac{1 - \epsilon_t}{\epsilon_t} \right) $$

**Proof:**

$$ Z_t = \sum_{i:y_i \neq h_t(x_i)} D_t(i) e^{\alpha_t} + \sum_{i:y_i = h_t(x_i)} D_t(i) e^{-\alpha_t} $$

$$ = \epsilon_t e^{\alpha_t} + (1 - \epsilon_t) e^{-\alpha_t} $$

$$ = 2 \sqrt{\epsilon_t (1 - \epsilon_t)} = \sqrt{1 - (1 - 2 \epsilon_t)^2} $$
Dumb classifiers made Smart

Training error of final classifier is bounded by:

\[
\frac{1}{m} \sum_{i=1}^{m} \delta(H(x_i) \neq y_i) \leq \prod_t Z_t = \prod_t \sqrt{1 - (1 - 2\epsilon_t)^2} \\
\leq \exp \left( -2 \sum_{t=1}^{T} (1/2 - \epsilon_t)^2 \right)
\]

Using \(1-x \leq e^{-x}\)

If each classifier is (at least slightly) better than random \(\epsilon_t < 0.5\)

AdaBoost will achieve zero **training error** exponentially fast (in number of rounds \(T\)) !!

What about test error?
Boosting results – Digit recognition

[Schapire, 1989]

• Boosting often, but not always
  – Robust to overfitting
  – Test set error decreases even after training error is zero
Margin Based Bounds

[Schapire, Freund, Bartlett, Lee’98]

$$\text{error}_{true}(H) \leq \Pr [\text{margin}_f(x,y) \leq \theta] + \tilde{O} \left( \sqrt{\frac{d}{m\theta^2}} \right)$$

With high probability

Boosting increases the margin very aggressively since it concentrates on the hardest examples.

If margin is large, more weak learners agree and hence more rounds does not necessarily imply that final classifier is getting more complex.

Bound is independent of number of rounds T!

Boosting can still overfit if margin is too small or weak learners are too complex
AdaBoost and AdaBoost.MH on Train (left) and Test (right) data from Irvine repository. [Schapire and Singer, ML 1999]
Boosting and Logistic Regression

Logistic regression equivalent to minimizing log loss

\[ \sum_{i=1}^{m} \ln(1 + \exp(-y_if(x_i))) \]

\[ f(x) = w_0 + \sum_{j} w_j x_j \]

Boosting minimizes similar loss function!!

\[ \frac{1}{m} \sum_{i=1}^{m} \exp(-y_if(x_i)) = \prod_{t} Z_t \]

\[ f(x) = \sum_{t} \alpha_t h_t(x) \]

Weighted average of weak learners

Both smooth approximations of 0/1 loss!
Boosting and Logistic Regression

Logistic regression:
• Minimize log loss
  \[ \sum_{i=1}^{m} \ln(1 + \exp(-y_i f(x_i))) \]
• Define
  \[ f(x) = \sum_j w_j x_j \]
  where \( x_j \) predefined features
  (linear classifier)
• Jointly optimize over all weights \( w_0, w_1, w_2 \ldots \)

Boosting:
• Minimize exp loss
  \[ \sum_{i=1}^{m} \exp(-y_i f(x_i)) \]
• Define
  \[ f(x) = \sum_t \alpha_t h_t(x) \]
  where \( h_t(x) \) defined dynamically to fit data
  (not a linear classifier)
• Weights \( \alpha_t \) learned per iteration \( t \) incrementally
Hard & Soft Decision

Weighted average of weak learners

\[ f(x) = \sum_{t} \alpha_t h_t(x) \]

Hard Decision/Predicted label:

\[ H(x) = \text{sign}(f(x)) \]

Soft Decision:
(based on analogy with logistic regression)

\[ P(Y = 1|X) = \frac{1}{1 + \exp(f(x))} \]
Effect of Outliers

**Good 😊:** Can identify outliers since focuses on examples that are hard to categorize

**Bad 😞:** Too many outliers can degrade classification performance dramatically increase time to convergence
Bagging

Related approach to combining classifiers:

1. Run independent weak learners on bootstrap replicates (sample with replacement) of the training set
2. Average/vote over weak hypotheses

### Bagging vs. Boosting

<table>
<thead>
<tr>
<th>Bagging</th>
<th>Boosting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resamples data points</td>
<td>Reweights data points (modifies their distribution)</td>
</tr>
<tr>
<td>Weight of each classifier is the same</td>
<td>Weight is dependent on classifier’s accuracy</td>
</tr>
<tr>
<td>Only variance reduction</td>
<td>Both bias and variance reduced – learning rule becomes more complex with iterations</td>
</tr>
</tbody>
</table>

[Breiman, 1996]
Boosting Summary

• Combine weak classifiers to obtain very strong classifier
  – Weak classifier – slightly better than random on training data
  – Resulting very strong classifier – can eventually provide zero training error

• AdaBoost algorithm

• Boosting v. Logistic Regression
  – Similar loss functions
  – Single optimization (LR) v. Incrementally improving classification (B)

• Most popular application of Boosting:
  – Boosted decision stumps!
  – Very simple to implement, very effective classifier