Recap

• A matching for a bipartite graph is called stable if it doesn’t contain any rogue couples, i.e. a man and a woman who would prefer each other over their matched partners.

• Given a list of men and women along with their preference lists, the Traditional Matching Algorithm (TMA) produces a stable matching in polynomial time. The algorithm is optimal for the gender that proposes (if men are proposing, each man will get his best possible partner).

• A circuit family is an infinite sequence of circuits \( \langle C_i \rangle \), where \( C_i \) is a circuit with \( i \) input bits. We say that a circuit family \( \langle C_i \rangle \) decides a language \( L \subseteq \Sigma^* \) if the set of strings accepted by \( C_i \) is exactly \( \{ s : s \in L \text{ and } |s| = i \} \).

Gates has 3 floors

Show that any boolean function \( f : \{0, 1\}^n \to \{0, 1\} \) (i.e. a function that takes in \( n \) input bits and outputs 1 bit) can be computed by a circuit of depth at most 3 (This means that the longest path from any input bit to the output should have at most 3 gates). Your gates may have any number of inputs.

What is the size (in big-O) of such a circuit in the worst case?

Bounds on Circuit Size

Let \( x_1, x_2, \ldots, x_n \) be input bits \( (n \geq 2) \). We use the convention that truth assignments are either 0 or 1. We are interested in computing the following Boolean function:

\[
H(x_1, x_2, \ldots, x_n) = \begin{cases} 
1 & \text{if at least 2 of the } x_i\text{'s are assigned 1}, \\
0 & \text{if fewer than 2 of the } x_i\text{'s are assigned 1}.
\end{cases}
\]

Prove there is a circuit computing \( H \) that uses at most \( C \cdot n \) gates. Here \( C \) should be some fixed positive number, like 3 or 4 or 10. (Your \( C \) should work for every choice of \( n \).) Your circuit can use any type of gate with fan-in at most 2 (though perhaps you will only need AND and OR gates?). If it helps you, you may assume that \( n \) is a power of 2.

[(Bonus. Lower bounds are hard. Prove that any circuit computing \( H \) must have at least \( 2n - 3 \) gates)]

It’s All the Same to Me

(a) Suppose that, in a stable marriage scenario, a man was the last choice for every women, and a woman was the last choice for every man. Is it necessary that in a stable matching, the man and women are paired together?

(b) Give a polynomial time algorithm to decide if a given instance of the stable matching problem has a unique solution.

I Couldn’t Find a Suitable Marriage Pun

Hall’s Marriage Theorem states that a bipartite graph \( G(X, Y, E) \) has a perfect matching iff \( |X| = |Y| \) and for every \( S \subseteq X, |N(S)| \geq |S| \). Prove this properly.