A Theory of Multi-Layer Flat Refractive Geometry

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Imaging with Refractions
Source: Shortis et al. SPIE 2007
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Multi-Layer Flat Refraction

Agrawal, Ramalingam, Taguchi, Chari

Underwater Imaging
BP Oil Disaster, 5000 feet under sea level
Deepsea Challenger submersible

Stereo Cameras, LED lights
Imaging through refractions

- Not the same as pinhole imaging
- Pinhole model (central approximation) is not valid
Calibration

Source: Shortis et al. SPIE 2007
Multi-Layer Flat Refractive Systems

Camera  Flat Refracting Layers  Scene

Light Path
- Unknown Orientation of Layers $n$
- Unknown Layer Thickness $d_0, d_1, \ldots, d_{k-1}$
- Unknown Refractive Indices $\mu_0, \mu_1, \ldots, \mu_{k-1}$
- Unknown Pose of the Checkerboard $(R, t)$
Related Work

- Treibitz et al. CVPR 2008
  - Single Refracting Layer
  - Known refractive index
  - Known distance of checkerboard
  - Optimize over one parameter $d$
    - Known internal camera calibration

- This paper
  - Multiple layers, unknown refractive indices
  - 2K parameters for K layers
  - Unknown Pose of calibrating object (6 parameters)
  - Unknown Orientation of layers (2)
  - 8 + 2K parameters

  - Single layer  10 parameters
  - Two Layers   12 parameters
Flat Refraction Constraint (FRC)

Transformed 3D point \( \text{RP+}t \) should lie on the outgoing ray \( v_k \).

\[ q \leftrightarrow \text{RP+}t \ (2D-3D \ correspondence) \]
FRC for Single Layer

- Non-Linear Equation in 10 unknowns
- Difficult to solve
- Complexity increases with each additional layer
Modeling Flat Refractions

- Pinhole Model is not good
  - Non-single view point camera
  - Well-known in photogrammetry (Kotowski 1988)
  - Treibitz et al. CVPR 2008

- Flat Refraction corresponds to Axial (non-central) camera
  - All outgoing rays pass through an axis
  - Axis: Camera ray parallel to layer orientation $\mathbf{n}$
Flat Refraction == Axial Camera

Transformed 3D point $(\mathbf{RP} + \mathbf{t})$ should also lie on the plane of refraction

$$(\mathbf{RP} + \mathbf{t})^T(\mathbf{A} \times \mathbf{v}_0) = 0$$
Key Idea: Coplanarity Constraint

- Transformed 3D point \((RP+t)\) should lie on plane of refraction
  - Weaker constraint than FRC

- Axis A, Camera ray \(v_0\)

\[
(RP + t)^T (A \times v_0) = 0
\]

- **Independent** of number of layers, layer distances and their refractive indices

- Allows estimating axis and pose **independently** of other calibration parameters
Coplanarity Constraint

\[(RP + t)^T (A \times v_0) = 0\]

\[E = [A]_\times R \text{ and } s = A \times t.\]

\[v_0^T EP + v_0^T s = 0\]

- Translation along axis vanishes in s
- 5 out of 6 pose parameters can be computed
11 Point Linear Algorithm

\[ v_0^T E_{3 \times 3} P + v_0^T s_{3 \times 1} = 0 \]

Using 11 2D-3D correspondences, we get 11 by 12 matrix \( B \)

\[
\begin{bmatrix}
(P(1)^T \otimes v_0(1)^T) & v_0(1)^T \\
\vdots & \vdots \\
(P(11)^T \otimes v_0(11)^T) & v_0(11)^T
\end{bmatrix}
\begin{bmatrix}
E(:) \\
\mathbf{s}
\end{bmatrix} = 0
\]

SVD based solution
Similarity with 5-point Relative Pose Problem

- $E = [A]_x R$, where $A$ is the axis and $R$ is unknown rotation

- For relative pose between two cameras
  - Essential matrix $E = [t]_x R$, where $t$ is the translation
  - 5-point algorithm [Nister 2004]

- We can map our problem to the 5-point Relative Pose problem
8-Point Axis Estimation Algorithm

\[ \nu_0^T E P + \nu_0^T s = 0 \]

- Using 8 correspondences, we get 8 by 12 matrix \( \mathbf{B} \)
- Solution lies in 4 dimensional sub-space

\[
\begin{bmatrix}
E(:,)
\end{bmatrix} = \lambda_1 \mathbf{V}_1 + \lambda_2 \mathbf{V}_2 + \lambda_3 \mathbf{V}_3 + \lambda_4 \mathbf{V}_4
\]

\[
E(:,) = \lambda_1 \mathbf{V}_1(1:9) + \lambda_2 \mathbf{V}_2(1:9) + \lambda_3 \mathbf{V}_3(1:9) + \mathbf{V}_4(1:9)
\]

Feed subspace vectors to Nister’s Solver and obtain \( \lambda_i \)
8-Point Axis Estimation Algorithm

- Compute Axis from $E$ as left null-singular vector
  \[ A^T E = 0 \]

- Compute Rotation matrices from $E$
  - Hartley and Zisserman, Multiview Geometry
  - Twisted pair ambiguity
  - Similar to Relative Pose problem
Obtaining Remaining Calibration Parameters

- Coplanarity Constraint
  - Obtain axis $A$, rotation $R$, and $s = A \times t$

- Remaining calibration parameters
  - Translation along axis $t_A$
  - Layer Thickness $d_i$, $i = 1$ to $k$
  - Layer Refractive Indices $\mu_i$, $i = 1$ to $k$
Known Refractive Indices

- Ray directions of $v_1, \ldots, v_k$ can be computed using Snell’s Law
- Layer Thicknesses $d_i$ and $t_A$ can be computed \textit{linearly}
Linear System

\[
\begin{align*}
&v_k \times v_0 \quad v_k \times v_1 \quad \cdots \quad v_k \times v_{k-1} \quad v_k \times A \\
&d_0 \\
&d_{k-1} \\
&t_A
\end{align*}
\]

= \mathbf{u}

\[
\begin{align*}
A
\end{align*}
\]
Light Path Triangulation

• Steger and Kutulakos, IJCV 2008

• Triangulation is not possible for more than 2 refractions

• General Shapes

• Theoretically possible for multi-layer flat refractions
  – Partial knowledge of shape
  – Flat layers, parallel to each other
Case 1: Single Layer

\[
\begin{bmatrix}
v_1 \times v_0 \\
v_1 \times A
\end{bmatrix}
= 
\begin{bmatrix}
d_0 \\
A
\end{bmatrix}
\]
Case 2: Two Layers

\[
\begin{pmatrix}
v_2 \times v_0 & v_2 \times v_1 & v_2 \times A
\end{pmatrix}
\begin{pmatrix}
d_0 \\
d_1 \\
t_A
\end{pmatrix}
= u
\]
Special Case: Looking **through** a medium

- Camera and Object are in the same refractive medium
- Example
  - Looking through a thick glass slab
  - (Air – Glass – Air)
  - Final refracted ray $v_2$ is parallel to camera ray $v_0$
Special Case: Two Layers

\[ v_2 \otimes v_0 \quad v_2 \times v_1 \quad v_2 \times A \quad \begin{array}{c} \times \cr t_A \end{array} \quad d_0 \quad d_1 \quad = \quad u \]
Special Case: Looking **through** a medium

- Camera and Object are in the same refractive medium
- Distance to the refractive medium $d_0$ cannot be estimated
  - Kutulakos and Steger
- Thickness of the medium $d_1$ can be estimated
- Pose estimation can be done
Multiple Layers

• If two layers i and j have same refractive indices
  \[ \mu_i = \mu_j \]

• Then only the combined layer thickness \( d_i + d_j \) can be estimated
Summary of Calibration

- **Step 1**: Compute Axis, Rotation and $s$
  - Using 11 pt or 8 pt algorithm
- **Step 2**: Compute layer thickness and $t_A$
  - Solve a linear system

- **Unknown Refractive Indices**
  - Step 1 remains the same
  - Step 2
    - Solve 6th degree equation for Single Layer
    - Solve 6th degree equation for Air-Medium-Air
  - Too difficult to solve general two layer case
Analytical Forward Projection

- Projection of 3D point onto the image plane?
- Required for minimizing re-projection error
  - bundle-adjustment in SfM
  - Refine calibration parameters

- Perspective projection equations
- \( x = \frac{P_x}{P_z}, \ y = \frac{P_y}{P_z} \)
Analytical Forward Projection

- Single Layer
  - 4th degree equation
Analytical Forward Projection

- Two Layers
  - Air – Medium – Air \( 4^{\text{th}} \) degree equation
  - General Case \( 12^{\text{th}} \) degree equation
Real Experiment using fish tank

Air – Water - Air
Calibration

- Unknown Thickness of Tank
- Unknown Orientation of Tank
- Unknown Pose of Checkerboards
Captured Photo
Photo without tank
## Results

- Thickness of tank measured using ruler = 260 mm

<table>
<thead>
<tr>
<th></th>
<th>Estimated Rotation of Checkerboard</th>
<th>Estimated Translation of Checkerboard</th>
<th>Estimated Tank Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground truth</td>
<td>131.3, 1.2, 84.0</td>
<td>−237.5, −128.8, 455.8</td>
<td>260</td>
</tr>
<tr>
<td>Pinhole Model</td>
<td>130.2, 1.4, 83.8</td>
<td>−217.7, −120.7, <strong>372.1</strong></td>
<td></td>
</tr>
<tr>
<td>Ours (using all planes)</td>
<td>131.3, 1.2, 84.1</td>
<td>−237.1, −128.1, 453.1</td>
<td><strong>255.69</strong></td>
</tr>
<tr>
<td>Using Single Plane</td>
<td>131.4, 1.3, 84.0</td>
<td>−239.7, −129.2, 456.3</td>
<td><strong>272.81</strong></td>
</tr>
</tbody>
</table>
Reprojected 3D Points

Red: Ours
Green: Pinhole Model
Summary

• Multi-Layer Flat Refractions == Axial Camera
  - Pinhole camera model is not a good approximation
  - Calibration algorithm
  - Coplanarity Constraints

• Applicable to
  - Spherical Ball Refraction (Agrawal et al. ECCV 2010)
  - Catadioptric Cameras (quadric mirrors)
  - Radial distortion correction (Hartley-Kang PAMI 2007)

• Analytical Forward Projection
A Theory of Multi-Layer Flat Refractive Geometry

Known 3D Point

Perspective Camera

Solve 4th degree equation

Known 3D Point

Perspective Camera

Solve 12th degree equation
Additional Slides
Related Work

• Calibration of Axial Cameras
  – Ramalingam, Sturm and Lodha, ACCV 2006
    • Requires checkerboard in three positions
  – Tardiff et al. PAMI 2009
    • Models each distortion circle separately

• This paper
  – Calibration using single checkerboard
    • Plane based calibration
  – Global Model
Relationship with Hartley-Kang Algorithm

- Parameter-free radial distortion correction
  - PAMI 2007

- Similar formulation as our coplanarity constraint
  - 8 point algorithm can be applied