COMPUTATIONAL IMAGING

Berthold K.P. Horn
What is Computational Imaging?

• Computation inherent in image formation
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(1) Computing is getting faster and cheaper
—precision physical apparatus is not
What is Computational Imaging?

• Computation inherent in image formation

(1) Computing is getting faster and cheaper — precision physical apparatus is not

(2) Can’t refract or reflect some radiation
What is Computational Imaging?

- Computation inherent in image formation

1. Computing is getting faster and cheaper
   - precision physical apparatus is not

2. Can’t refract or reflect some radiation

3. Detection is at times inherently coded
Computational Imaging System
Examples of Computational Imaging:

(1) Synthetic Aperture Imaging
(2) Coded Aperture Imaging
(3) Exact Cone Beam Reconstruction
(4) Diaphanography—Diffuse Tomography
SYNTHETIC APERTURE IMAGING

Traditional approach:
- Coupling of resolution, DOF, FOV to NA
- Precision imaging — “flat” illumination

with: Michael Mermelstein, Jekwan Ryu, Stanley Hong, and Dennis Freeman
Objective Lens Parameter Coupling
Synthetic Aperture Imaging

Traditional approach:
- Coupling of resolution, DOF, FOV to NA
- Precision imaging — “flat” illumination

New approach:
- Precision illumination — Simple imaging
- Multiple images — Textured illumination
Synthetic Aperture Imaging

- Precision illumination — Simple imaging
- Multiple images — Textured illumination
- Image detail in response to textures
- Non-uniform samples in FT space
SAM M6
Creating Interference Pattern

\( N \) intersecting coherent beams

low NA lens

sample
Reflective Optics M6
Creating Interference Pattern
Fourier Transform of Texture Pattern
Interference Pattern Texture
Synthetic Aperture Microscopy

- Interference of many Coherent Beams
- Amplitude and Phase Control of Beams
Amplitude and Phase Control

Milestone reached: generated fifty beams with off-the-shelf AOM
Synthetic Aperture Microscopy

- Interference of many Coherent Beams
- Amplitude and Phase Control of Beams
- On the fly calibration
- Non-uniform inverse FT Least Squares
Wavenumber Calibration using FT
Hough Transform Calibration
Least Squares Match in FT
Polystyrene Micro Beads (1 µm)
(2) CODED APERTURE IMAGING

- Can’t refract or reflect gamma rays
- Pinhole — tradeoff resolution and SNR

with: Richard Lanza, Roberto Accorsi, Klaus Ziock, and Lorenzo Fabris.
Coded Aperture Imaging

- Can’t refract or reflect gamma rays
- Pinhole — tradeoff resolution and SNR
- Multiple pinholes
- Complex masks can “cast shadows”
Coded Aperture Principle
Decoding Method Rationale
Coded Aperture Imaging

- Can’t refract or reflect gamma rays
- Pinhole — tradeoff resolution and SNR
- Complex masks can “cast shadows”

- Decoding by Correlation
- Special Masks with Flat Power Spectrum
Mask Design — Inverse Systems

\[ h(x,y) \otimes h'(x,y) = \delta(x,y) \]

\[ H(u,v) \cdot H'(u,v) = 1 \]
Masks — XRT Coarse
Mask Design — 1D

Definition: $q$ is a quadratic residue $\pmod{p}$ if $\exists \ n \ \text{s.t.} \ n^2 \equiv q \pmod{p}$

Legendre symbol

\[
\left( \frac{a}{p} \right) = \begin{cases} 
1 & \text{if } a \text{ is quadratic residue} \\
-1 & \text{otherwise}
\end{cases}
\]

Correlation with zero shift $(p - 1)/2$
Correlation with non-zero shift $(p - 1)/4$
Mask Design

- Auto Correlation

\[ a(i) = \frac{(p - 1)}{4} (1 + \delta(i)) \]

- Power Spectrum

\[ A(j) = \frac{(p - 1)}{4} (\delta(j) + 1) \]
Coded Aperture Extensions

• Imaging Nearby Objects
• Mask / Countermask Combination
  * Dynamic Reconstruction
Coded Aperture Application

- Detection of Fissile Material
- Large Area Detector Myth
- Signal *and* Background Amplified
Large Area Alone Doesn’t Help
Imaging and Large Area Do!
Coded Aperture Example

- Imaging — $1/R$ instead of $1/R^2$
Coded Aperture Detector Array
Computational Imaging System
Dynamic Reconstruction

Three weak, distant radioactive sources

Reconstruction Animation
Coded Aperture Applications

- Detection of Fissile Material
- Imaging — $1/R$ instead of $1/R^2$
- Increasing Gamma Camera Resolution
- Replacing Rats with Mice
(4) EXACT CONE BEAM ALGORITHM

- Faster Scanning—Fewer Motion Artifacts
- Lower Exposure—Uniform Resolution

with: Xiaochun Yang
Exact Cone Beam Reconstruction

- Faster Scanning—Fewer Motion Artifacts
- Lower Exposure—Uniform Resolution

- Parallel Beam $\rightarrow$ Fan Beam
- Planar Fan $\rightarrow$ Cone Beam
Parallel Beam to Fan Beam

Coordinate Transform in 2D Radon Space
Cone Beam Geometry — 3D
Radon’s Formula

- In 2D: \( \sim \) derivatives of line integrals
- In 3D: derivatives of plane integrals
- Can’t get plane integrals from projections

\[
\int \left( \int f(r, \theta) \, dr \right) \, d\theta
\]

\[
\int \int \frac{1}{r} f(x, y) \, dx \, dy
\]
Radon’s Formula in 3D

\[ f(x) = -\frac{1}{8\pi^2} \int_{S^2} \left. \frac{\partial^2 R f(l, \beta)}{\partial l^2} \right|_{l=x \cdot \beta} d\beta \]

where

\[ R f(l, \beta) = \int f(x) \delta(x \cdot \beta - l) dV \]
Grangeat’s Trick

$$\frac{\partial}{\partial z} \iint f(x, y, z) \, dx \, dy = \frac{\partial}{\partial \theta} \iint f(r, \phi, \theta) \, dr \, d\phi$$
Exact Cone Beam Reconstruction

- Data Sufficiency Condition
- Good “Orbit” for Radiation Source
Radon Space — 2D
Circular Orbit is Inadequate (3D)
Data Insufficiency
Good Source Orbit
(3) DIAPHANOGRAPHY

(Diffuse Optical Imaging)

- Highly Scattering — Low Absorption
- Many Sources — Many Detectors

with: Xiaochun Yang, Richard Lanza, David Boas and Anna Custo.
Diaphanography

- Randomization of Direction

- *Scalar* Flux Density
Diffusion Approximation

\[-\nabla \cdot (\nu_d \nabla \Phi) + \mu_a c \Phi = q_0\]

where

\[\nu_d = \frac{c}{3(\mu_a + (1 - f)\mu_s)}\]

\(\mu_a\) is the absorption coefficient
\(\mu_s\) is the scattering coefficient
\(f\) is the anistropy factor
Diffusion Approximation

\[ \nabla^2 \Phi - k \Phi = -\frac{1}{\nu_d} q_0 \]

where \( k = 3\mu_a (\mu_a + (1 - f)\mu_s) \)

Green’s function (\( k \) constant, no boundary):

\[ \Phi = \frac{1}{r} e^{-\sqrt{k}r} \]
Diaphanography

- Approximation: Diffusion Equation

\[ \Delta v(x, y) + \rho(x, y)c(x, y) = 0 \]

\( v(x, y) \) flux density
\( \rho(x, y) \) scattering coefficient
\( c(x, y) \) absorption coefficient

- Forward: given \( c(x, y) \) find \( v(x, y) \)
3 × 3 node grid example

Here we have three input nodes (1, 2, 3), three output nodes (7, 8, 9), and three interior nodes (4, 5, 6). In three experiments we apply currents to each of the input nodes in turn, each time reading out all of the output nodes, yielding a total of nine measurements. We try and recover the nine leakage conductances to ground from each of the nine nodes.

It is natural to partition the conductance matrix as follows given that \( I_4, I_5, I_6, I_7, I_8, \) and \( I_9 \) are always zero, and that we do not measure \( V_1, V_2, V_3, V_4, V_5, \) and \( V_6 \).

\[
\begin{pmatrix}
G_{11} & G_{12} & G_{13} \\
\vdots & \vdots & \vdots \\
G_{31} & G_{32} & G_{33}
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_7 \\
V_8 \\
V_9
\end{pmatrix}
= 
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_7 \\
I_8 \\
I_9
\end{pmatrix}
\]

In detail

\[
\begin{pmatrix}
g'_1 & -g_{12} & -g_{13} & -g_{14} & 0 & 0 & 0 & 0 & 0 \\
-g_{12} & g'_2 & -g_{23} & 0 & -g_{25} & 0 & 0 & 0 & 0 \\
-g_{13} & -g_{23} & g'_3 & 0 & 0 & g_{36} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-g_{14} & 0 & 0 & g'_4 & -g_{45} & -g_{46} & -g_{47} & 0 & 0 \\
0 & -g_{25} & 0 & -g_{45} & g'_5 & -g_{56} & 0 & -g_{58} & 0 \\
0 & 0 & -g_{36} & -g_{46} & -g_{56} & g'_6 & 0 & 0 & -g_{69} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -g_{47} & 0 & 0 & g'_7 & -g_{78} & -g_{79} \\
0 & 0 & 0 & 0 & -g_{58} & 0 & -g_{78} & g'_8 & -g_{89} \\
0 & 0 & 0 & 0 & 0 & -g_{69} & -g_{79} & -g_{89} & g'_9
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_7 \\
V_8 \\
V_9
\end{pmatrix}
= 
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_7 \\
I_8 \\
I_9
\end{pmatrix}
\]

where \( g'_1 = (g_1 + g_{12} + g_{13} + g_{14}) \), \( g'_2 = (g_2 + g_{12} + g_{23} + g_{25}) \), \( g'_3 = (g_3 + g_{13} + g_{23} + g_{36}) \), \( g'_4 = (g_4 + g_{14} + g_{45} + g_{46} + g_{47}) \), \( g'_5 = (g_5 + g_{25} + g_{45} + g_{56} + g_{47}) \), \( g'_6 = (g_6 + g_{36} + g_{46} + g_{56} + g_{69}) \), \( g'_7 = (g_7 + g_{47} + g_{78} + g_{79}) \), \( g'_8 = (g_8 + g_{58} + g_{78} + g_{89}) \), and \( g'_9 = (g_9 + g_{69} + g_{79} + g_{89}) \).

We note that \( G_{13} \) and \( G_{31} \) are all zeros, and \( G_{12} = G_{21} \), and \( G_{23} = G_{32} \) are diagonal. Also, the sub-matrices appearing on the diagonal are of Toeplitz form. Toeplitz matrices can be inverted in order \( N^2 \) (instead of order \( N^3 \)).
### 3 × 3 node grid example

Here we have three input nodes (1, 2, 3), three output nodes (7, 8, 9), and three interior nodes (4, 5, 6). In three experiments we apply currents to each of the input nodes in turn, each time reading out all of the output nodes, yielding a total of nine measurements. We try and recover the nine leakage conductances to ground from each of the nine nodes.

It is natural to partition the conductance matrix as follows given that $I_4, I_5, I_6, I_7, I_8,$ and $I_9$ are always zero, and that we do not measure $V_1, V_2, V_3, V_4, V_5,$ and $V_6$.

\[
\begin{pmatrix}
G_{11} & G_{12} & G_{13} \\
\vdots & \ddots & \vdots \\
G_{21} & G_{22} & G_{23} \\
\vdots & \ddots & \vdots \\
G_{31} & G_{32} & G_{33}
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
\vdots \\
V_4 \\
V_5 \\
\vdots \\
V_7 \\
V_8 \\
V_9
\end{pmatrix} =
\begin{pmatrix}
I_1 \\
I_2 \\
\vdots \\
I_4 \\
I_5 \\
\vdots \\
I_7 \\
I_8 \\
I_9
\end{pmatrix}
\]

In detail

\[
\begin{pmatrix}
g'_1 & -g_{12} & -g_{13} & -g_{14} & 0 & 0 & 0 \\
-g_{12} & g'_2 & -g_{23} & 0 & -g_{25} & 0 & 0 \\
-g_{13} & -g_{23} & g'_3 & 0 & 0 & g_{36} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-g_{14} & 0 & 0 & g'_4 & -g_{45} & -g_{46} & -g_{47} \\
0 & -g_{25} & 0 & -g_{45} & g'_5 & -g_{56} & 0 \\
0 & 0 & -g_{36} & -g_{46} & -g_{56} & g'_6 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -g_{47} & 0 & 0 & g'_7 & -g_{78} & -g_{79} \\
0 & 0 & 0 & 0 & -g_{58} & 0 & -g_{78} & g'_8 & -g_{89} \\
0 & 0 & 0 & 0 & 0 & -g_{69} & -g_{79} & -g_{89} & g'_9
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_4 \\
V_5 \\
\vdots \\
V_7 \\
V_8 \\
V_9
\end{pmatrix} =
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_4 \\
I_5 \\
\vdots \\
I_7 \\
I_8 \\
I_9
\end{pmatrix}
\]

where $g'_1 = (g_1 + g_{12} + g_{13} + g_{14}), g'_2 = (g_2 + g_{12} + g_{23} + g_{25}), g'_3 = (g_3 + g_{13} + g_{23} + g_{36}), g'_4 = (g_4 + g_{14} + g_{45} + g_{46} + g_{47}), g'_5 = (g_5 + g_{25} + g_{45} + g_{56} + g_{47}), g'_6 = (g_6 + g_{36} + g_{46} + g_{56} + g_{69}), g'_7 = (g_7 + g_{47} + g_{78} + g_{79}), g'_8 = (g_8 + g_{58} + g_{78} + g_{89}),$ and $g'_9 = (g_9 + g_{69} + g_{79} + g_{89})$.

We note that $G_{13}$ and $G_{31}$ are all zeros, and $G_{12} = G_{21}$, and $G_{23} = G_{32}$ are diagonal. Also, the sub-matrices appearing on the diagonal are of Toeplitz form. Toeplitz matrices can be inverted in order $N^2$ (instead of order $N^3$).
We can write the inverse as follows:

\[
\begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{13} \\
\vdots & \ddots & \ddots & \vdots \\
C_{21} & C_{22} & \cdots & C_{23} \\
\vdots & \ddots & \ddots & \vdots \\
C_{31} & C_{32} & \cdots & C_{33}
\end{pmatrix}
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
\vdots \\
I_4 \\
\vdots \\
I_7 \\
I_8 \\
I_9
\end{pmatrix}
= 
\begin{pmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_4 \\
\vdots \\
V_7 \\
V_8 \\
V_9
\end{pmatrix}
\]

We can use the formula for the inverse of matrix partitioned into four parts twice on this matrix partitioned into nine parts. But it may be a bit much to expect to easily obtain explicit formulae the way we did for the \(2 \times 2\) case...

Note that we are only really interested in the bottom left corner \((C_{31})\) of the inverse, given that \(I_4, I_5, I_6, I_7, I_8,\) and \(I_9\) are always zero, and that we do not measure \(V_1, V_2, V_3, V_4, V_5,\) and \(V_6\). Each experiment yields three measurements and thus three equations of the form

\[
C_{31} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_7 \\ V_8 \\ V_9 \end{pmatrix}.
\]

By performing three experiments we can find all nine elements of the matrix \(C_{31}\). Each of these is a polynomial in the unknown leakage conductances \(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8,\) and \(g_9\) (or rather, we can cross-multiply to obtain nine such polynomials).

The part of the inverse of this conductance matrix that we need is the lower left corner, \(C_{31}\). Using the decomposition rule for partitioned \(2 \times 2\) matrices twice, we get

\[
C_{31} = G_{33}^{-1}G_{32}(G_{22} - G_{23}G_{33}^{-1}G_{21})^{-1}G_{21}(G_{11} - G_{12}(G_{22} - G_{23}G_{33}^{-1}G_{21})^{-1}G_{21})^{-1}
\]

Note that the term \((G_{22} - G_{23}G_{33}^{-1}G_{21})^{-1}G_{21}\) appears twice. This can be exploited to save on computation.
UPDATING RULE - "ASSIGNING BLAME"

FORWARD SOLUTION

\[ c^{(k)}(r) \]

\[ I(r_1, r_2) \]

\[ I'(r_1, r_2) \]
Diaphanography

- “Invert” Diffusion Equation

- Regions of Influence
(a) Total fluence [TD]

(b) Deviation of fluence

```
-20  -15  -10  -5
-20  -15  -10  -5
-20  -15  -10  -5
```

```
0.01  0.1  1.0
0.01  0.1  1.0
0.01  0.1  1.0
```

```
0.001  0.01  0.1  1.0
0.001  0.01  0.1  1.0
0.001  0.01  0.1  1.0
```

```
20 mm  30 mm  40 mm
20 mm  30 mm  40 mm
20 mm  30 mm  40 mm
```

```
0  0.5  1  1.5  2  2.5  3
0  0.5  1  1.5  2  2.5  3
0  0.5  1  1.5  2  2.5  3
```

```
0  0.5  1  1.5  2  2.5  3
0  0.5  1  1.5  2  2.5  3
0  0.5  1  1.5  2  2.5  3
```

```
0  0.5  1  1.5  2  2.5  3
0  0.5  1  1.5  2  2.5  3
0  0.5  1  1.5  2  2.5  3
```
COMPUTATIONAL IMAGING

(1) Synthetic Aperture Imaging
(2) Coded Aperture Imaging
(3) Exact Cone Beam Reconstruction
(4) Diaphanography—Diffuse Tomography
COMPUTATIONAL IMAGING
Synthetic Aperture Lithography

- Create pattern — controlled interference
  - Example: Two Dots
  - Example: Straight Line

- Destructive interference “safe zone”
  - Example: Bessel Ring
SAM M4
Interference Pattern Texture
Fourier Transform of Texture Pattern
Uneven Fourier Sampling
Amplitude and Phase Control
Cover image
Resolution Enhancement

• Reflective Optics Illumination

Vacuum UV — Short Wavelength
Resolution Enhancement

- Reflective Optics Illumination
  Vaccum UV — Short Wavelength

- Fluorescence Mode

  Resolution Determined by Illumination
Masks — Fresnel Camera

Fig. 1. X-ray star camera.
Masks — XRT Fine

[Image of a detailed mask pattern]
Masks — Hexagonal
Maximizing SNR

\[ \min \sum_{i=1}^{n} w_i^2 \quad \text{subject to} \quad \sum_{i=1}^{n} w_i = 1 \]

yields \( w_i = \frac{1}{n} \)
Spatially Varying Background

![Graph showing spatially varying background intensity](image-url)
Coded Aperture Extensions

- Artifacts due to Finite Distance
- Mask / Countermask Combination
- Multiple Detector Array Positions
- “Synthetic Aperture” Radiography
Dynamic Reconstruction
Exact Cone Beam Reconstruction

- Data Sufficiency Condition
- Good “Orbit” for Radiation Source
- Practical Issue: Spiral CT Scanners
- Practical Issue: “Long Body” Problem
Diaphanography

- Approximation: Diffusion Equation
- Leaky Resistive Sheet Analog (2D)