The relationship between water depth $h(\mathbf{x}, t)$ and distortion function $\mathbf{w}(\mathbf{x}, t)$

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July 9, 2009

1 Problem Setting

A scene $I_g(\mathbf{x})$ is settled underwater, and a sequence of video $I(\mathbf{x}, t)$ is taken from above water. Each video frame records a distorted version of the static image:

$$I(\mathbf{x}, t) = I_g(\mathbf{x} + \mathbf{w}(\mathbf{x}, t)) \tag{1}$$

where $\mathbf{w}(\mathbf{x}, t)$ is the unknown distortion over time.

The problem is, can we recover the original image $I_g(\mathbf{x})$ using a sequence of video $I(\mathbf{x}, t)$? Note that both the state of water and the underwater scene is unknown.

1.1 First-Order Approximation

According to Snell's law, we can write the distortion to be:

$$\mathbf{w}(\mathbf{x}, t) = \alpha \nabla h(\mathbf{x}, t) \tag{2}$$

where $h(\mathbf{x}, t)$ is the height of the water surface at time t, and α is a constant related to average water height and relative refraction index.

Eqn. 2 is derived using first-order approximation.

Firstly, we consider the normal vector $\hat{\mathbf{n}}$ of water surfaceat $\mathbf{x} = (x, y)$. It is the cross product of the two tangent vectors $\mathbf{t}_x = (1, 0, h_x)$ and $\mathbf{t}_y = (0, 1, h_y)$:

$$\hat{\mathbf{n}} = \frac{\mathbf{t}_x \times \mathbf{t}_y}{\|\mathbf{t}_x \times \mathbf{t}_y\|} = \frac{1}{\sqrt{h_x^2 + h_y^2 + 1}}(-h_x, -h_y, 1) \tag{3}$$

The viewing vector is $\mathbf{v}_{\text{in}} = (0, 0, -1)$. According to the Snell's Law (Fig. 1), the resulting ray direction $\mathbf{v}_{\text{out}}$ lies on the plane spanned by $\mathbf{v}_{\text{in}}$ and $\hat{\mathbf{n}}$, hence can be linear represented by these two vectors. Since $\mathbf{v}_{\text{in}}$ has zero x and y components, the x and y components of $\mathbf{v}_{\text{out}} = (\mathbf{v}_{\text{out,x}}, \mathbf{v}_{\text{out,y}}, \mathbf{v}_{\text{out,z}})$ is linear to the corresponding components of $\hat{\mathbf{n}}$. Hence, we have the following relationship:

$$\frac{\mathbf{v}_{\text{out,x}}}{\mathbf{v}_{\text{out,y}}} = \frac{h_x}{h_y} \tag{4}$$

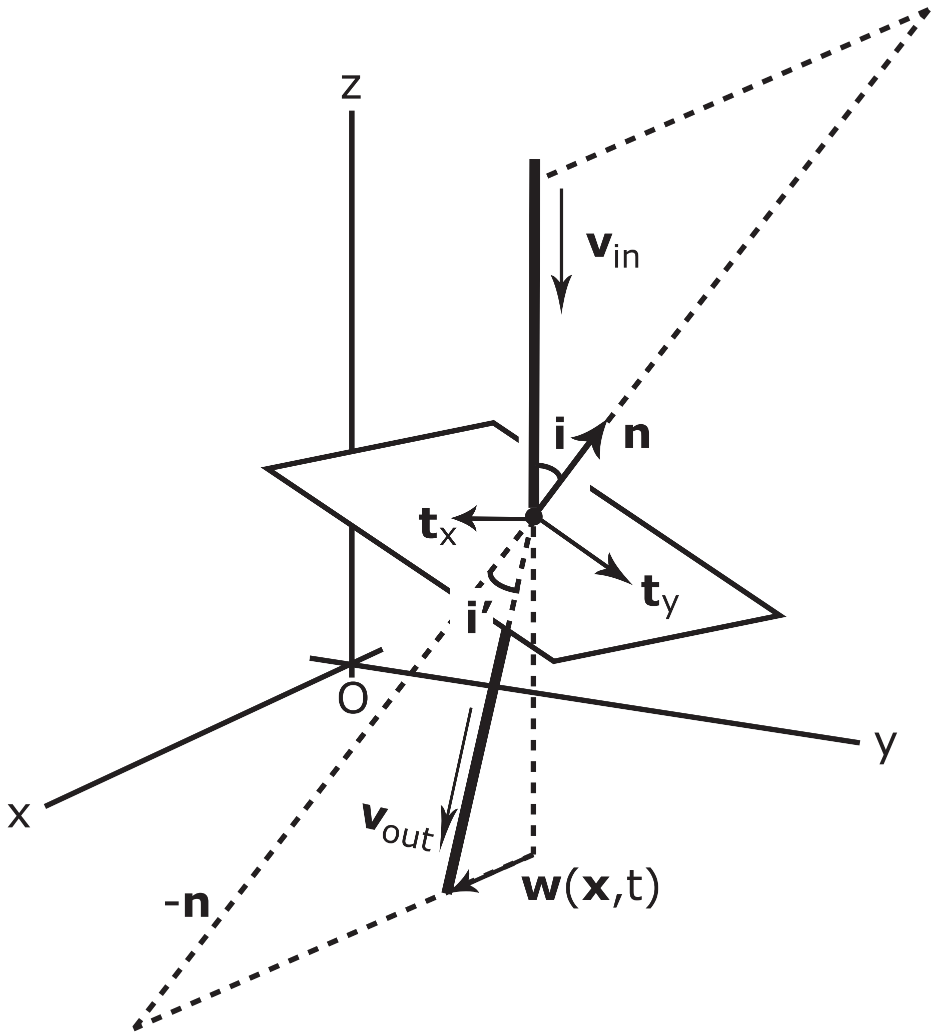

Figure 1: Snell's Law

As a result, its 2-D projection $\mathbf{w}(\mathbf{x}, t)$ follows the direction of (h_x, h_y), i.e. $\mathbf{w}(\mathbf{x}, t) = \lambda \nabla h(\mathbf{x}, t)$.

On the other hand, the magnitude of $\mathbf{w}(\mathbf{x}, t)$ is determined by the Snell's Law:

$$\frac{\sin i}{\sin i'} = n \tag{5}$$

if the average water depth is h_0, then $\|\mathbf{w}\| = h_0 \tan(i - i')$. If only small fluctuation is considered, then $\sin i \approx \tan i \approx i$, and $\|\nabla h\| \ll 1$. As a result, we have:

$$\begin{aligned}
\|\mathbf{w}\| &= h_0 \tan(i - i') && (6)\\
&\approx h_0 \left(1 - \frac{1}{n}\right) \sin i && (7)\\
&= h_0 \left(1 - \frac{1}{n}\right) \sqrt{1 - (\mathbf{v}_{\text{in}} \cdot \hat{\mathbf{n}})^2} && (8)\\
&= h_0 \left(1 - \frac{1}{n}\right) \frac{\|\nabla h\|}{\sqrt{\|\nabla h\|^2 + 1}} && (9)\\
&\approx h_0 \left(1 - \frac{1}{n}\right) \|\nabla h\| && (10)
\end{aligned}$$

So the distortion vector $\mathbf{w}(\mathbf{x}, t) = \alpha \nabla h(\mathbf{x}, t)$, with $\alpha = h_0(1 - \frac{1}{n}) > 0$ (The sign of α can be determined by simple intuition).