Epipolar Geometry

16-385 Computer Vision
Carnegie Mellon University (Kris Kitani)
Tie tiny threads on HERB and pin them to your eyeball

What would it look like?
You see points on HERB

What does the second observer see?
You see points on HERB

Second person sees lines
This is Epipolar Geometry
Epipolar geometry
Epipolar geometry

Image plane

Baseline

Point $p$
Epipolar geometry

Epipole (projection of o' on the image plane)

Baseline

Image plane

o

p

e

o'

e'
Epipolar geometry

Epipolar plane

Image plane

Epipole
(projection of o' on the image plane)
Epipolar geometry

Epipolar line (intersection of Epipolar plane and image plane)

Epipole (projection of o' on the image plane)
Potential matches for $x$ lie on the epipolar line $l'$.
The point \( x \) (left image) maps to a ___________ in the right image.

The baseline connects the ___________ and ____________

An epipolar line (left image) maps to a __________ in the right image

An epipole \( e \) is a projection of the ________________ on the image plane.

All epipolar lines in an image intersect at the ________________
Converging cameras

Where is the epipole in this image?
It's not always in the image

Where is the epipole in this image?

It's not always in the image
Parallel cameras

Where is the epipole?
Parallel cameras

epipole at infinity
Forward moving camera
Forward moving camera
Where is the epipole?

What do the epipolar lines look like?
Epipole has same coordinates in both images. Points move along lines radiating from “Focus of expansion”
The epipolar constraint is an important concept for stereo vision.

**Task:** Match point in left image to point in right image

How would you do it?
The epipolar constraint is an important concept for stereo vision

**Task:** Match point in left image to point in right image

Want to avoid search over entire image

(if the images have been rectified) Epipolar constrain reduces search to a single line
Essential Matrix

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Recall: Epipolar constraint

Potential matches for $x$ lie on the epipolar line $l'$. 
The epipolar geometry is an important concept for stereo vision.

**Task:** Match point in left image to point in right image.

*How would you do it?*
The epipolar constraint is an important concept for stereo vision

**Task:** Match point in left image to point in right image

Epipolar constrain reduces search to a single line

*How do you compute the epipolar line?*
Essential Matrix

The Essential Matrix is a $3 \times 3$ matrix that encodes epipolar geometry.
Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

\[ Ex = l' \]
Epipolar Line

The equation of the epipolar line in vector form is given by:

\[ ax + by + c = 0 \]

If the point \( x \) is on the epipolar line \( l \), then

\[ x^\top l = ? \]

Where \( l = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \).
Epipolar Line

\[ ax + by + c = 0 \]  \hspace{2cm} \text{in vector form} \hspace{2cm} l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}

If the point \( x \) is on the epipolar line \( l \) then

\[ x^\top l = 0 \]
Recall: Dot Product

\[ c = a \times b \]

\[ c \cdot a = 0 \quad c \cdot b = 0 \]
So if $\mathbf{x}^\top l = 0$ and $\mathbf{E}\mathbf{x} = l'$ then

$$\mathbf{x'}^\top \mathbf{E}\mathbf{x} = ?$$
So if $\mathbf{x}^\top \mathbf{l} = 0$ and $\mathbf{E} \mathbf{x} = \mathbf{l}'$ then

$$\mathbf{x}'^\top \mathbf{E} \mathbf{x} = 0$$
Motivation

The Essential Matrix is a 3 x 3 matrix that encodes **epipolar geometry**

Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.
Essential Matrix vs Homography

What’s the difference between the essential matrix and a homography?
Essential Matrix vs Homography

What’s the difference between the essential matrix and a homography?

They are both 3 x 3 matrices but …
Essential Matrix vs Homography

What’s the difference between the essential matrix and a homography?

They are both $3 \times 3$ matrices but …

$$l' = Ex$$

Essential matrix maps a **point** to a **line**

$$x' = Hx$$

Homography maps a **point** to a **point**
Where does the Essential matrix come from?
\[ x' = R(x - t) \]
$x' = R(x - t)$

Does this look familiar?
Camera-camera transform just like world-camera transform

\[ x' = R(x - t) \]
These three vectors are coplanar

\[ \mathbf{x}, \mathbf{t}, \mathbf{x}' \]
If these three vectors are coplanar \( \mathbf{x}, t, \mathbf{x}' \) then

\[
\mathbf{x}^\top (t \times \mathbf{x}) = ?
\]
If these three vectors are coplanar $\mathbf{x}, \mathbf{t}, \mathbf{x}'$ then

$$\mathbf{x}^\top (\mathbf{t} \times \mathbf{x}) = 0$$
Recall: Cross Product

Vector (cross) product
takes two vectors and returns a vector perpendicular to both

\[ c = a \times b \]

\[ c \cdot a = 0 \quad c \cdot b = 0 \]
If these three vectors are coplanar $x, t, x'$ then

$$(x - t)^\top (t \times x) = ?$$
If these three vectors are coplanar \( x, t, x' \) then

\[
(x - t) \mathbf{T} (t \times x) = 0
\]
putting it together

rigid motion

$$x' = R(x - t)$$

coplanarity

$$(x - t)^\top (t \times x) = 0$$

$$(x'\top R)(t \times x) = 0$$
putting it together

**Rigid motion**

\[ x' = R(x - t) \]

**Coplanarity**

\[ (x - t)^\top (t \times x) = 0 \]

\[ (x'\top R)(t \times x) = 0 \]

\[ (x'\top R)([t \times] x) = 0 \]
\[ \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \]

Can also be written as a matrix multiplication

\[ \mathbf{a} \times \mathbf{b} = [\mathbf{a}] \times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \]

Skew symmetric
putting it together

rigid motion

\[ x' = R(x - t) \]

coplanarity

\[ (x - t)^T (t \times x) = 0 \]

\[ (x' \, ^T \, R)(t \times x) = 0 \]

\[ (x' \, ^T \, R)([t \times] x) = 0 \]

\[ x' \, ^T \,(R[t \times ] )x = 0 \]
putting it together

rigid motion

\[ x' = R(x - t) \]

co-planarity

\[ (x - t)\top (t \times x) = 0 \]

\[ (x'\top R)(t \times x) = 0 \]

\[ (x'\top R)([t \times] x) = 0 \]

\[ x'\top (R[t \times]) x = 0 \]

\[ x'\top E x = 0 \]
putting it together

rigid motion

\[ x' = R(x - t) \]

coplanarity

\[ (x - t)^\top (t \times x) = 0 \]

\[ (x'\top R)(t \times x) = 0 \]

\[ (x'\top R)([t \times]x) = 0 \]

\[ x'\top (R[t \times])x = 0 \]

\[ x'\top Ex = 0 \]

Essential Matrix

[Longuet-Higgins 1981]
properties of the E matrix

Longuet-Higgins equation

\[ x'^\top E x = 0 \]

(points in normalized coordinates)
properties of the E matrix

Longuet-Higgins equation
\[ x'^\top E x = 0 \]

Epipolar lines
\[ x^\top l = 0 \]
\[ l' = E x \]
\[ x'^\top l' = 0 \]
\[ l = E^T x' \]

(points in normalized coordinates)
properties of the E matrix

Longuett-Higgins equation

\[ x'\top E x = 0 \]

Epipolar lines

\[ x \top l = 0 \quad x' \top l' = 0 \]
\[ l' = E x \quad l = E^T x' \]

Epipoles

\[ e' \top E = 0 \quad E e = 0 \]

(points in normalized coordinates)
How do you generalize to uncalibrated cameras?
Fundamental Matrix

16-385 Computer Vision
Recall: Epipolar constraint

Potential matches for $x$ lie on the epipolar line $l'$. 
Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

\[ E \mathbf{x} = l' \]

**Assumption:**
points aligned to camera coordinate axis (calibrated camera)
The Fundamental matrix is a generalization of the Essential matrix, where the assumption of calibrated cameras is removed.
The Essential matrix operates on image points expressed in **normalized coordinates**
(points have been aligned (normalized) to camera coordinates)

\[
\hat{x}'^\top E \hat{x} = 0
\]

Writing out the epipolar constraint in terms of image coordinates

\[
\hat{x} = K^{-1} x
\]

\[
\text{camera point}
\]

\[
\hat{x}' = K^{-1} x'
\]

\[
\text{image point}
\]

Writing out the epipolar constraint in terms of image coordinates

\[
x'\top K'_{-1}^\top E K^{-1} x = 0
\]

\[
x'\top (K'_{-1}^\top E K^{-1}) x = 0
\]

\[
x'\top F x = 0
\]
Same equation works in image coordinates!

\[ x'\top Fx = 0 \]

it maps pixels to epipolar lines
properties of the $E$ matrix

**Longuet-Higgins equation**

\[ x'^\top E x = 0 \]

**Epipolar lines**

\[ x^\top l = 0 \quad x'^\top l' = 0 \]
\[ l' = E x \quad l = E^T x' \]

**Epipoles**

\[ e'^\top E = 0 \quad E e = 0 \]

(points in **image** coordinates)
Breaking down the fundamental matrix

\[ F = K'^{-\top} E K^{-1} \]

\[ F = K'^{-\top} [t \times] R K^{-1} \]

Depends on both intrinsic and extrinsic parameters
Breaking down the fundamental matrix

\[ F = K^{-	op} EK^{-1} \]

\[ F = K'^{-	op} [t \times] RK^{-1} \]

Depends on both intrinsic and extrinsic parameters

*How would you solve for \( F \)?*

\[ x'_m^\top F x_m = 0 \]