Theoretical Bounds in Distortion Estimation Algorithm

Yuandong Tian
Robotics Institute
Carnegie Mellon University
yuandong@andrew.cmu.edu

Srinivasa Narasimhan
Robotics Institute
Carnegie Mellon University
srinivas@cs.cmu.edu

March 10, 2010

1 Proof of the pullback bound (Eqn. 4) in [1]

Denote the pull-back operation H that maps a distorted image $I_{\mathbf{p}}$ using the parameters $\mathbf{q}$ to a new (possibly less distorted) image $H(I_{\mathbf{p}}, \mathbf{q})$. In the following we shall prove

$$\|H(I_{\mathbf{p}}, \mathbf{q}) - I_{\mathbf{p}-\mathbf{q}}\| \leq R\|\mathbf{p} - \mathbf{q}\| \tag{1}$$

for the following setting of H:

	Forward Case	Backward Case
Generating function	$G = G_{\mathrm{F}}$	$G = G_{\mathrm{B}}$
Pull-back operation	$H = G_{\mathrm{B}}$	$H = G_{\mathrm{F}}$

Table 1: The generating function and pull-back operation in the forward and backward case.

Fig. 1 shows the intuition of Eqn. 1, in particular, the reason why $H(I_{\mathbf{p}}, \mathbf{p}) = T$ in both forward and backward cases. Although the intuition shown in Fig. 1 is valid for any warping function $W(\mathbf{x}, \mathbf{p})$, we only consider linear warping, i.e., $W(\mathbf{x}, \mathbf{p}) = \mathbf{x} + B(\mathbf{x})\mathbf{p}$ in the proof. Here $B(\mathbf{x}) = [\mathbf{b}_1(\mathbf{x}), \mathbf{b}_2(\mathbf{x}), \ldots, \mathbf{b}_d(\mathbf{x})]$ are a set of orthonormal warping bases.

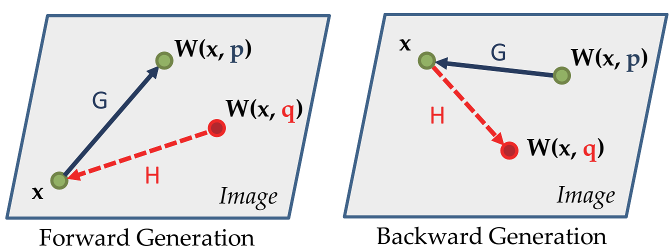

Figure 1: The mechanism of the pull-back operation H. In forward generation, a pixel $\mathbf{x}$ in the template is pushed to the position $W(\mathbf{x}, \mathbf{p})$ in the distorted image by the generating function G. The corresponding pull-back operation H do the opposite: it takes the pixel value at $W(\mathbf{x}, \mathbf{q})$ in the distorted image, and stores it at position $\mathbf{x}$ in the resulting image $H(I_{\mathbf{p}}, \mathbf{q})$. In the case of $\mathbf{p} = \mathbf{q}$, the pixel pushed by G is the same pixel pulled by H, yielding $H(I_{\mathbf{p}}, \mathbf{p}) = T$. Similarly we define H in backward generation.

In the following, we give the bound for backward warping case $G = G_{\mathrm{B}}$, in which the pull-back operation is $H = G_{\mathrm{F}}$. Similarly we can prove the forward case.

Theorem 1.1 (The Upper bound of the pull-back operation H**)** *Suppose the (backward) distorted image* $I_{\mathbf{p}-\mathbf{q}}$ *maps the pixel at* L_G *on the template image* T *to position* $\mathbf{y} \in \mathbb{R}^2$*, and the pulled-back image* $H(I_{\mathbf{p}}, \mathbf{q}) = G_{\mathrm{F}}(I_{\mathbf{p}}, \mathbf{q})$ *maps the pixel at* L_H *on the template image* T *to the same position* $\mathbf{y}$*. Then we have the following bound if there exists an* $\mathbf{x}$ *so that* $\mathbf{y} = \mathbf{x} + B(\mathbf{x})\mathbf{q}$ *(Or* $W(\mathbf{x}, \mathbf{q})$ *is onto):*

$$\|L_G - L_H\|_1 \leq R'\|\mathbf{p} - \mathbf{q}\|_1 \tag{2}$$

where $R' = 2B_0 \min(B_1\|\mathbf{q}\|_1, 2)$, $B_0 = \|B\|_\infty$ and $B_1 = \max_j \max_{\mathbf{x}} \max(\|\nabla b_j^x(\mathbf{x})\|_1, \|\nabla b_j^y(\mathbf{x})\|_1)$ is the gradient bound of basis $B(\mathbf{x})$ (Note: $\mathbf{b}_j(\mathbf{x}) = [b_j^x(\mathbf{x}); b_j^y(\mathbf{x})]$ is a column vector at each $\mathbf{x}$). Therefore, we have

$$\|H(I_{\mathbf{p}}, \mathbf{q}) - I_{\mathbf{p}-\mathbf{q}}\|_\infty \leq R\|\mathbf{p} - \mathbf{q}\|_1 \tag{3}$$

where $R = R'Q_1$ and $Q_1 = \max_{\mathbf{x}} \|\nabla T(\mathbf{x})\|_1$ is the gradient bound of the template T.

Proof According to Fig. 1, $H(I_{\mathbf{p}}, \mathbf{q})$ essentially moves the pixel $L_H \equiv \mathbf{x} + B(\mathbf{x})\mathbf{p}$ on the *template* T to the position $\mathbf{x} + B(\mathbf{x})\mathbf{q}$:

$$H : L_H \equiv \mathbf{x} + B(\mathbf{x})\mathbf{p} \longrightarrow \mathbf{x} + B(\mathbf{x})\mathbf{q} \tag{4}$$

This is valid for any $\mathbf{x} \in \mathbb{R}^2$. On the other hand, for the pixel $\mathbf{y}$ on distorted image $I_{\mathbf{p}-\mathbf{q}}$, it comes from the pixel $L_G \equiv \mathbf{y} + B(\mathbf{y})(\mathbf{p} - \mathbf{q})$ on the template T:

$$G : L_G \equiv \mathbf{y} + B(\mathbf{y})(\mathbf{p} - \mathbf{q}) \longrightarrow \mathbf{y} \tag{5}$$

Since $W(\mathbf{x}, \mathbf{q})$ is onto, there exists $\mathbf{x}$ so that $\mathbf{y} = \mathbf{x} + B(\mathbf{x})\mathbf{q}$, then Eqn. 5 becomes

$$G : L_G \equiv \mathbf{x} + B(\mathbf{x})\mathbf{q} + B(\mathbf{x} + B(\mathbf{x})\mathbf{q})(\mathbf{p} - \mathbf{q}) \longrightarrow \mathbf{x} + B(\mathbf{x})\mathbf{q} \tag{6}$$

Note the destination(right) part of Eqn. 4 and Eqn. 6 are the same ($\mathbf{y}$), while the difference between the source(left) part of Eqn. 4 and Eqn. 6 is:

$$L_G - L_H = [B(\mathbf{x} + B(\mathbf{x})\mathbf{q}) - B(\mathbf{x})]\,(\mathbf{p} - \mathbf{q}) \tag{7}$$

so we directly have the bound $\|L_G - L_H\|_1 \leq 4B_0\|\mathbf{p} - \mathbf{q}\|_1$ where $B_0 = \|B\|_\infty = \max_{\mathbf{x}} \max_{ij} B_{ij}(\mathbf{x})$. In addition, using intermediate value theorem, from Eqn. 7 there exists $\{\boldsymbol{\xi}_1^x, \boldsymbol{\xi}_2^x, \ldots, \boldsymbol{\xi}_d^x\}$ and $\{\boldsymbol{\xi}_1^y, \boldsymbol{\xi}_2^y, \ldots, \boldsymbol{\xi}_d^y\}$ on the 2D line segment $[\mathbf{x}, \mathbf{x} + B(\mathbf{x})\mathbf{q}]$ so that:

$$B^x(\mathbf{x} + B(\mathbf{x})\mathbf{q}) - B^x(\mathbf{x}) = \mathbf{q}^T B(\mathbf{x})^T \left[\nabla b_1^x(\boldsymbol{\xi}_1^x), \nabla b_2^x(\boldsymbol{\xi}_2^x), \ldots, \nabla b_d^x(\boldsymbol{\xi}_d^x)\right] \tag{8}$$

$$B^y(\mathbf{x} + B(\mathbf{x})\mathbf{q}) - B^y(\mathbf{x}) = \mathbf{q}^T B(\mathbf{x})^T \left[\nabla b_1^y(\boldsymbol{\xi}_1^y), \nabla b_2^y(\boldsymbol{\xi}_2^y), \ldots, \nabla b_d^y(\boldsymbol{\xi}_d^y)\right] \tag{9}$$

where $B^x(\mathbf{x}) = [b_1^x(\mathbf{x}), b_2^x(\mathbf{x}), \ldots, b_d^x(\mathbf{x})]$ and $B^y(\mathbf{x}) = [b_1^y(\mathbf{x}), b_2^y(\mathbf{x}), \ldots, b_d^y(\mathbf{x})]$ are the x and y component of $B(\mathbf{x})$. Then:

$$|L_G^x - L_H^x| \leq B_1 B_0 \|\mathbf{q}\|_1 \|\mathbf{p} - \mathbf{q}\|_1 \tag{10}$$

$$|L_G^y - L_H^y| \leq B_1 B_0 \|\mathbf{q}\|_1 \|\mathbf{p} - \mathbf{q}\|_1 \tag{11}$$

where $B_1 = \max_j \max_{\mathbf{x}} \max(\|\nabla b_j^x(\mathbf{x})\|_1, \|\nabla b_j^y(\mathbf{x})\|_1)$. Hence the bound. ∎

References

[1] Y. Tian and S. Narasimhan. A Globally Optimal Data-Driven Approach for Image Distortion Estimation. *CVPR*, 2010. 1