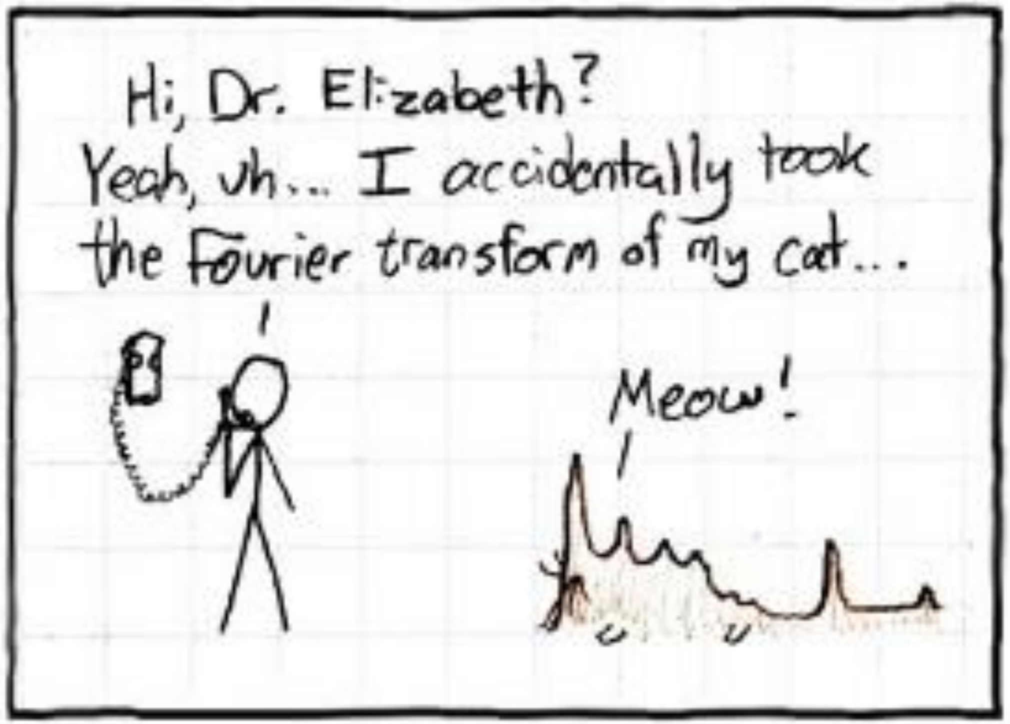
Fourier Transform and Frequency Domain



http://www.cs.cmu.edu/~16385/

16-385 Computer Vision
Spring 2018, Lecture 3 (part 2)

Overview of today’s lecture

- Some history.
- Fourier series.
- Frequency domain.
- Fourier transform.
- Frequency-domain filtering.
- Revisiting sampling.

Slide credits

Most of these slides were adapted from:

- Kris Kitani (15-463, Fall 2016).

Some slides were inspired or taken from:

- Fredo Durand (MIT).
- James Hays (Georgia Tech).

Some history

Who is this guy?

What is he famous for?

Jean Baptiste Joseph Fourier
(1768-1830)

What is he famous for?

Jean Baptiste Joseph Fourier
(1768-1830)

The Fourier series claim (1807):

‘Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.’

… and apparently also for the discovery of the greenhouse effect

Is this claim true?

Jean Baptiste Joseph Fourier
(1768-1830)

The Fourier series claim (1807):

> *'<u>Any</u> univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'*

Is this claim true?

Jean Baptiste Joseph Fourier
(1768-1830)

The Fourier series claim (1807):

> *'Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'*

Well, almost.

- The theorem requires additional conditions.
- Close enough to be named after him.
- Very surprising result at the time.

Is this claim true?

Jean Baptiste Joseph Fourier
(1768-1830)

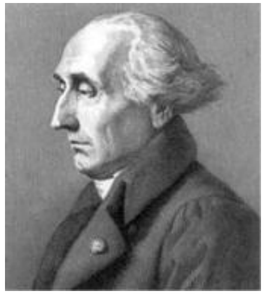
The Fourier series claim (1807):

> *'<u>Any</u> univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'*

Well, almost.

- The theorem requires additional conditions.
- Close enough to be named after him.
- Very surprising result at the time.

Malus



Lagrange

Legendre

Laplace

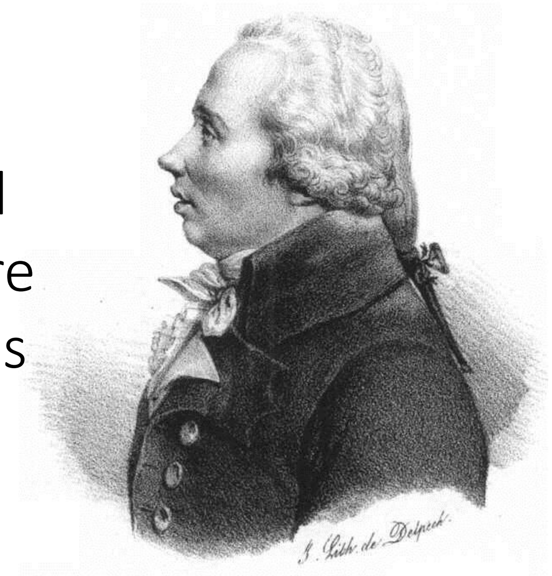
The committee examining his paper had expressed skepticism, in part due to not so rigorous proofs

Amusing aside

Only known portrait of Adrien-Marie Legendre

1820 watercolor caricatures of French mathematicians Adrien-Marie Legendre (left) and Joseph Fourier (right) by French artist Julien-Leopold Boilly

For two hundred years, people were misidentifying this portrait as him



Louis Legendre (same last name, different person)

Fourier series

Basic building block

$$A \sin(\omega x + \phi)$$

Fourier's claim: Add enough of these to get <u>any</u> *periodic* signal you want!

Basic building block

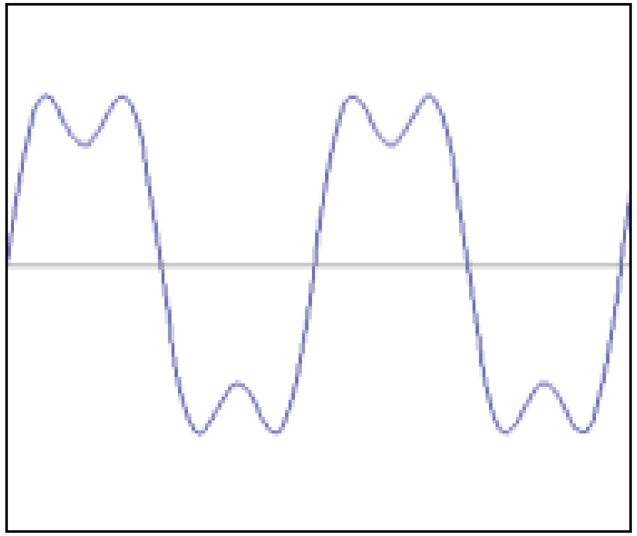
$$A\sin(\omega x + \phi)$$

amplitude
sinusoid
angular frequency
variable
phase

Fourier's claim: Add enough of these to get any *periodic* signal you want!

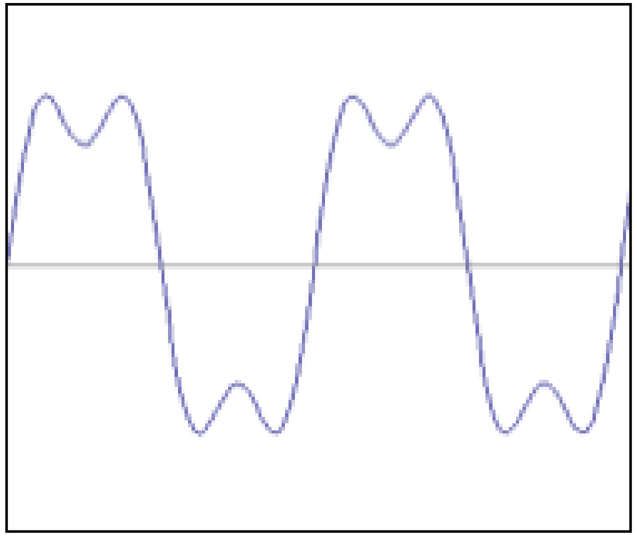
Examples

How would you generate this function?

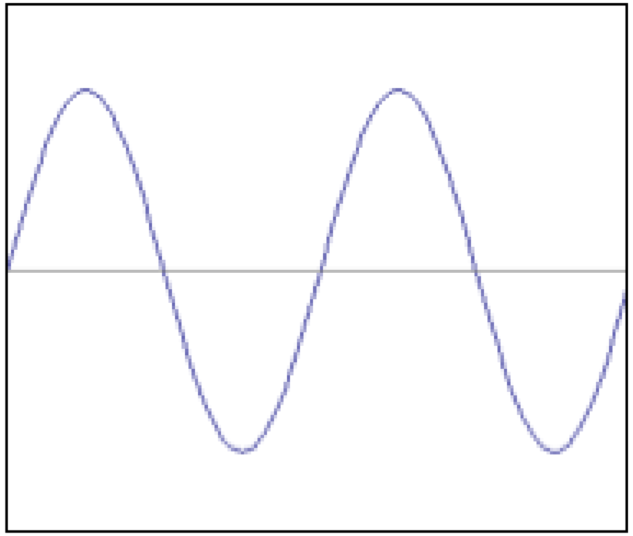
 = ? + ?

Examples

How would you generate this function?



=

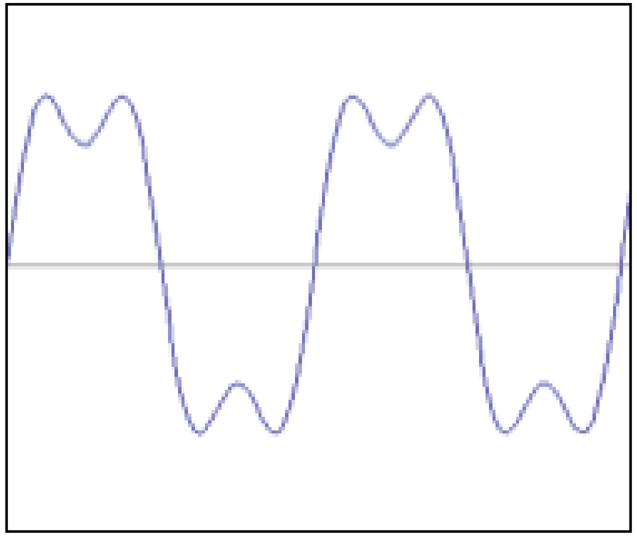


$\sin(2\pi x)$

\+ ?

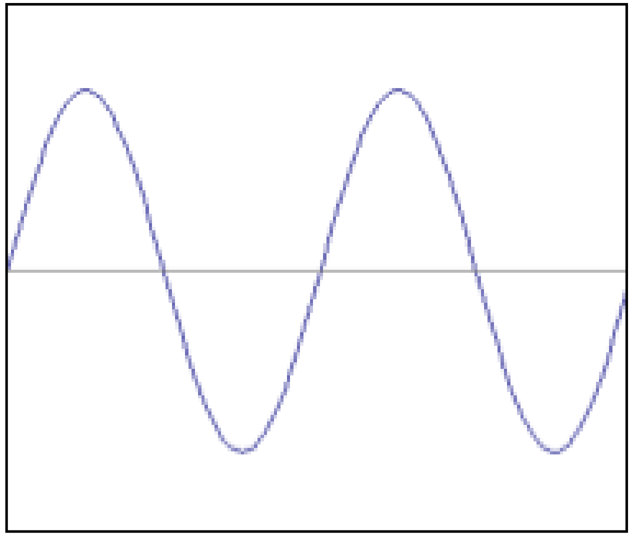
Examples

How would you generate this function?



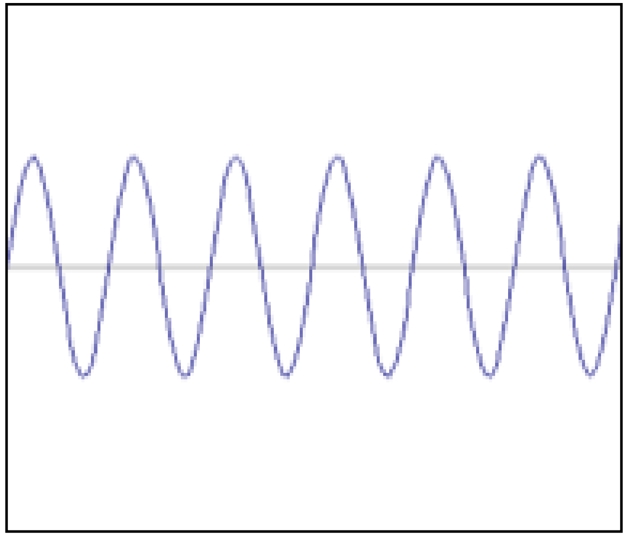
$$f(x) = \sin(2\pi x) + \frac{1}{3}\sin(2\pi 3x)$$

=



$$\sin(2\pi x)$$

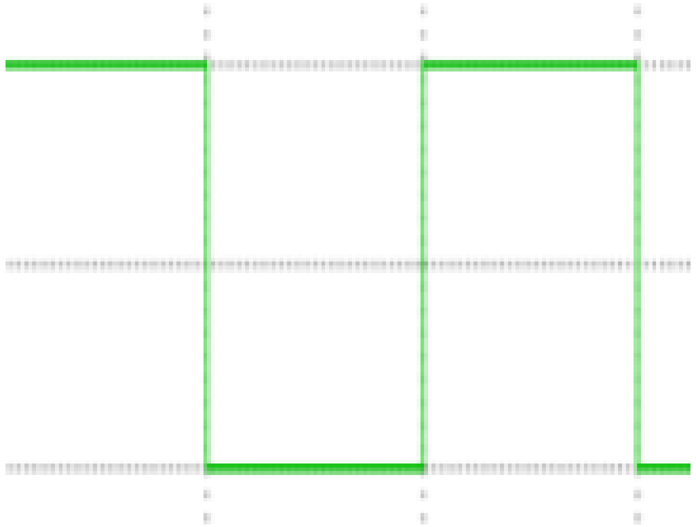
+



$$\frac{1}{3}\sin(2\pi 3x)$$

Examples

How would you generate this function?

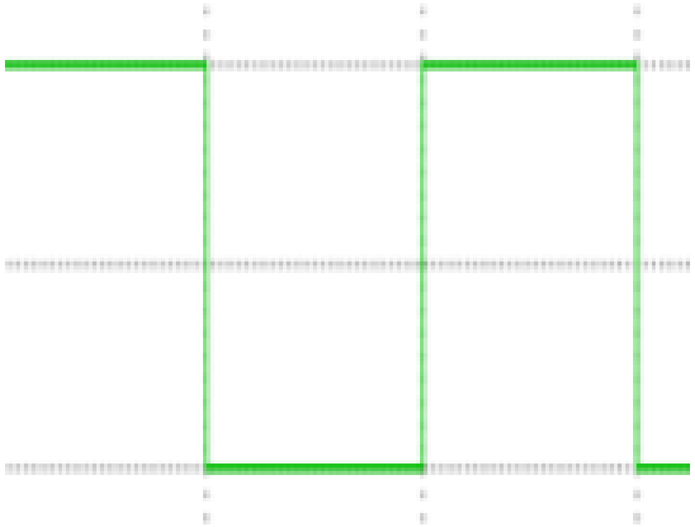


square wave

$$= \qquad ? \qquad + \qquad ?$$

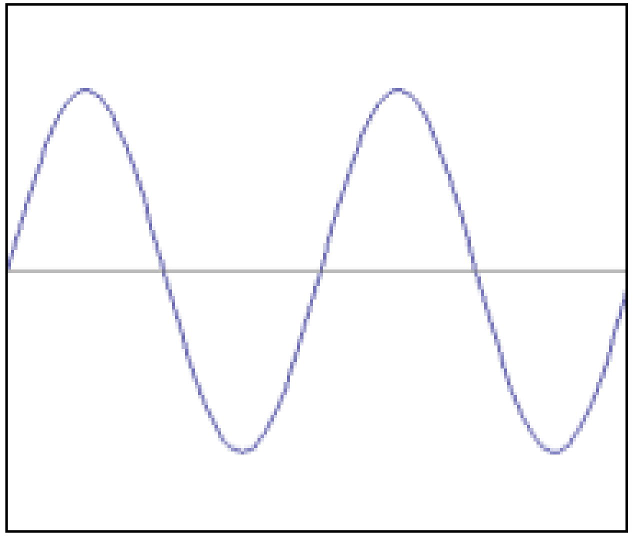
Examples

How would you generate this function?

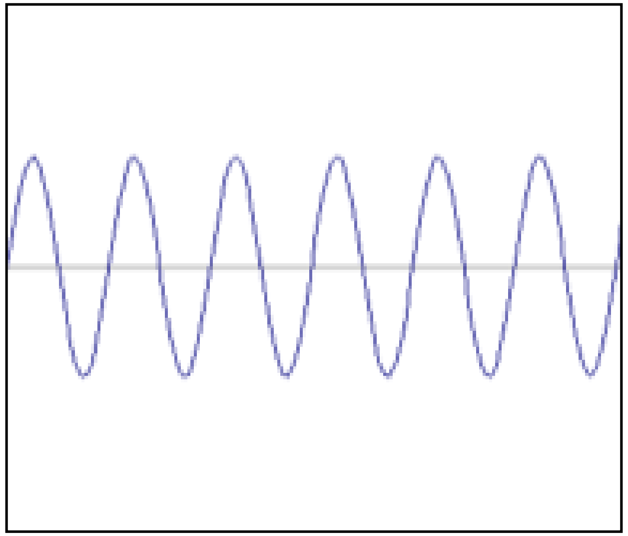


square wave

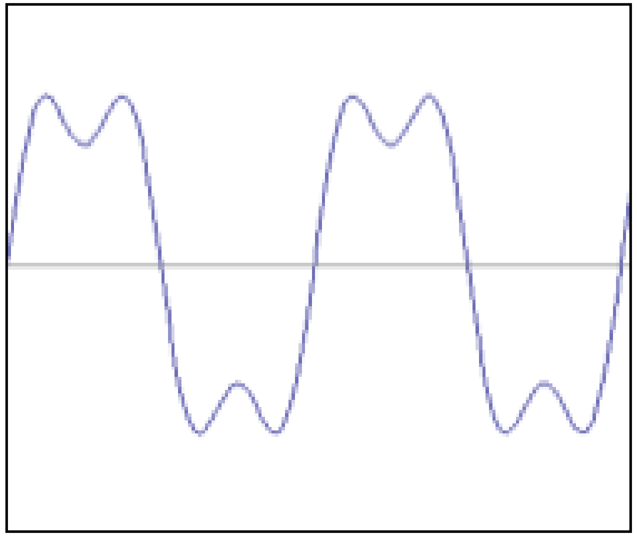
$\approx$



$+$

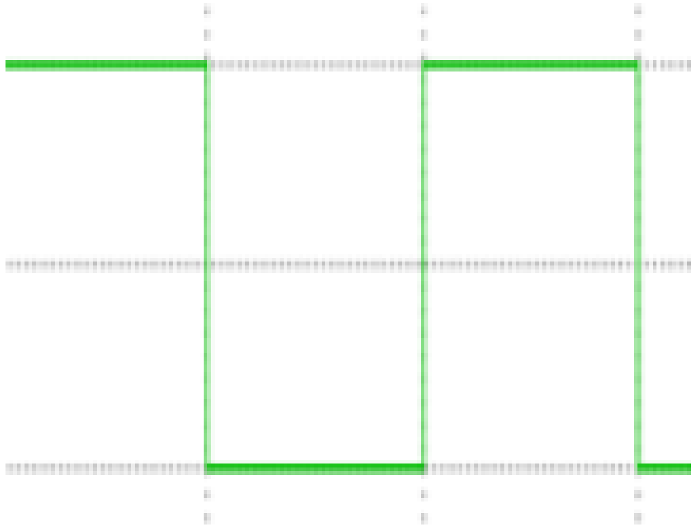


$=$



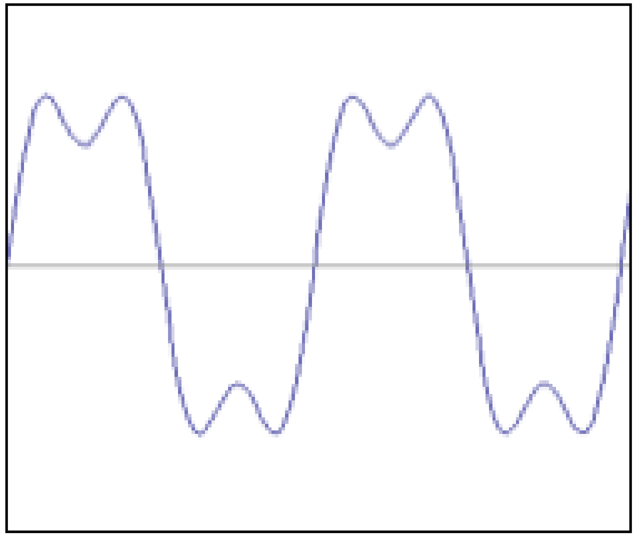
Examples

How would you generate this function?

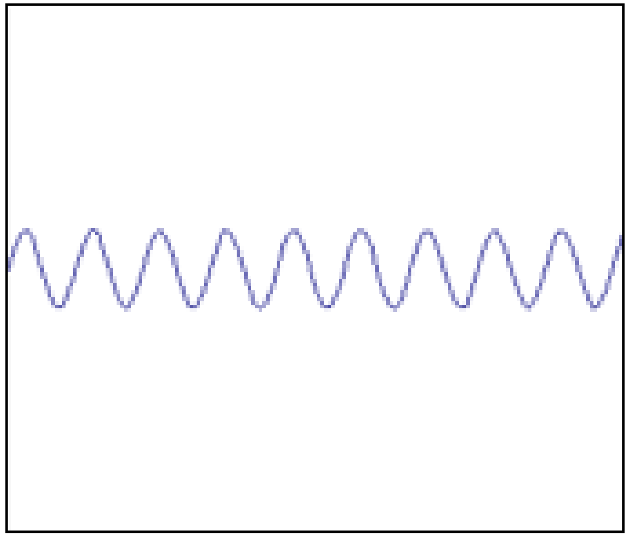


square wave

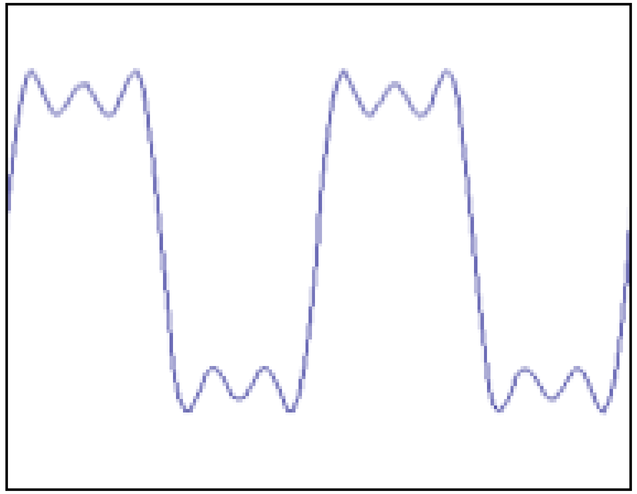
≈



+

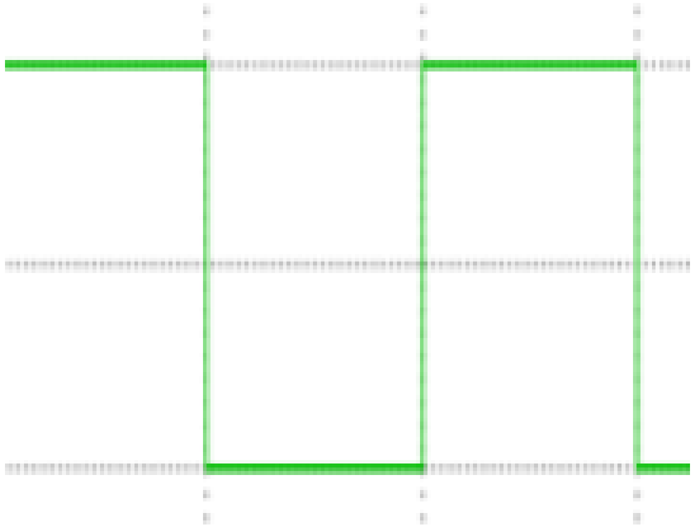


=



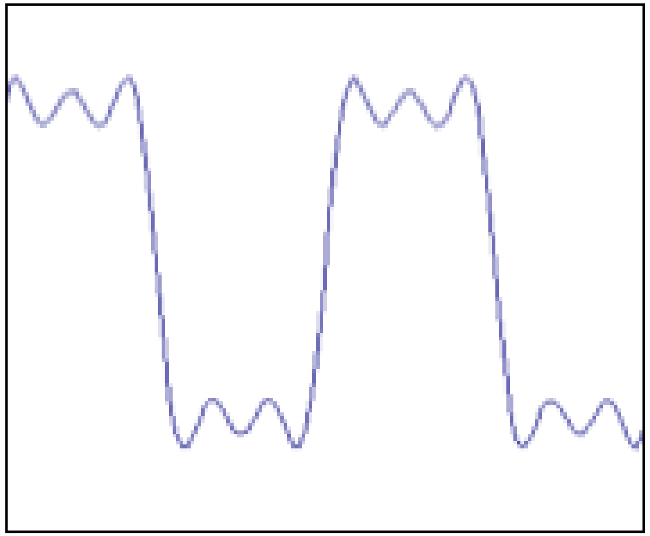
Examples

How would you generate this function?

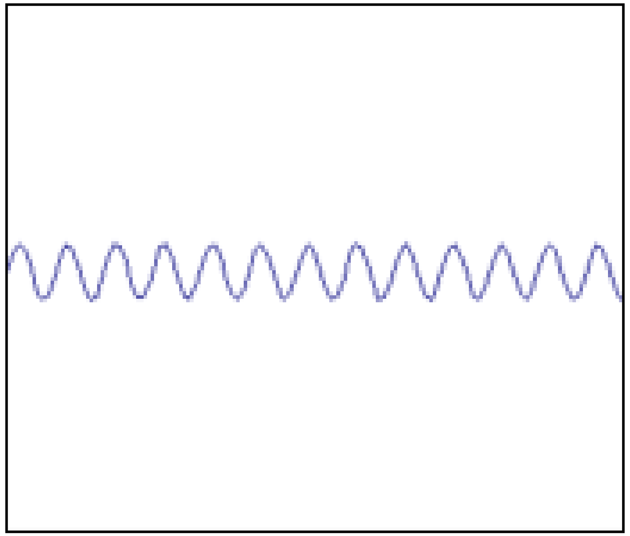


square wave

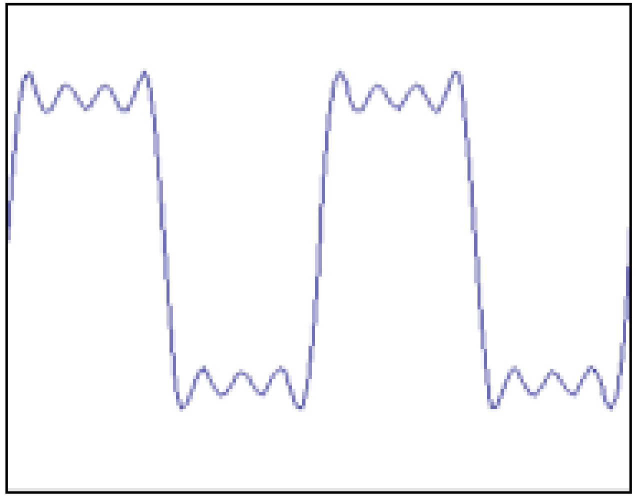
≈



+

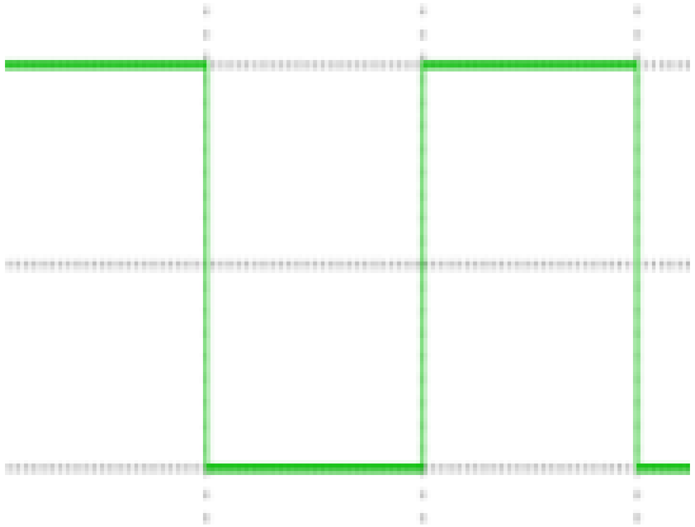


=



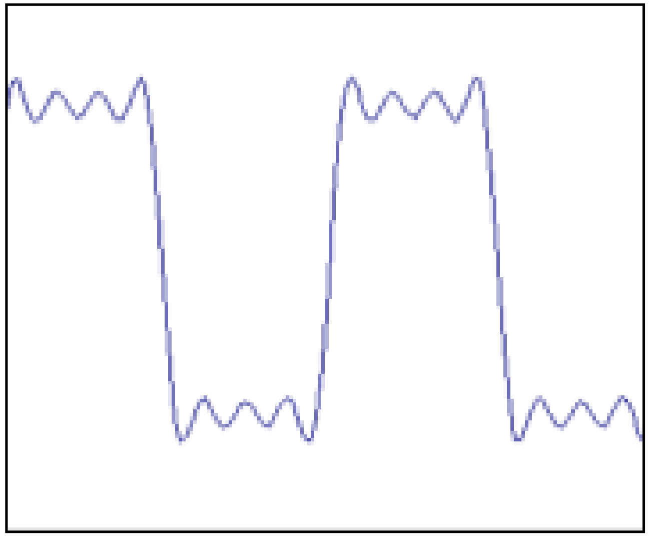
Examples

How would you generate this function?

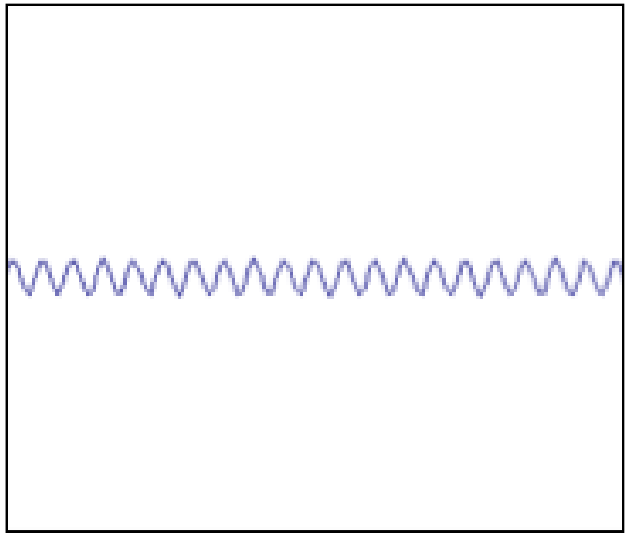


square wave

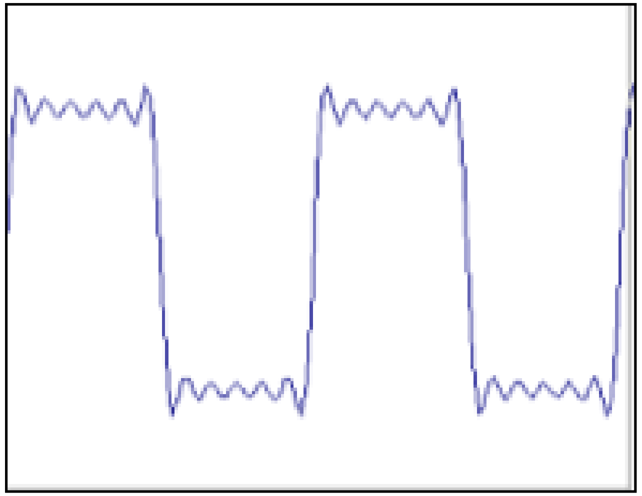
≈



+

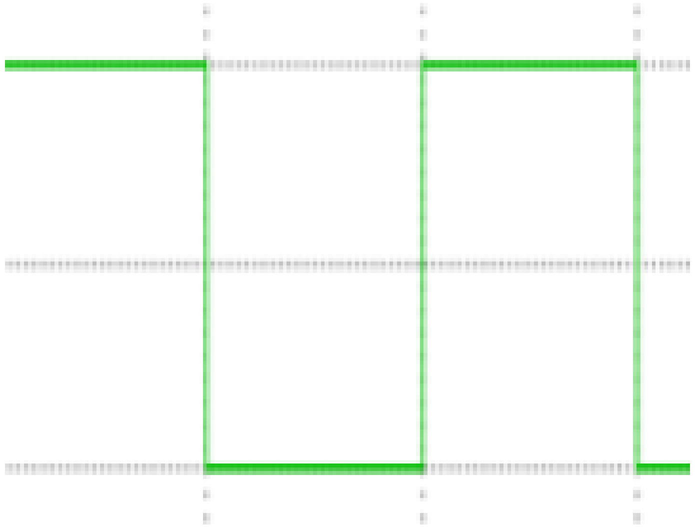


=



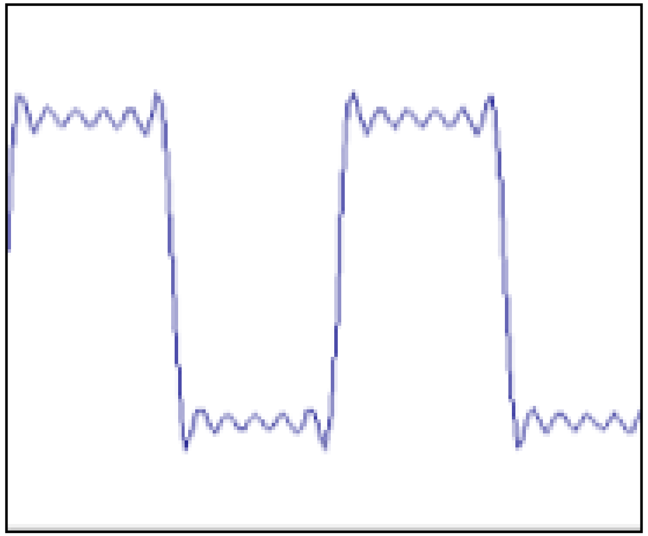
Examples

How would you generate this function?

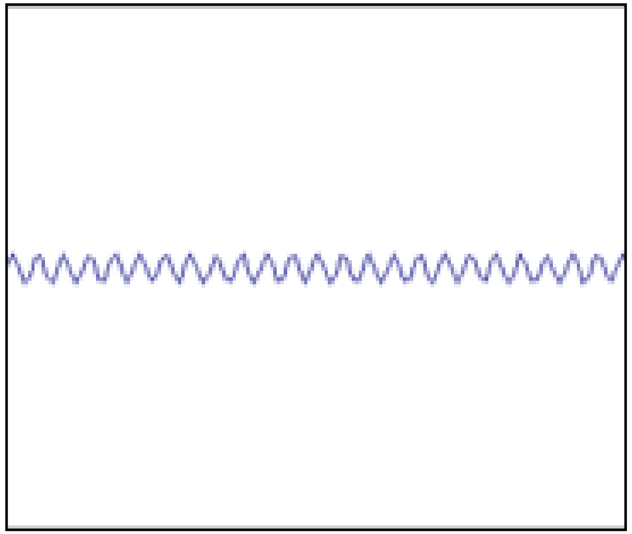


square wave

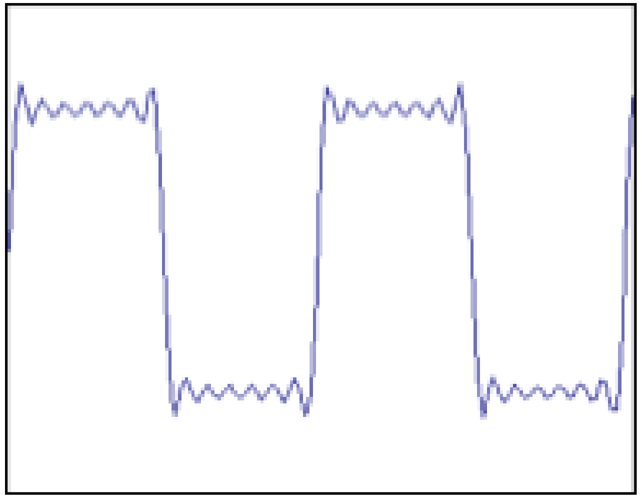
$\approx$



$+$

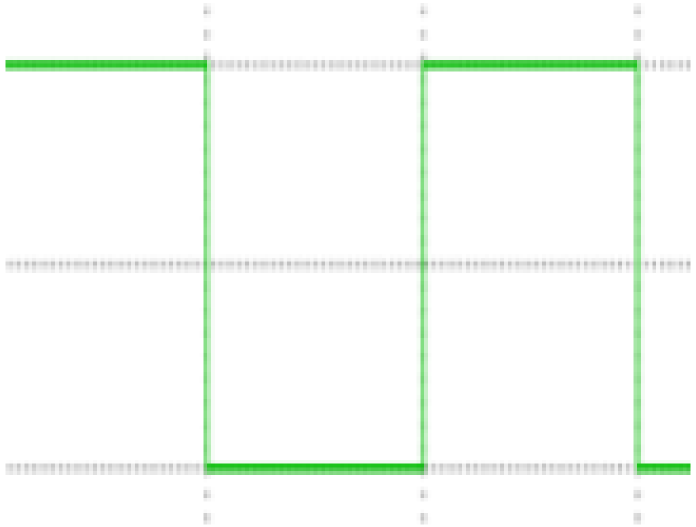


$=$



How would you express this mathematically?

Examples



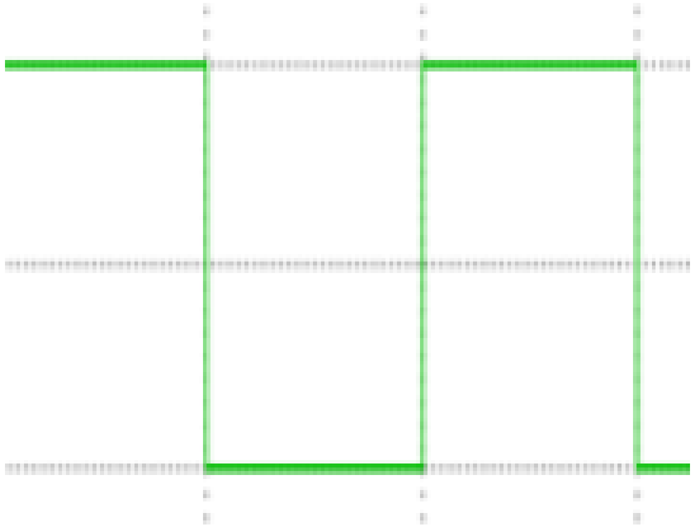
square wave

$$= \quad A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k x)$$

infinite sum of sine waves

How would could you visualize this in the frequency domain?

Examples

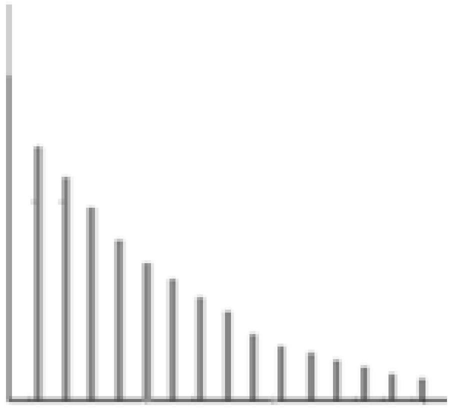


$$= \quad A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kx)$$

square wave

infinite sum of sine waves

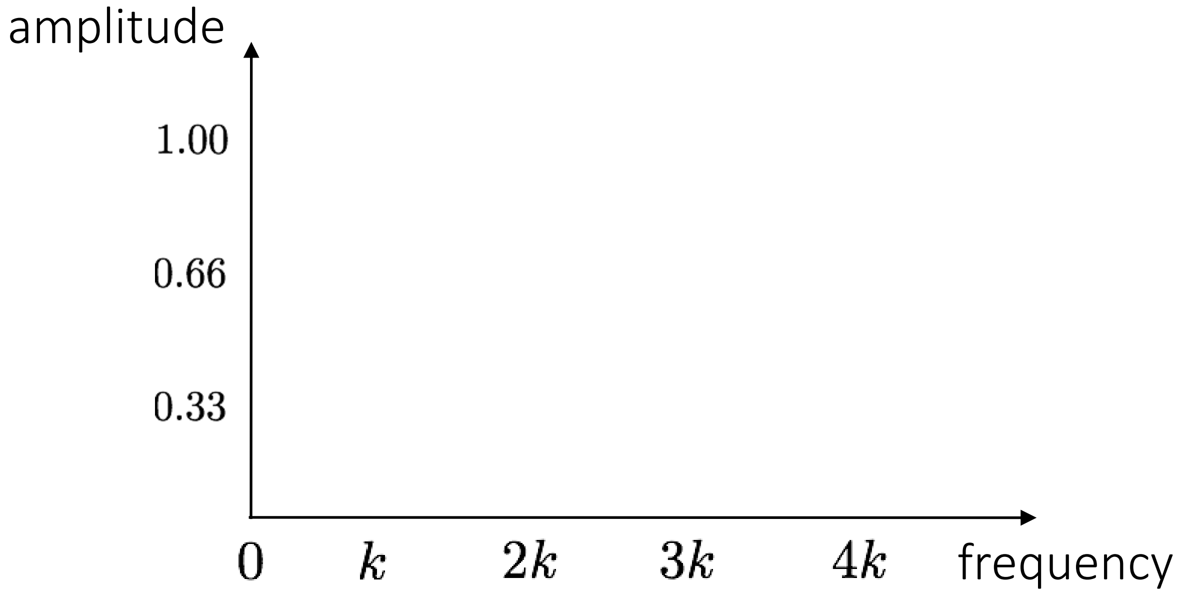
magnitude



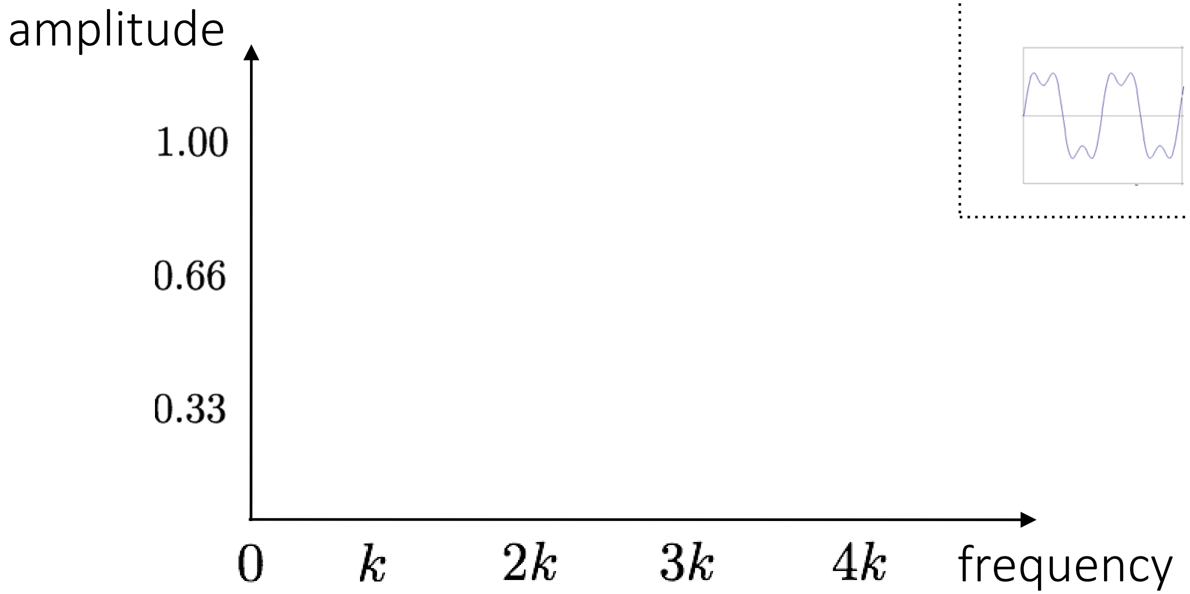
frequency

Frequency domain

Visualizing the frequency spectrum

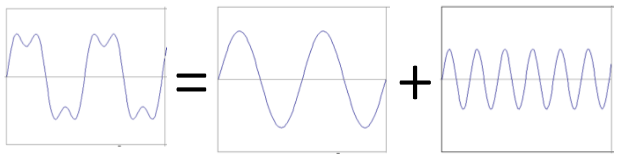


Visualizing the frequency spectrum



Recall the temporal domain visualization

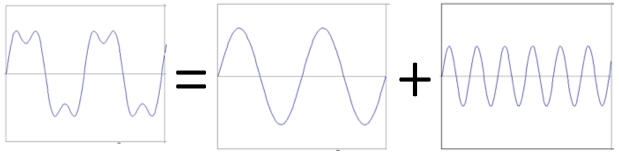
$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$

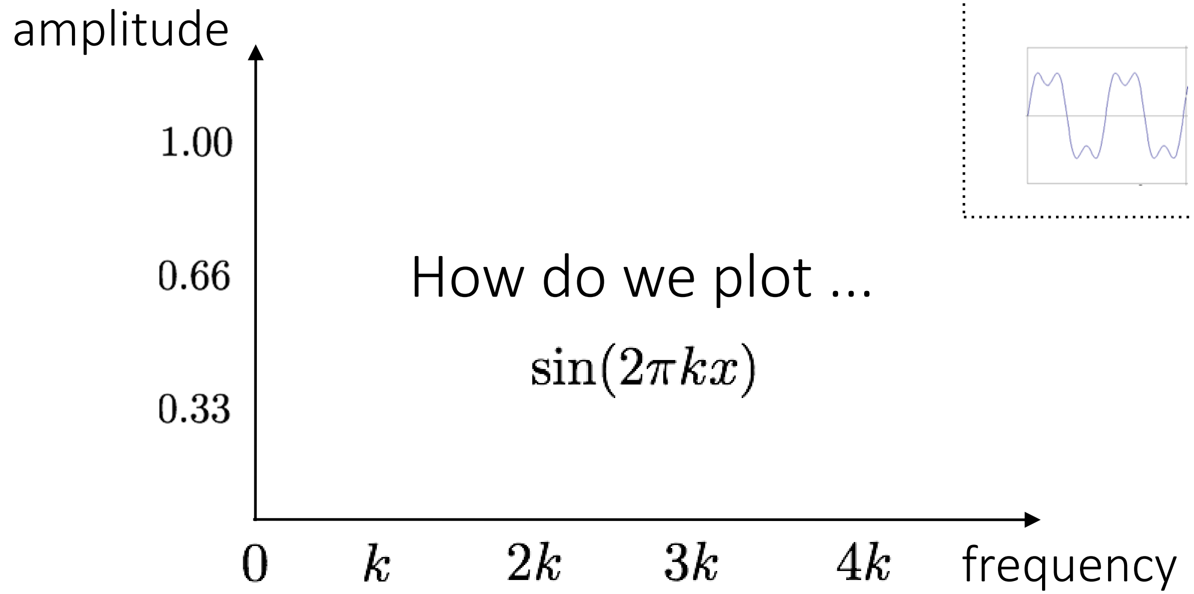


Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$

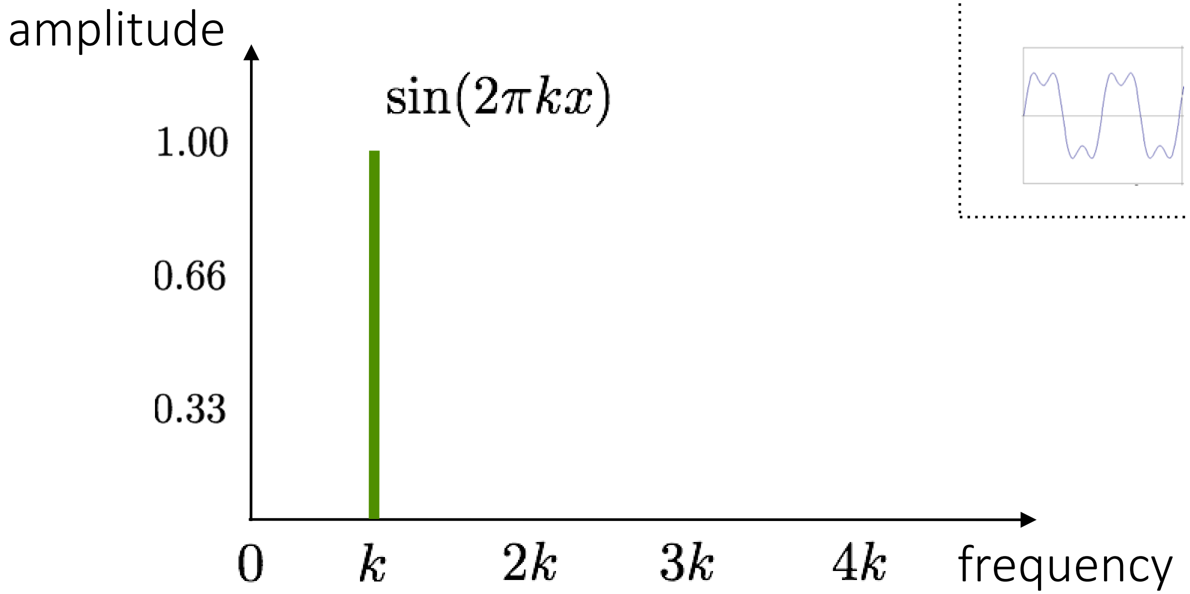




How do we plot ...

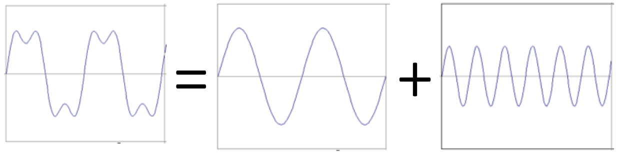
$$\sin(2\pi kx)$$

Visualizing the frequency spectrum

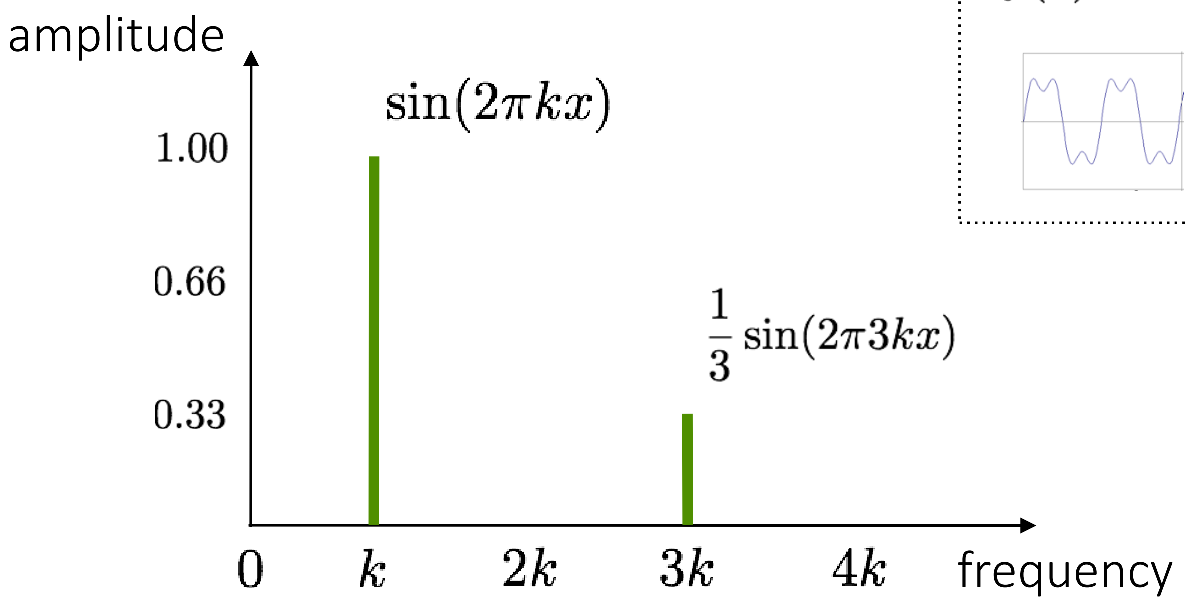


Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$

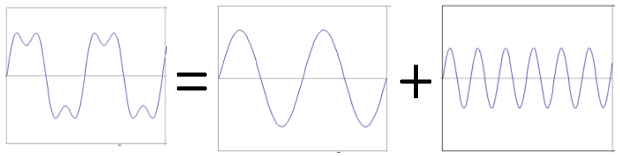


Visualizing the frequency spectrum



Recall the temporal domain visualization

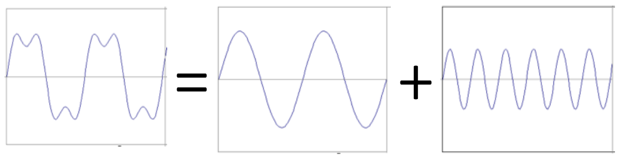
$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$

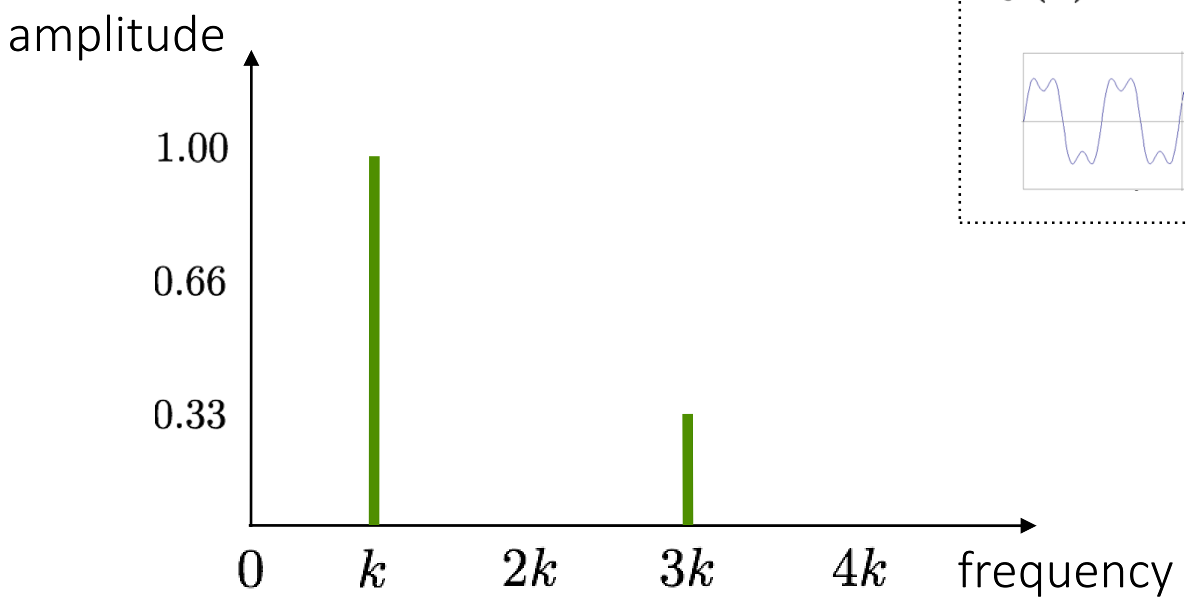


Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3}\sin(2\pi 3kx)$$





not visualizing the symmetric negative part

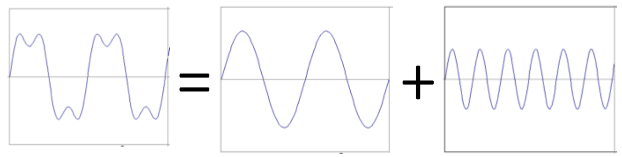
What is at zero frequency?

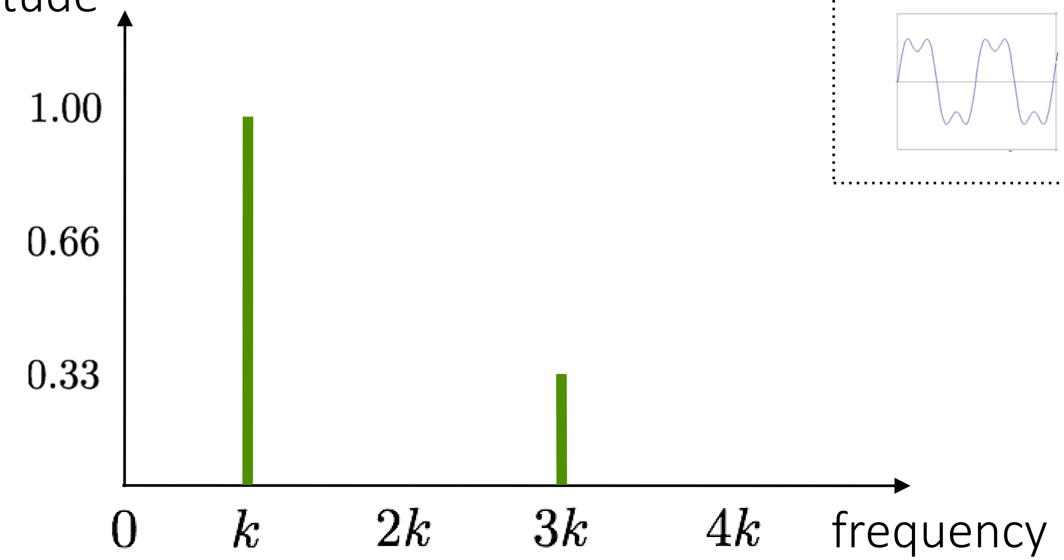
Need to understand this to understand the 2D version!

Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi k x) + \frac{1}{3}\sin(2\pi 3 k x)$$





not visualizing the symmetric negative part

signal average (zero for a sine wave with no offset)

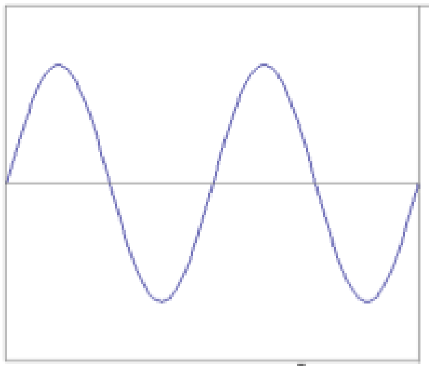
Need to understand this to understand the 2D version!

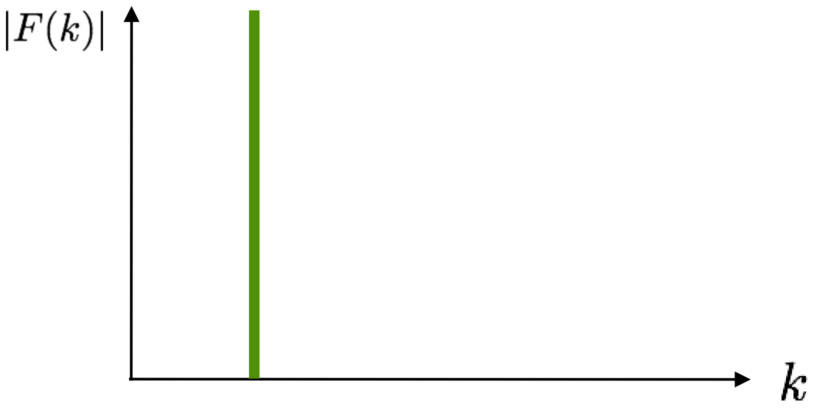
Examples

Spatial domain visualization

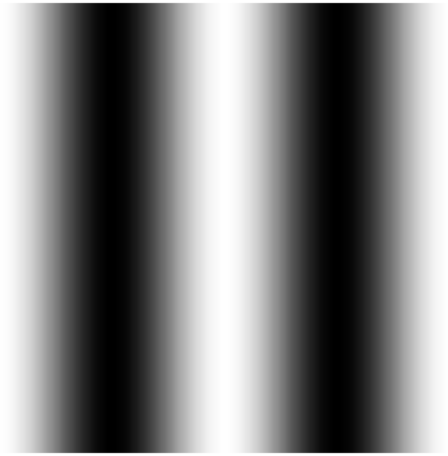
Frequency domain visualization

1D





2D

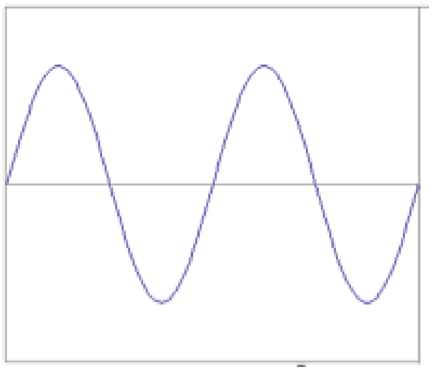


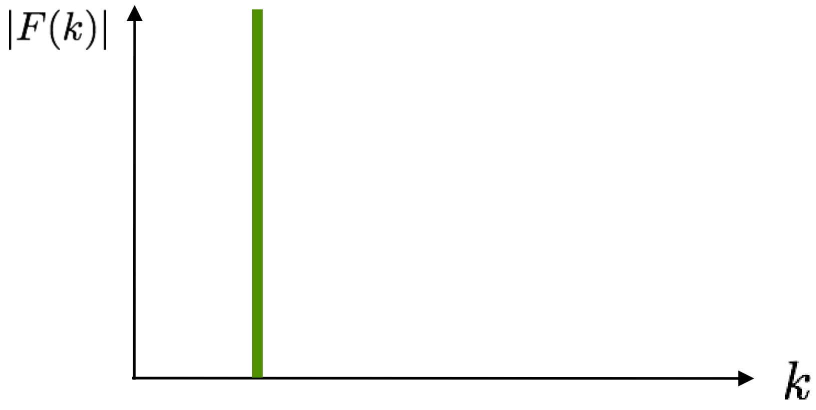
?

Examples

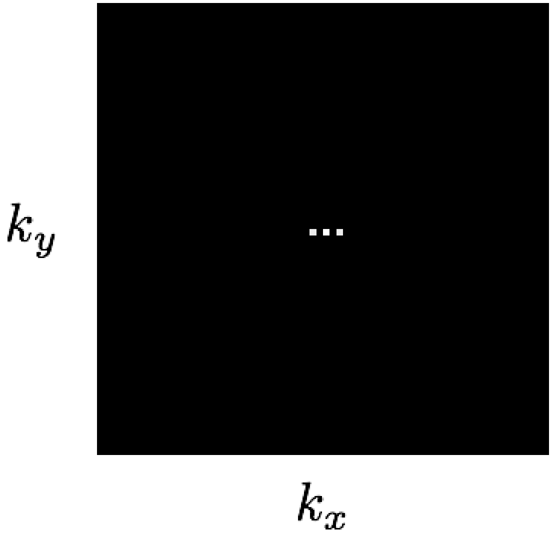
Spatial domain visualization | Frequency domain visualization

1D





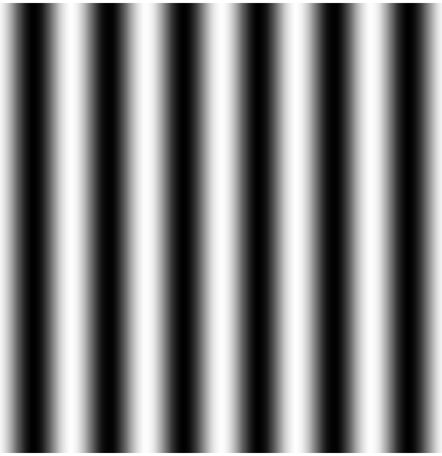
2D



What do the three dots correspond to?

Examples

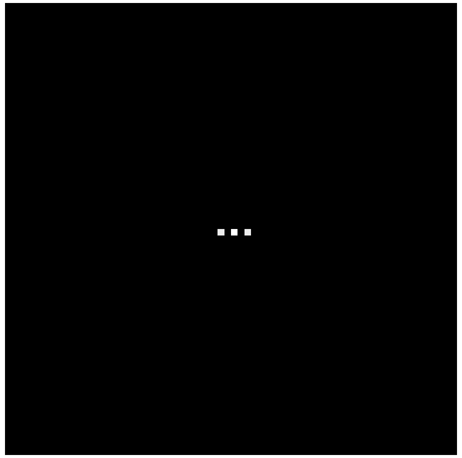
Spatial domain visualization



Frequency domain visualization

?

k_y

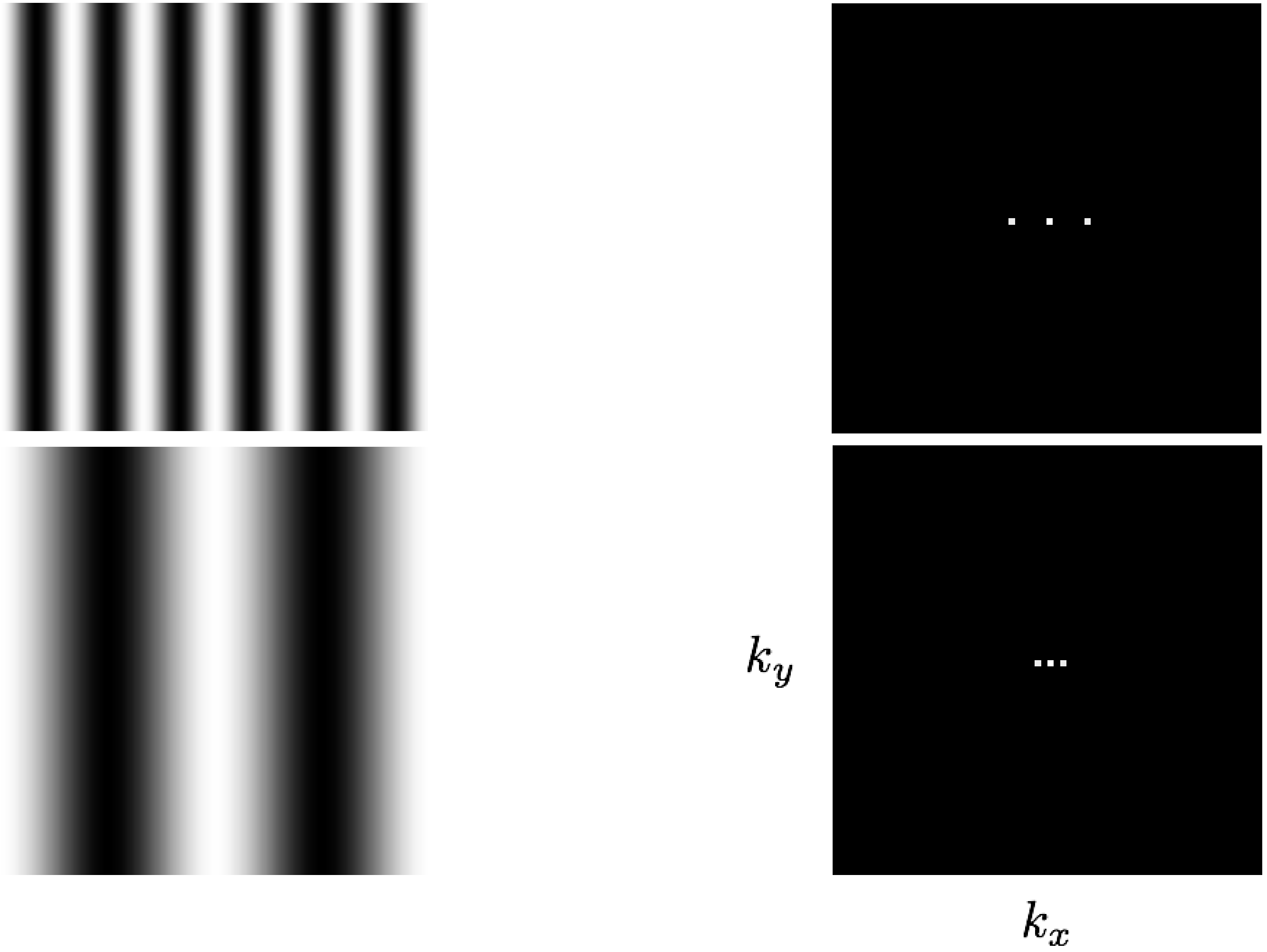


k_x

Examples

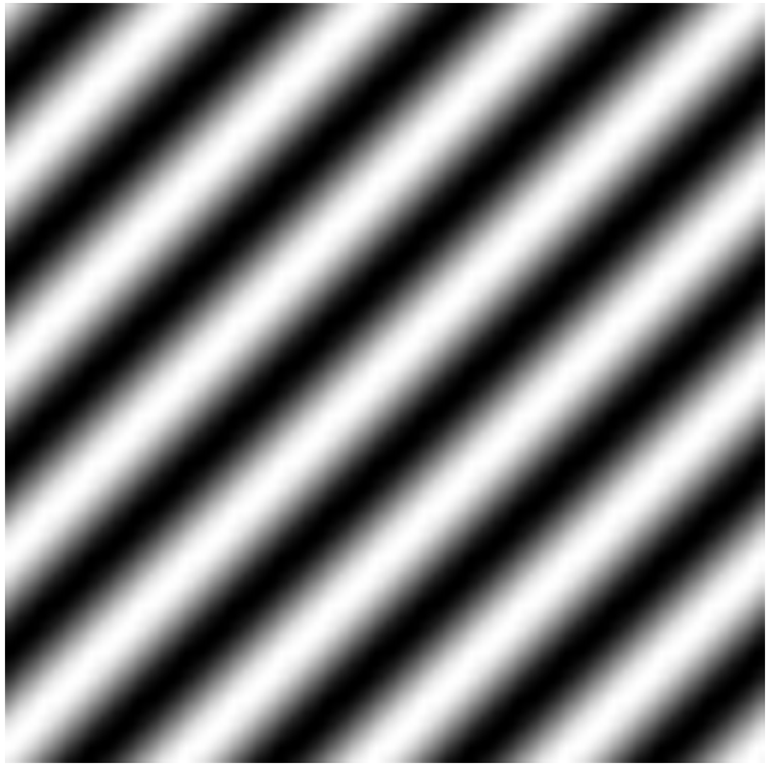
Spatial domain visualization

Frequency domain visualization



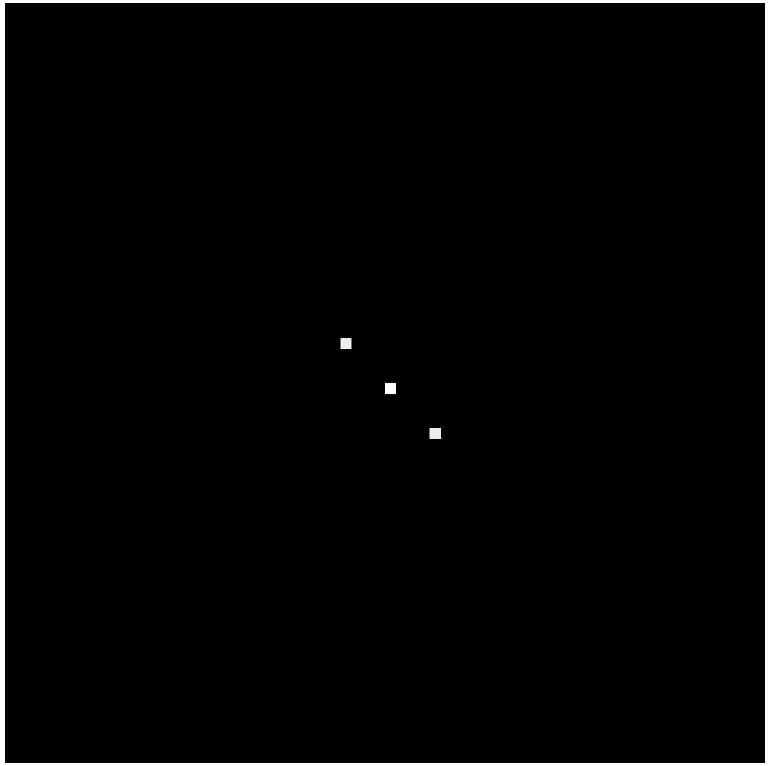
Examples

How would you generate this image with sine waves?



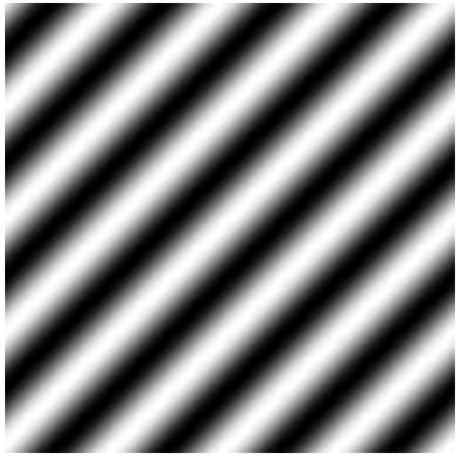
Examples

How would you generate this image with sine waves?

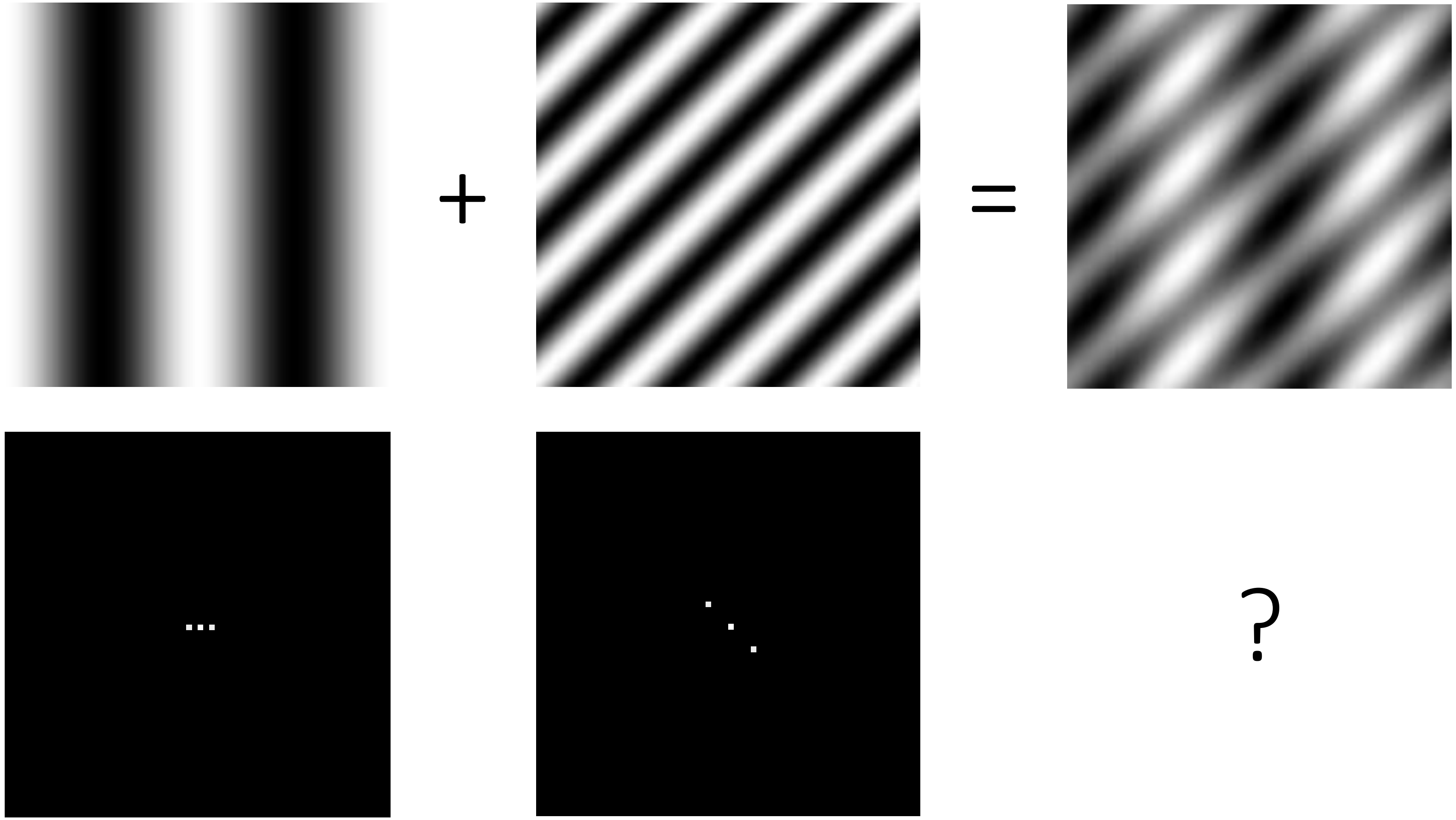


Has both an x and y components

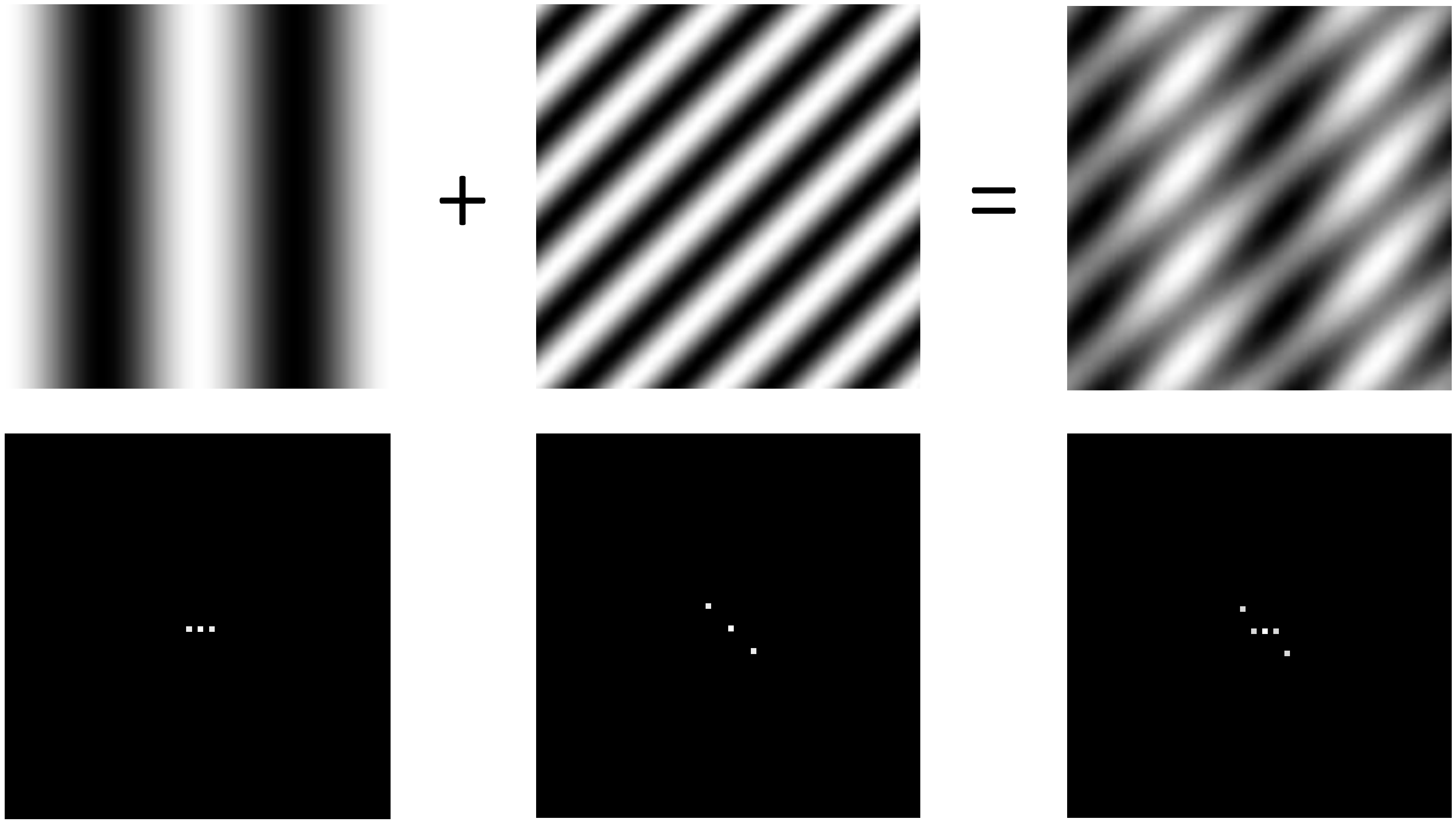
Examples

 +  = ?

Examples



Examples



Basic building block

$$A\sin(\omega x + \phi)$$

amplitude

sinusoid

angular frequency

variable

phase

What about non-periodic signals?

Fourier's claim: Add enough of these to get <u>any</u> *periodic* signal you want!

Fourier transform

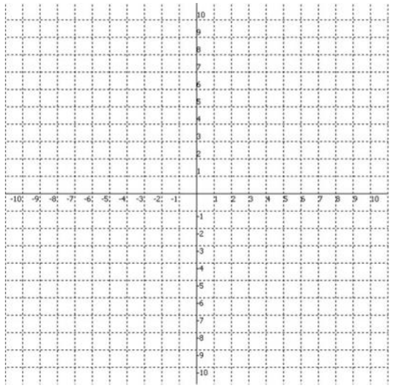
Recalling some basics

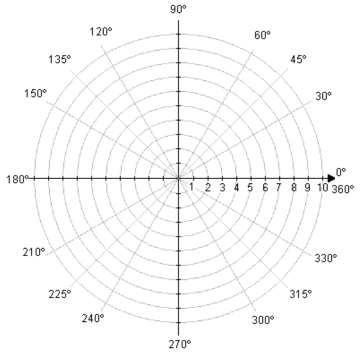
Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

what‘s this? what‘s this?



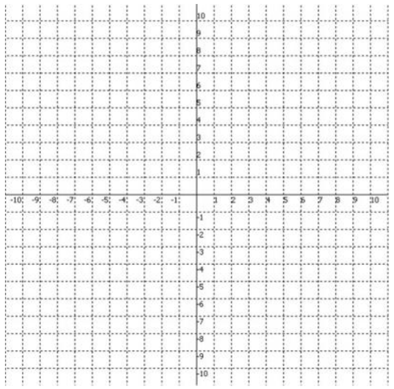


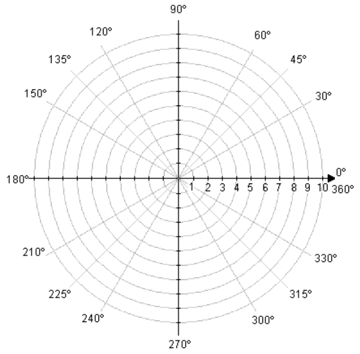
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$\underbrace{R}_{\text{real}} + \underbrace{jI}_{\text{imaginary}}$$





Recalling some basics

Complex numbers have two parts:

rectangular coordinates

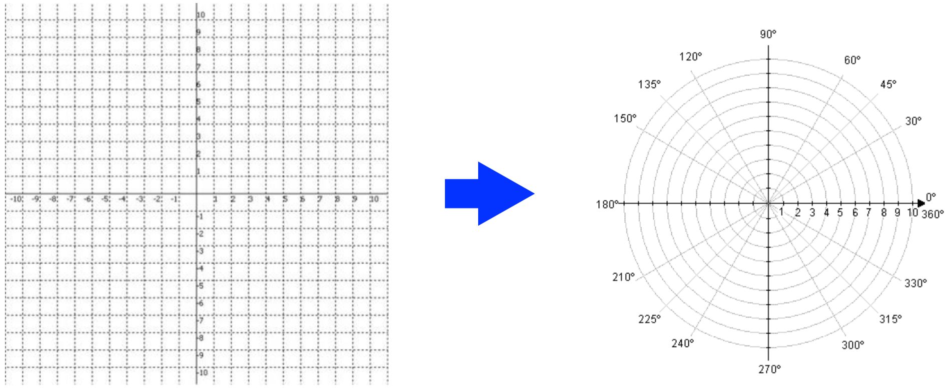
$$\underbrace{R}_{\text{real}} + \underbrace{jI}_{\text{imaginary}}$$

Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

how do we compute these?



polar transform

Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary

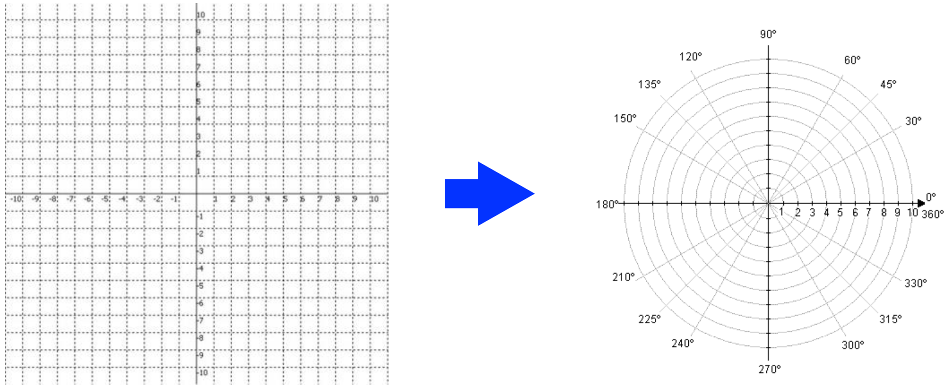
Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$



polar transform

Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary

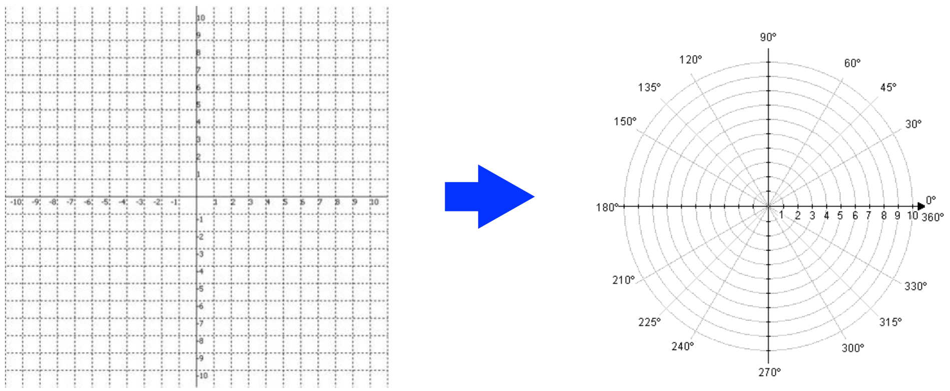
Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$



polar transform

How do you write these in exponential form?

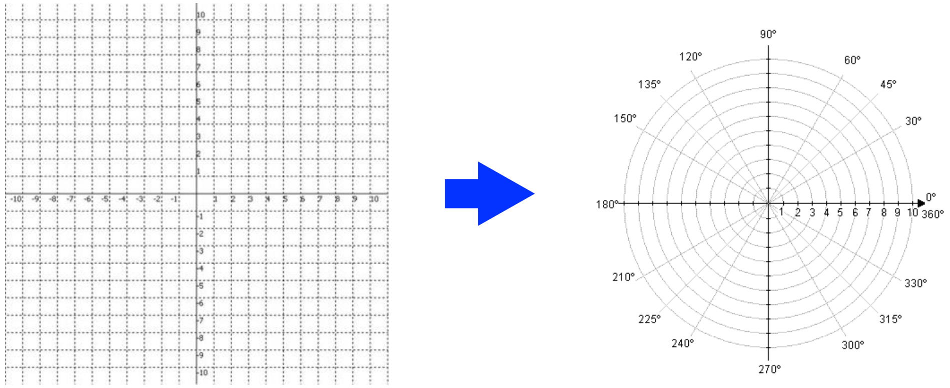
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}(\frac{I}{R}) \quad r = \sqrt{R^2 + I^2}$$

or equivalently

$$re^{j\theta}$$

how did we get this?

exponential form

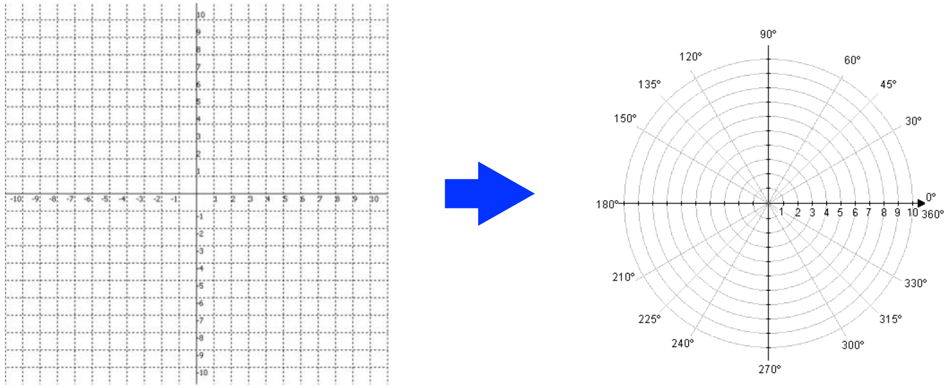
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

real imaginary



Alternative reparameterization:

polar coordinates

$$r(\cos\theta + j\sin\theta)$$

polar transform

$$\theta = \tan^{-1}(\frac{I}{R}) \quad r = \sqrt{R^2 + I^2}$$

or equivalently

$$re^{j\theta}$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

exponential form

This will help us understand the Fourier transform equations

Fourier transform

Fourier transform | inverse Fourier transform

continuous

$$F(k) = \int_{\infty}^{-\infty} f(x) e^{-j2\pi kx} dx$$

$$f(x) = \int_{\infty}^{-\infty} F(k) e^{j2\pi kx} dk$$

discrete

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$

$k = 0, 1, 2, \ldots, N-1$

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

$x = 0, 1, 2, \ldots, N-1$

Where is the connection to the ‘summation of sine waves’ idea?

Fourier transform

	Fourier transform	inverse Fourier transform
continuous	$$F(k) = \int_{\infty}^{-\infty} f(x) e^{-j2\pi kx} dx$$	$$f(x) = \int_{\infty}^{-\infty} F(k) e^{j2\pi kx} dk$$
discrete	$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$ $k = 0, 1, 2, \ldots, N-1$	$$\boxed{f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N} \quad x = 0, 1, 2, \ldots, N-1}$$

Where is the connection to the ‘summation of sine waves’ idea?

Fourier transform

Where is the connection to the 'summation of sine waves' idea?

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

sum over frequencies

$$f(x) = \sum_{k=0}^{N-1} F(k) \left\{ \cos(2\pi kx) + j\sin(2\pi kx) \right\}$$

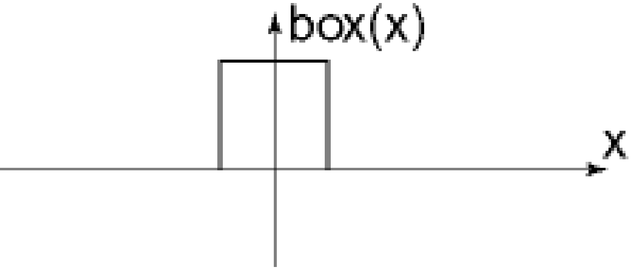
scaling parameter

wave components

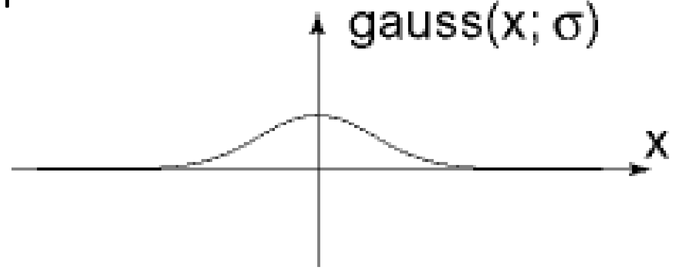
Fourier transform pairs

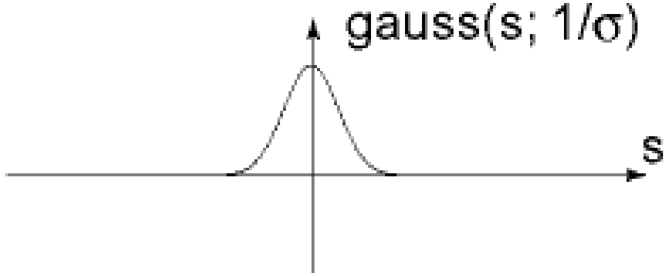
spatial domain

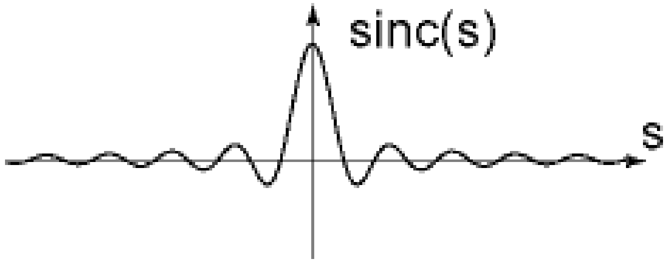
frequency domain

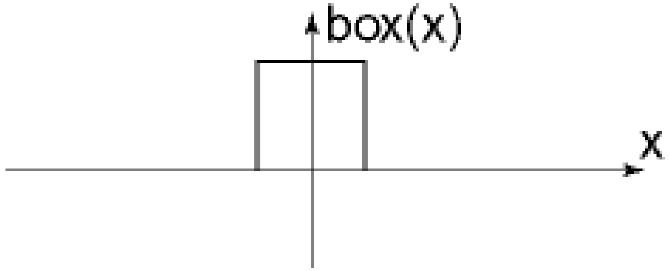


Note the symmetry: duality property of Fourier transform









Computing the discrete Fourier transform (DFT)

Computing the discrete Fourier transform (DFT)

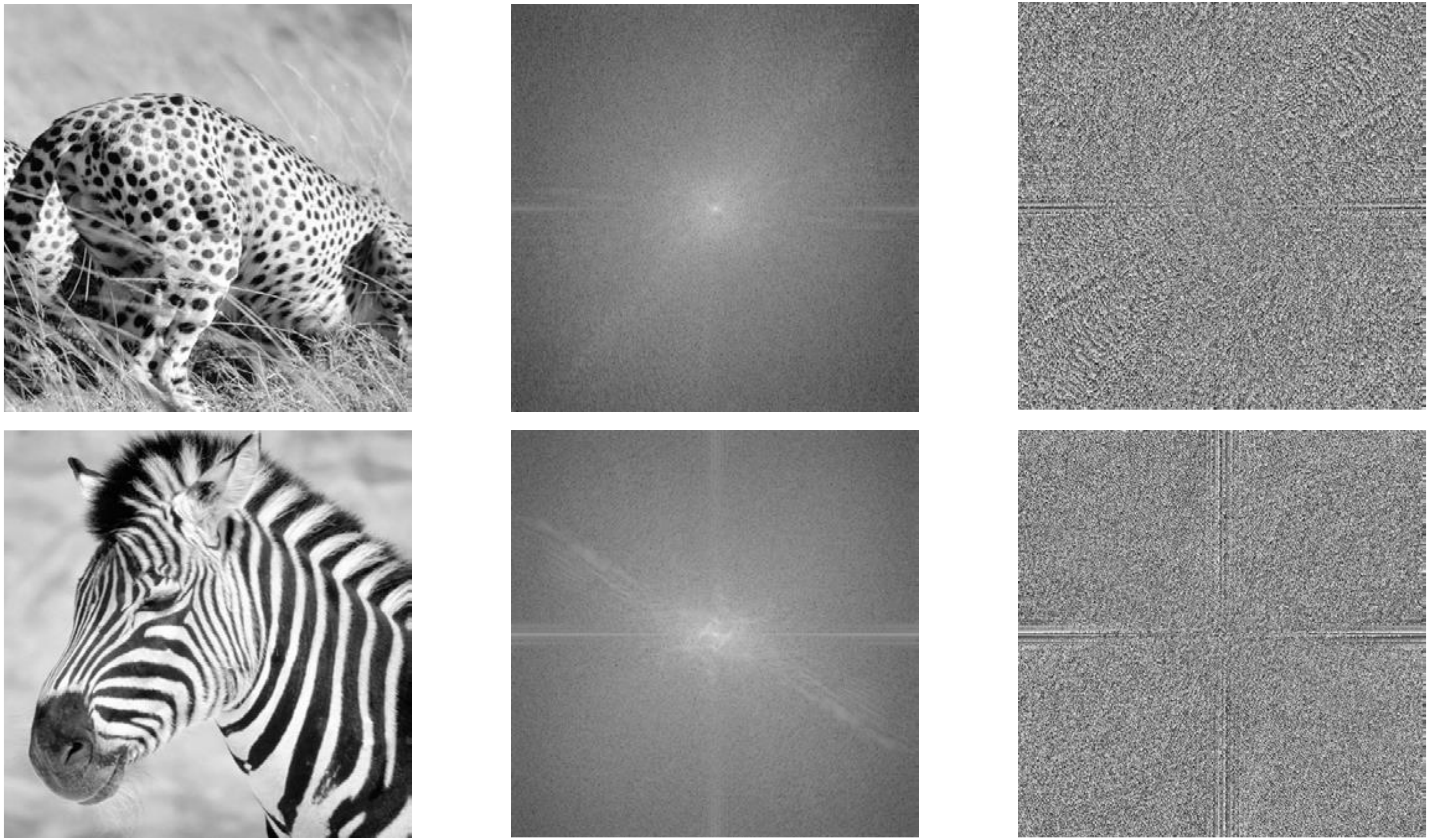
$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$ is just a matrix multiplication:

$$\boldsymbol{F} = \boldsymbol{W} \boldsymbol{f}$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & \ddots & \vdots \\ W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N-1) \end{bmatrix} \qquad W = e^{-j2\pi/N}$$

In practice this is implemented using the *fast Fourier transform* (FFT) algorithm.

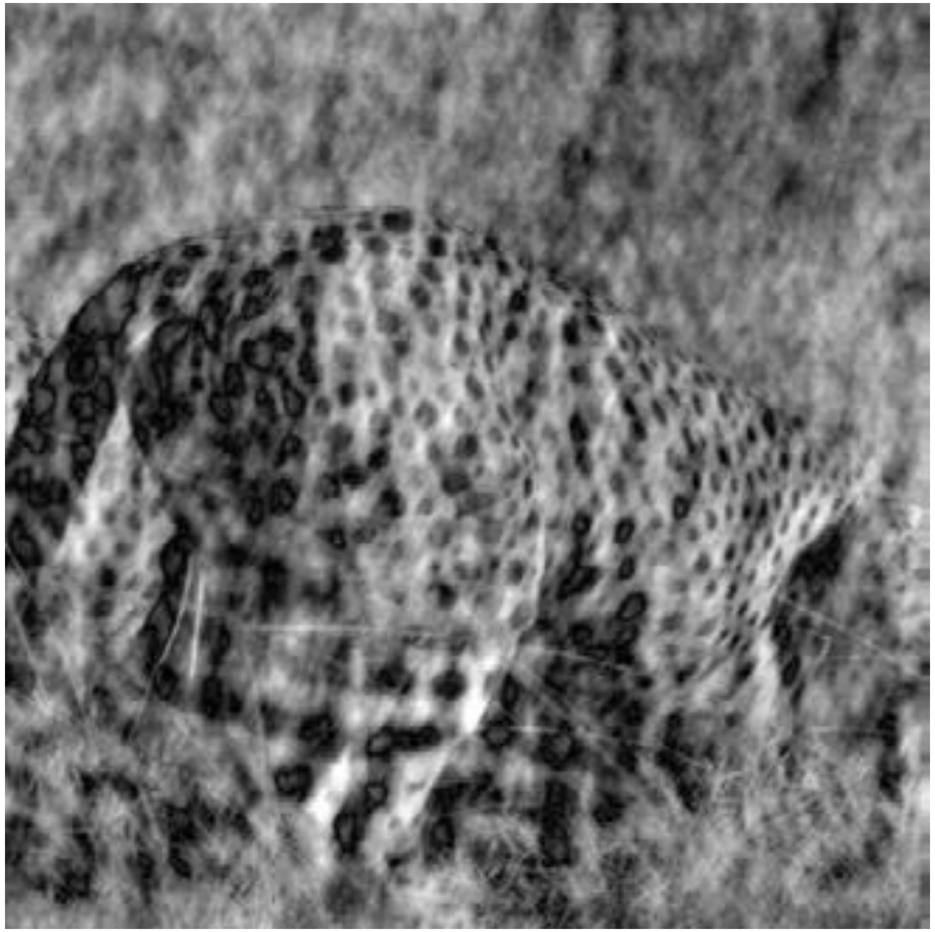
Fourier transforms of natural images



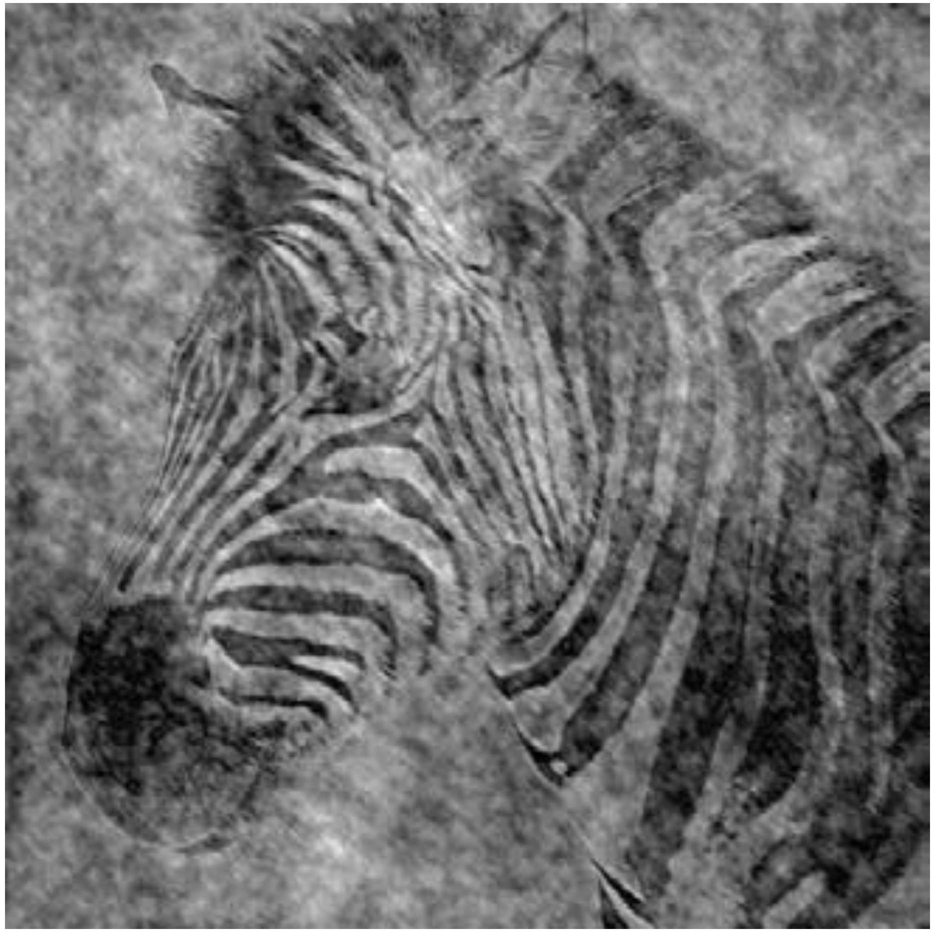
original amplitude phase

Fourier transforms of natural images

Image phase matters!



cheetah phase with zebra amplitude



zebra phase with cheetah amplitude

Frequency-domain filtering

Why do we care about all this?

The convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} * \mathcal{F}^{-1}\{h\}$$

Convolution in spatial domain is equivalent to multiplication in frequency domain!

What do we use convolution for?

Convolution for 1D continuous signals

Definition of linear shift-invariant filtering as convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy$$

filtered signal

filter

input signal

Using the convolution theorem, we can interpret and implement all types of linear shift-invariant filtering as multiplication in frequency domain.

Why implement convolution in frequency domain?

Frequency-domain filtering in Matlab

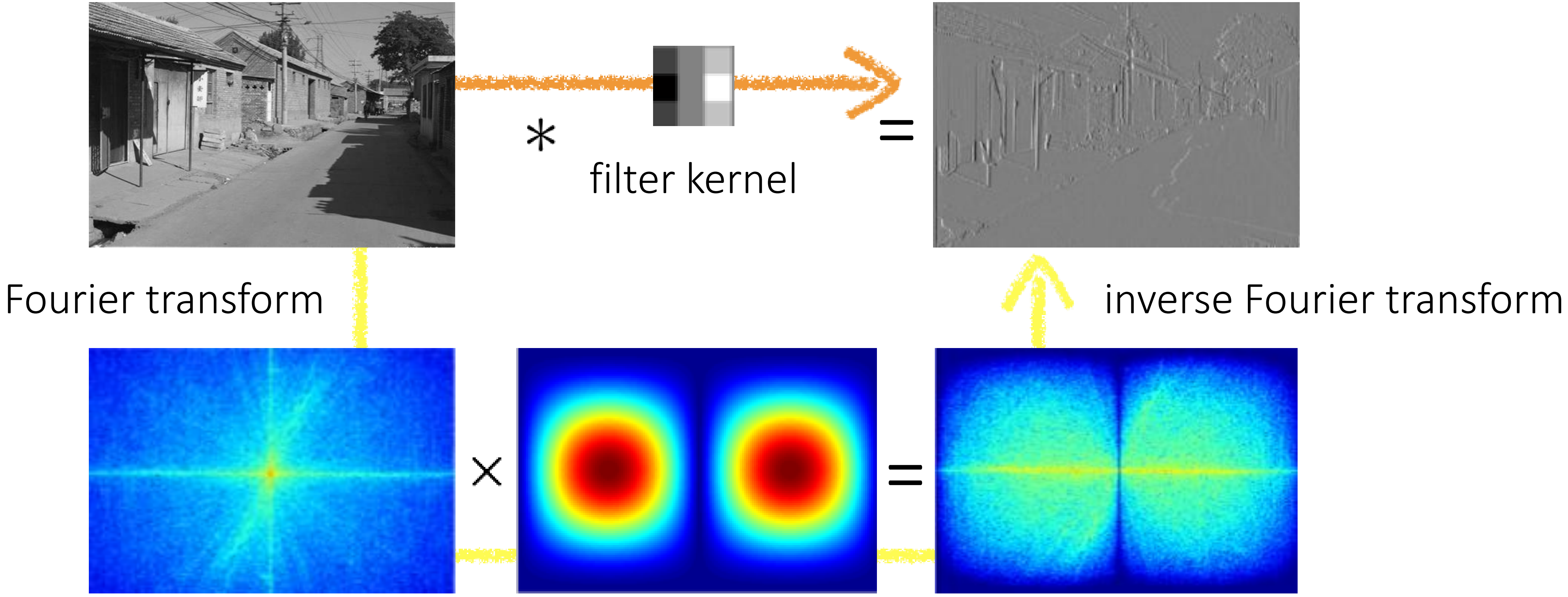
Filtering with `fft`:

```
im = double(imread('…'))/255;
im = rgb2gray(im); % "im" should be a gray-scale floating point image
[imh, imw] = size(im);

hs = 50; % filter half-size
fil = fspecial('gaussian', hs*2+1, 10);

fftsize = 1024; % should be order of 2 (for speed) and include padding
im_fft  = fft2(im,  fftsize, fftsize);                         % 1) fft im with
padding
fil_fft = fft2(fil, fftsize, fftsize);                         % 2) fft fil, pad to
same size as image
im_fil_fft = im_fft .* fil_fft;                                % 3) multiply fft
images
im_fil = ifft2(im_fil_fft);                                    % 4) inverse fft2
im_fil = im_fil(1+hs:size(im,1)+hs, 1+hs:size(im, 2)+hs); % 5) remove padding
```

Displaying with `fft`:

```
figure(1), imagesc(log(abs(fftshift(im_fft)))), axis image, colormap jet
```

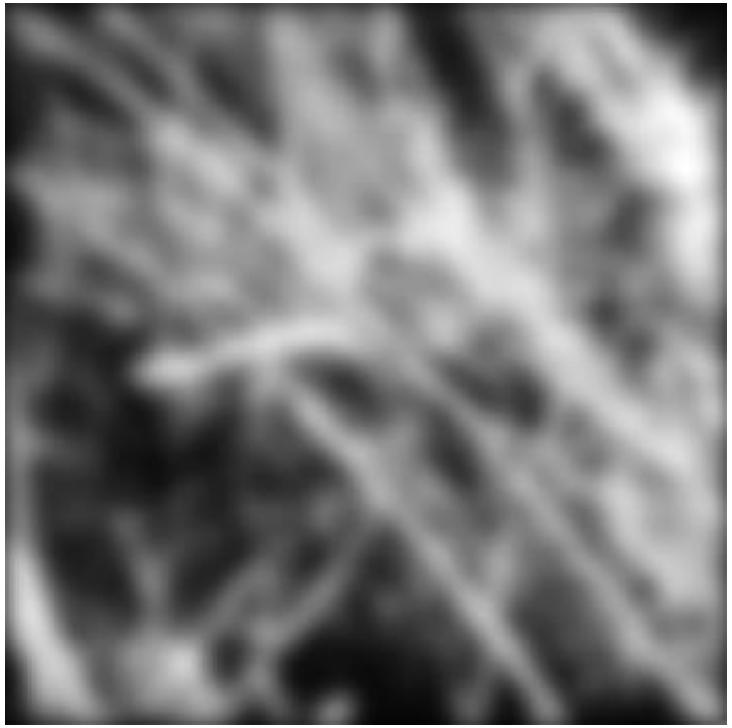
Spatial domain filtering



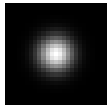
Frequency domain filtering

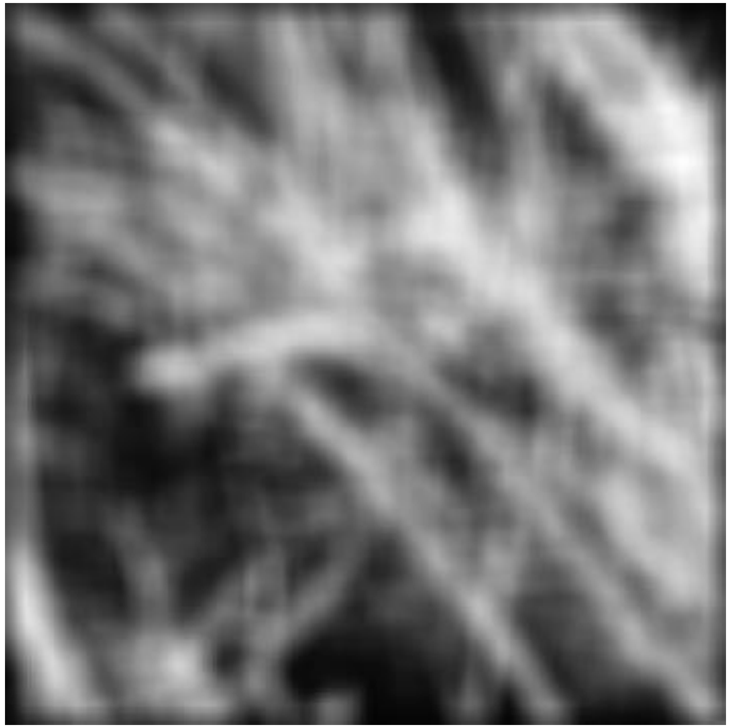
Revisiting blurring

Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

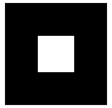


Gaussian filter

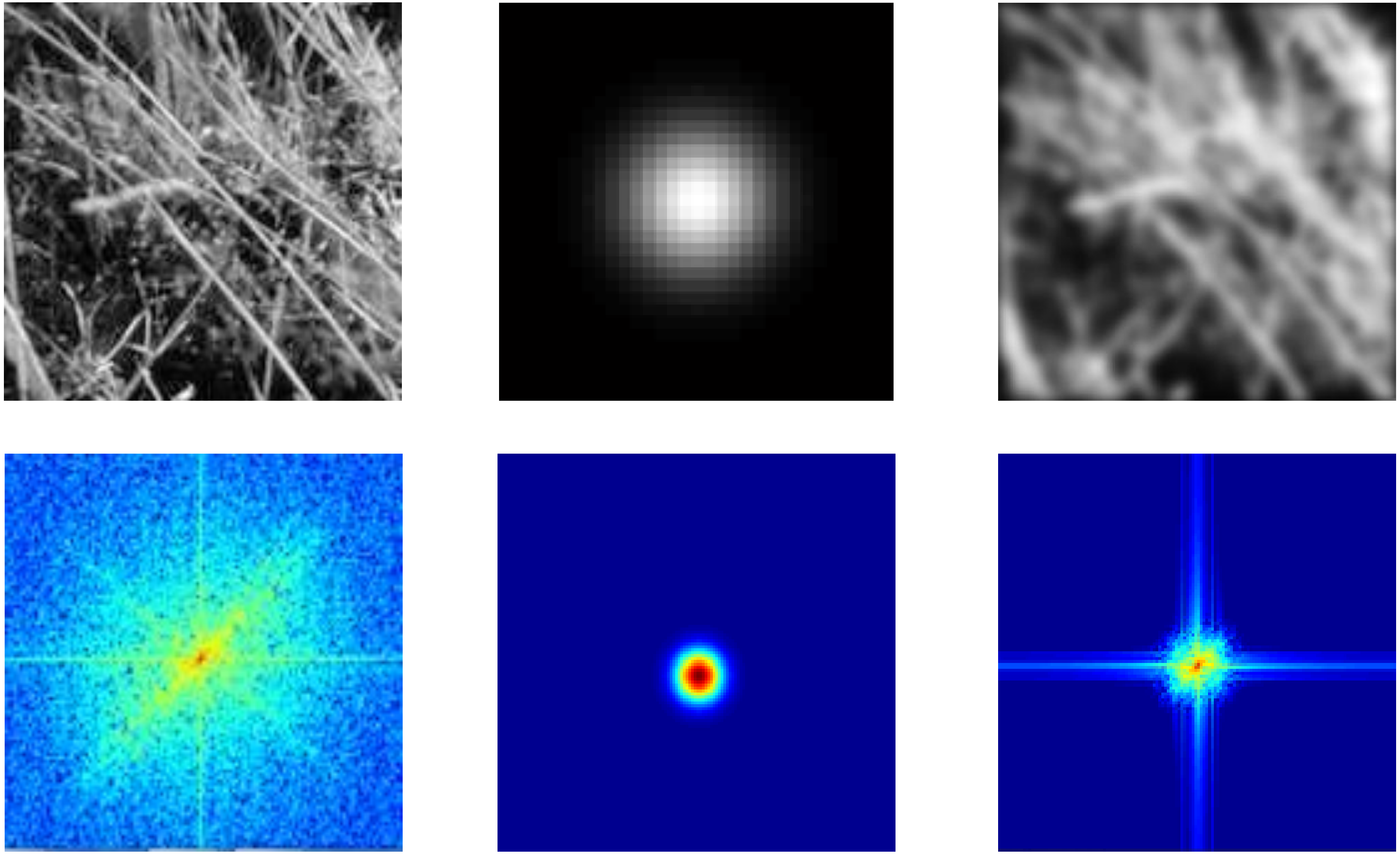




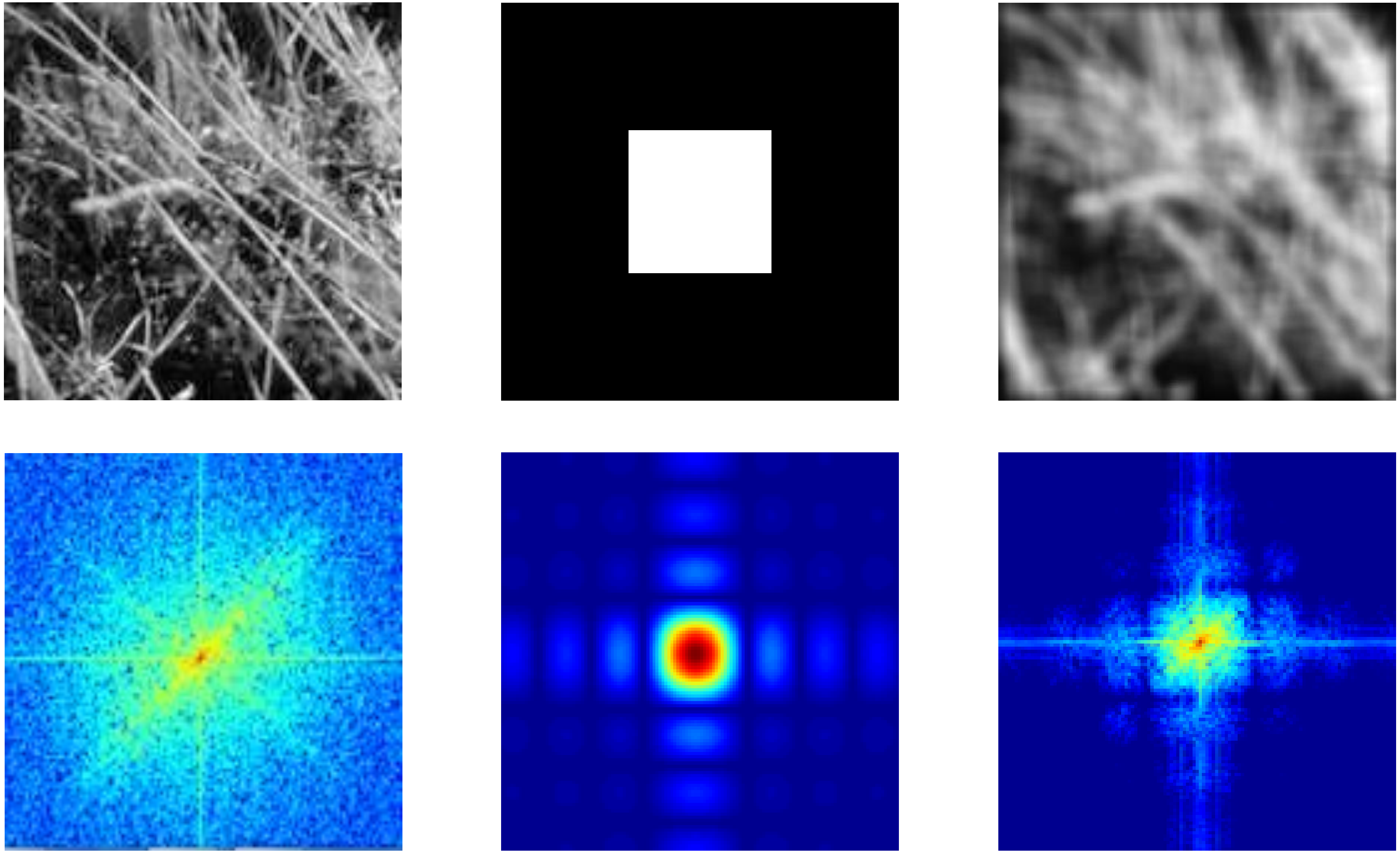
Box filter



Gaussian blur

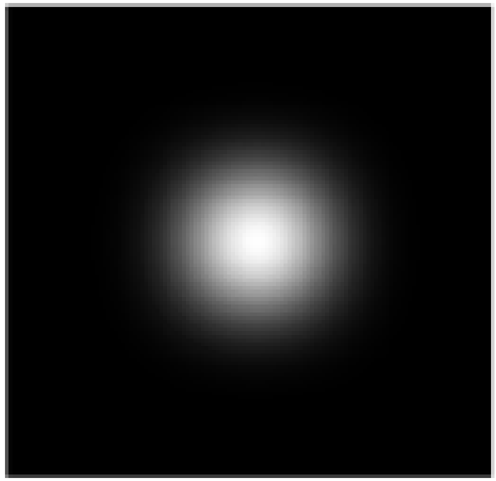


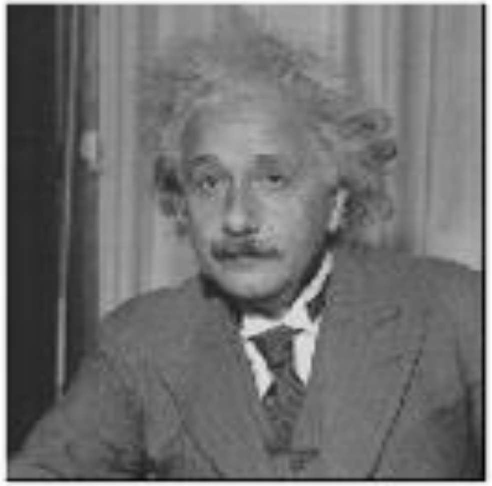
Box blur



More filtering examples

?



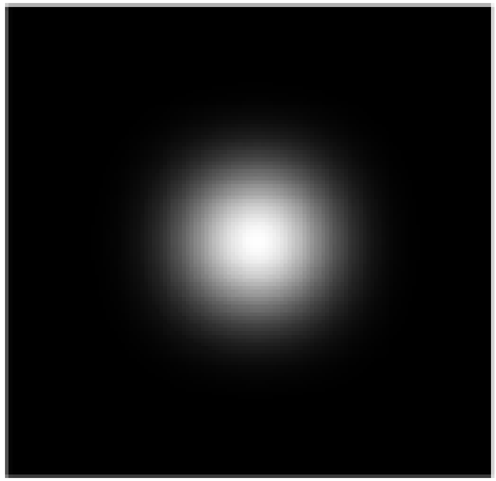


?

filters shown in frequency-domain

More filtering examples

low-pass



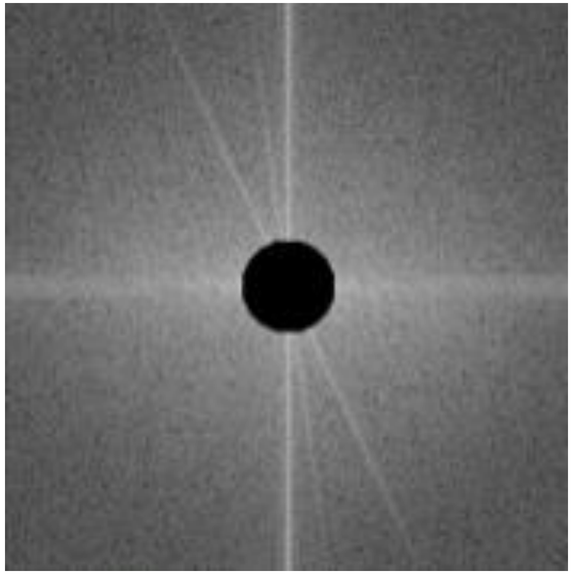
band-pass

filters shown in frequency-domain

More filtering examples

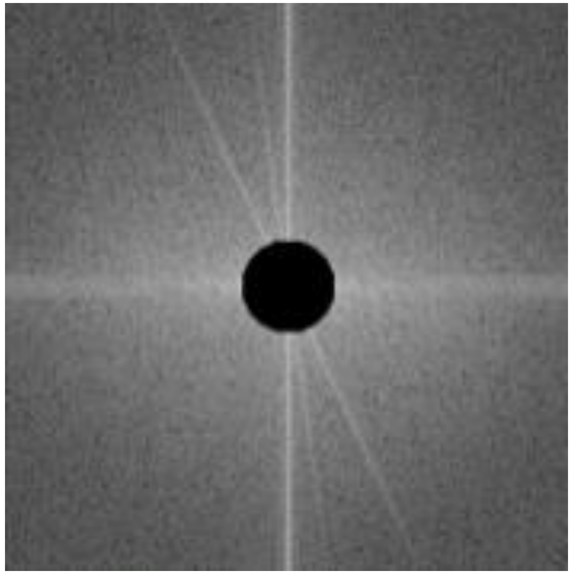
?

high-pass



More filtering examples

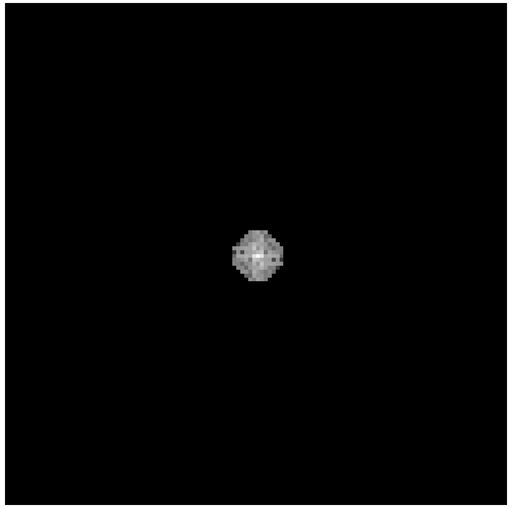
high-pass



More filtering examples

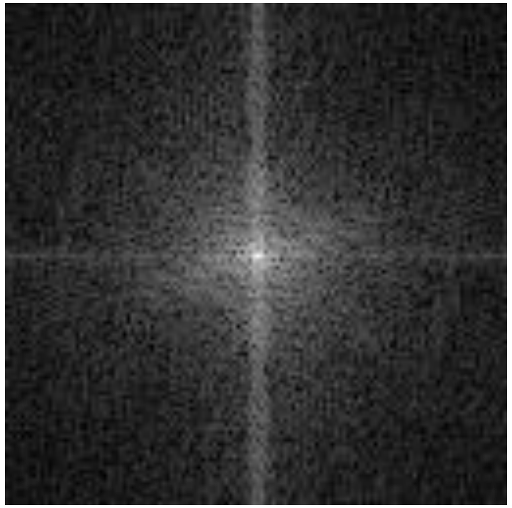
original image

low-pass filter



?

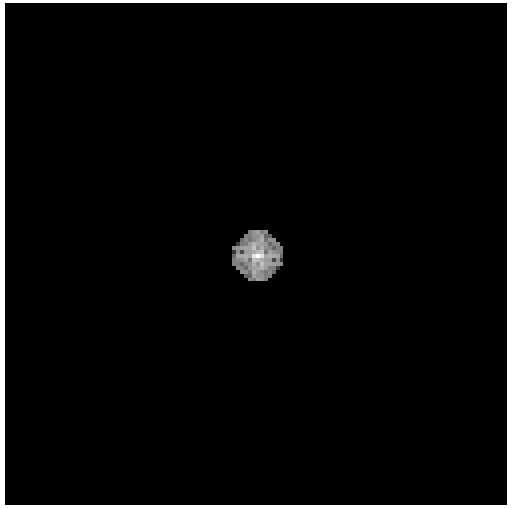
frequency magnitude



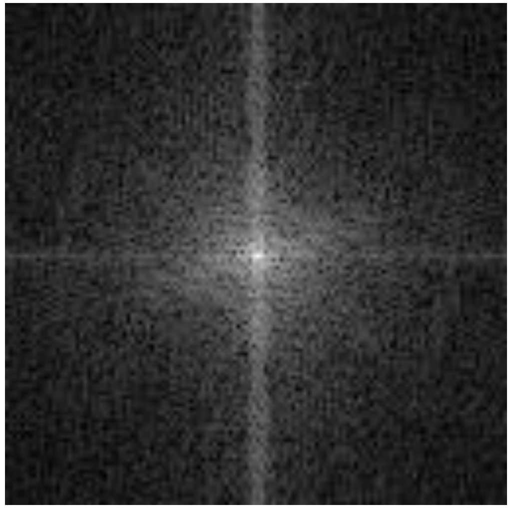
More filtering examples

original image

low-pass filter



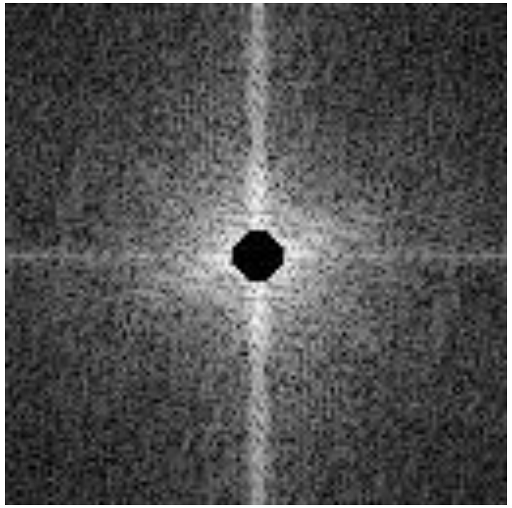
frequency magnitude



More filtering examples

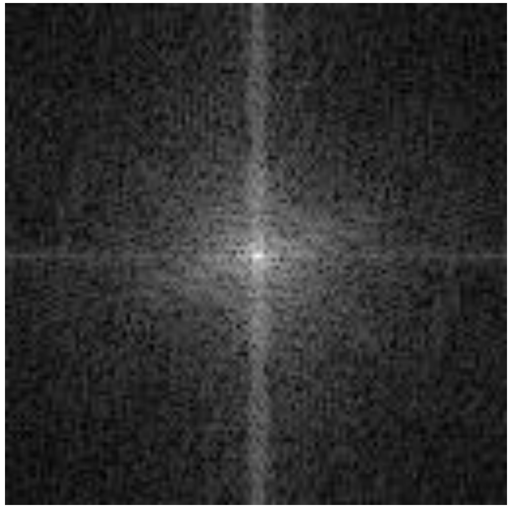
original image

high-pass filter



?

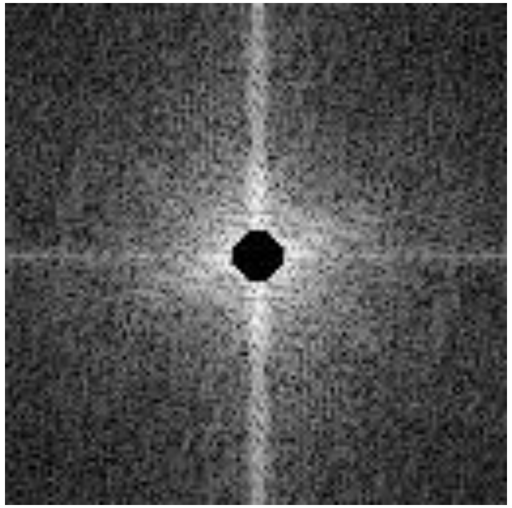
frequency magnitude



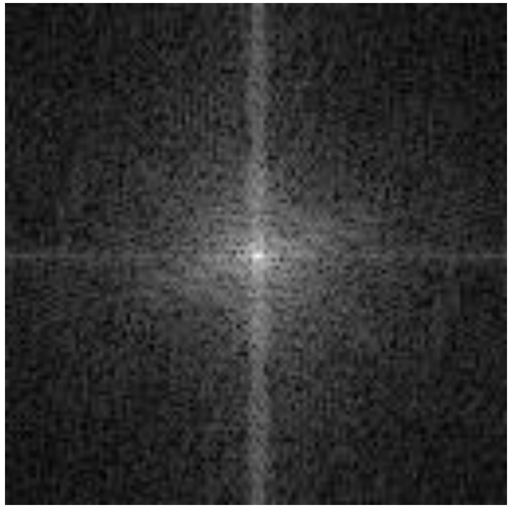
More filtering examples

original image

high-pass filter



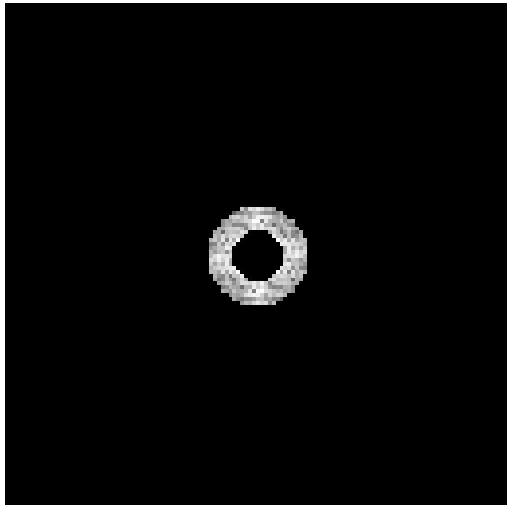
frequency magnitude

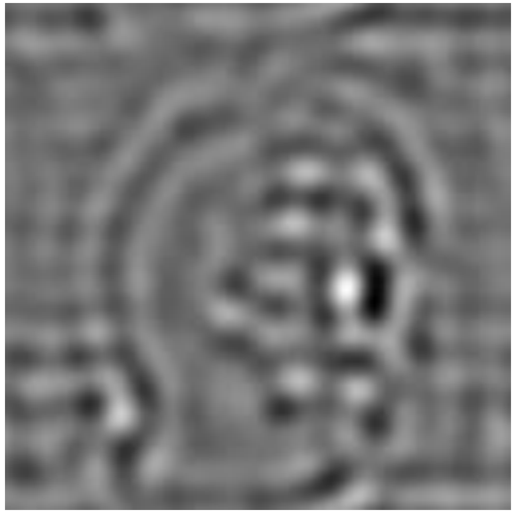


More filtering examples

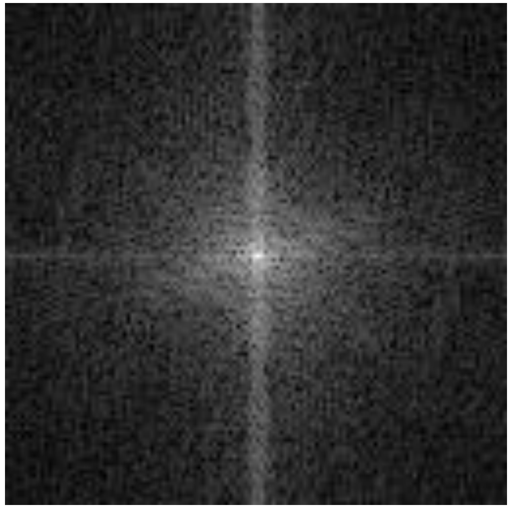
original image

band-pass filter





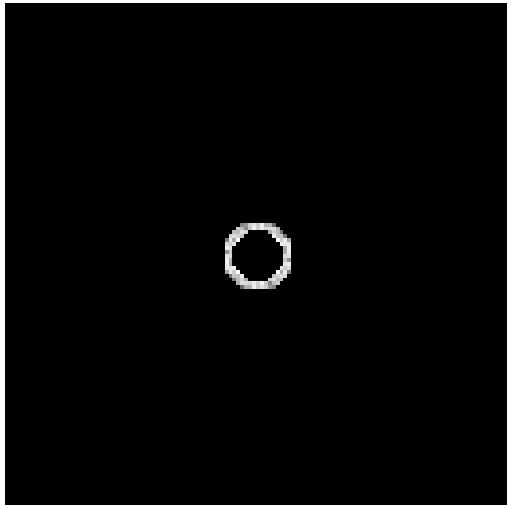
frequency magnitude

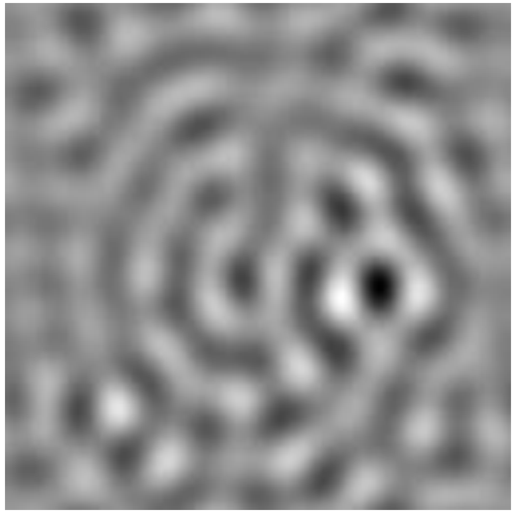


More filtering examples

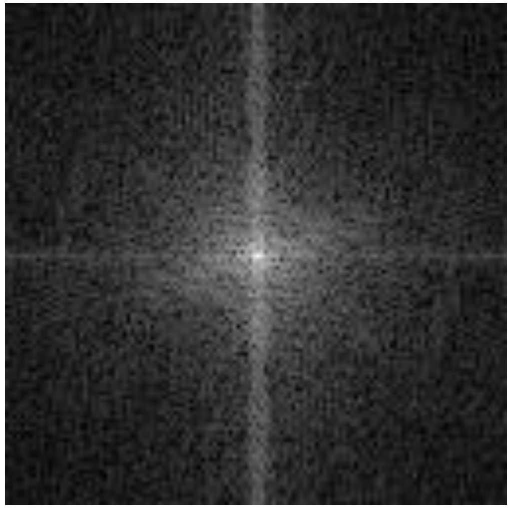
original image

band-pass filter





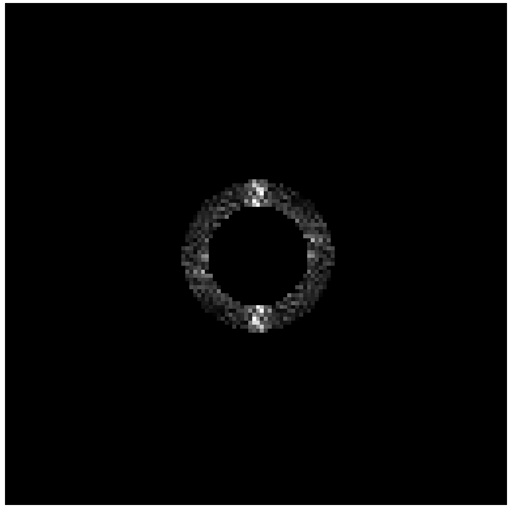
frequency magnitude



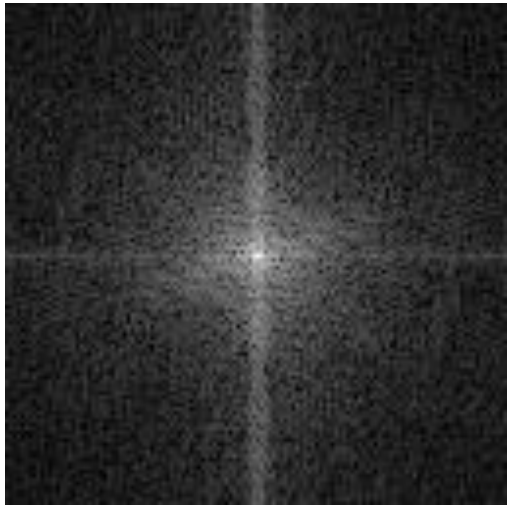
More filtering examples

original image

band-pass filter



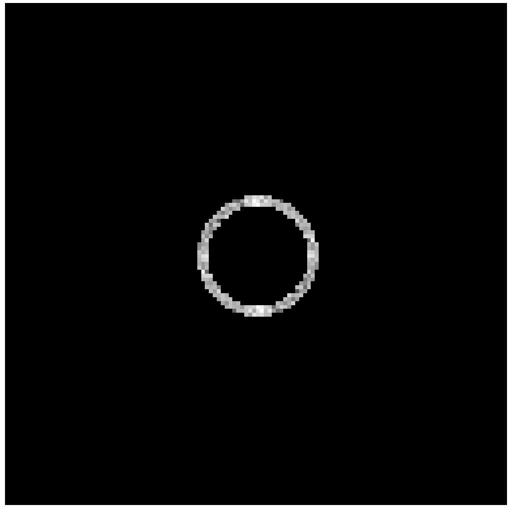
frequency magnitude

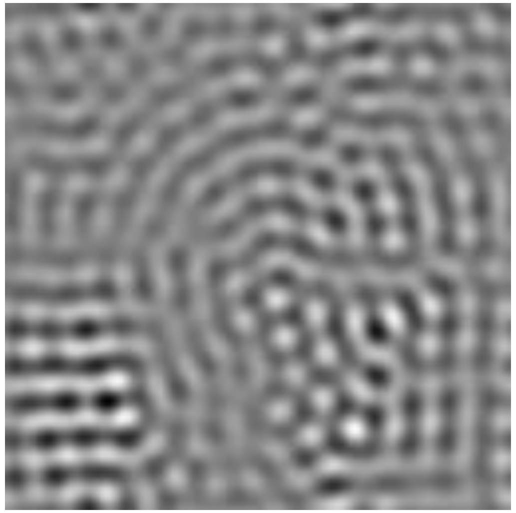


More filtering examples

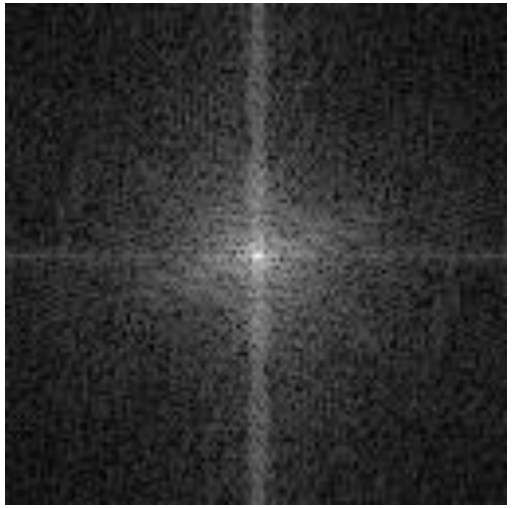
original image

band-pass filter





frequency magnitude



Revisiting sampling

The Nyquist-Shannon sampling theorem

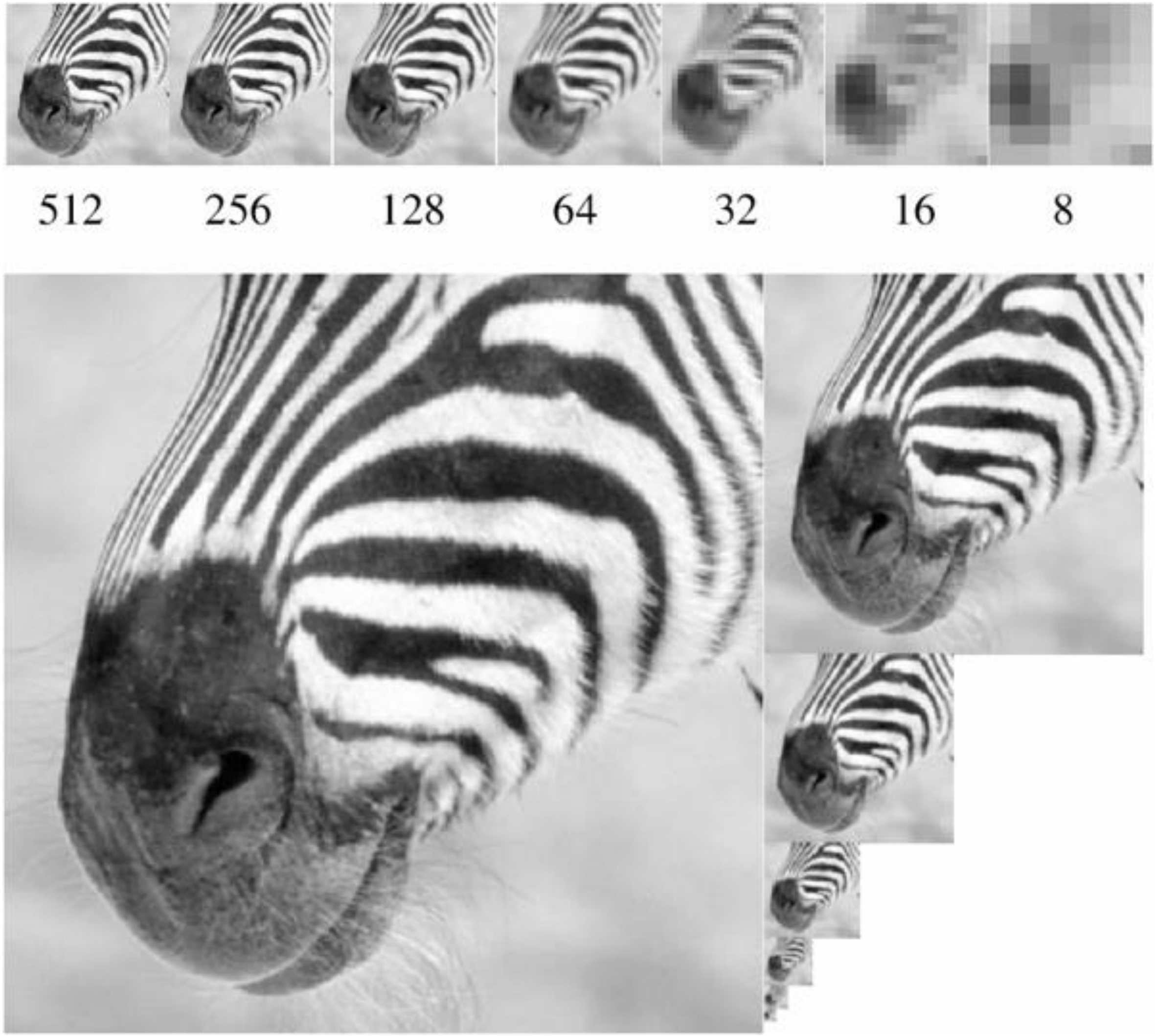
A continuous signal can be perfectly reconstructed from its discrete version using linear interpolation, if sampling occurred with frequency:

$$f_s \geq 2 f_{\max}$$

This is called the Nyquist frequency

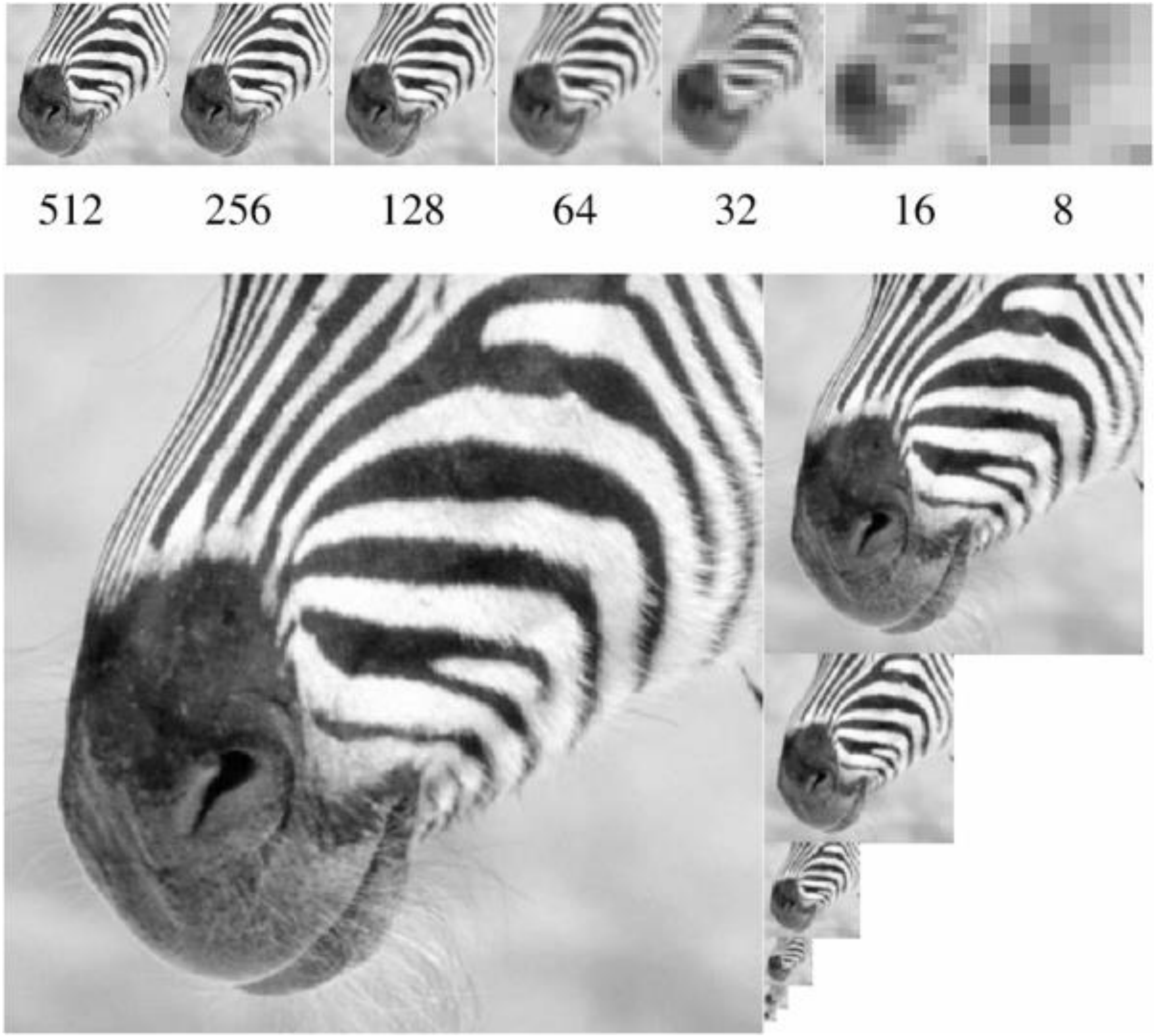
Equivalent reformulation: When downsampling, aliasing does not occur if samples are taken at the Nyquist frequency or higher.

Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

Gaussian pyramid

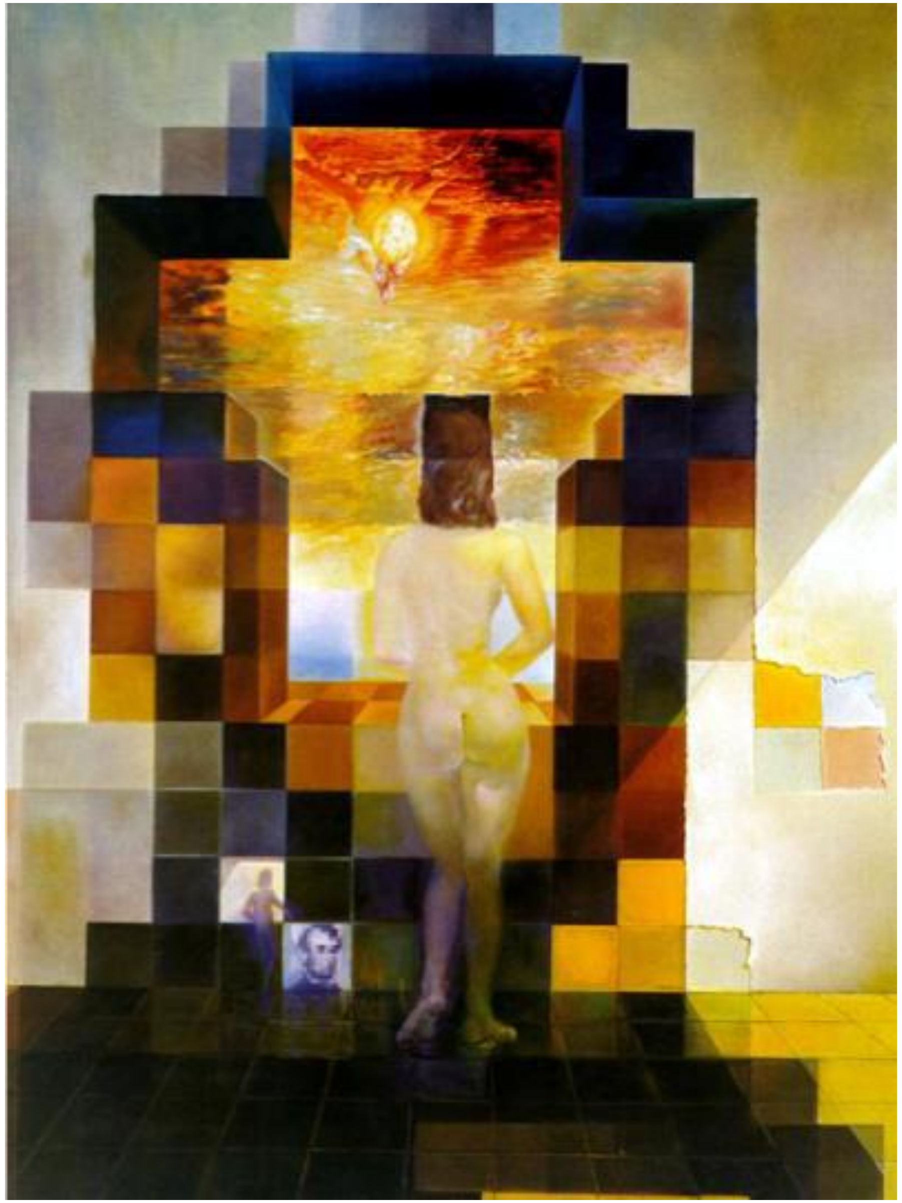


How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

- Gaussian blurring is low-pass filtering.
- By blurring we try to sufficiently decrease the Nyquist frequency to avoid aliasing.

How large should the Gauss blur we use be?

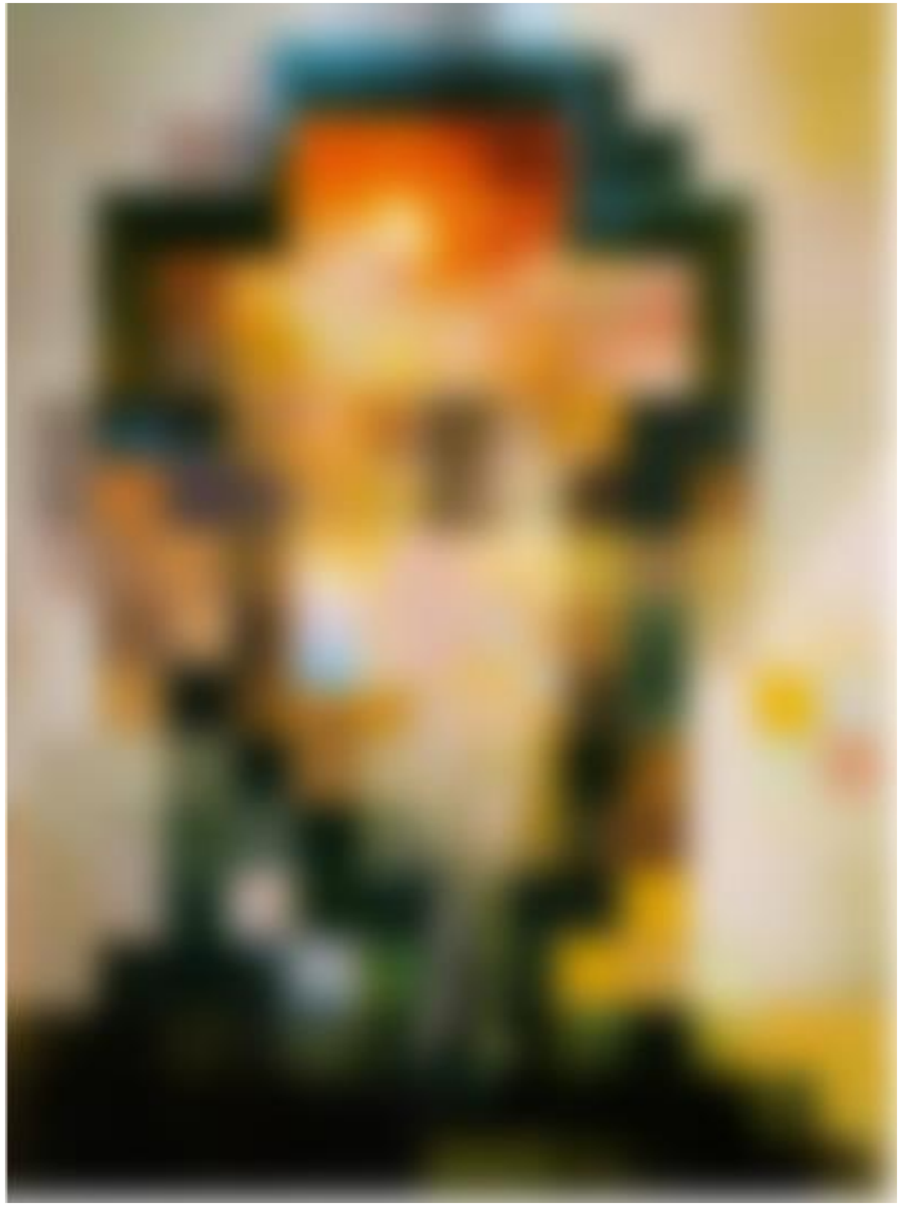
Frequency-domain filtering in human vision



Gala Contemplating the Mediterranean Sea Which at Twenty Meters Becomes the Portrait of Abraham Lincoln (Homage to Rothko)

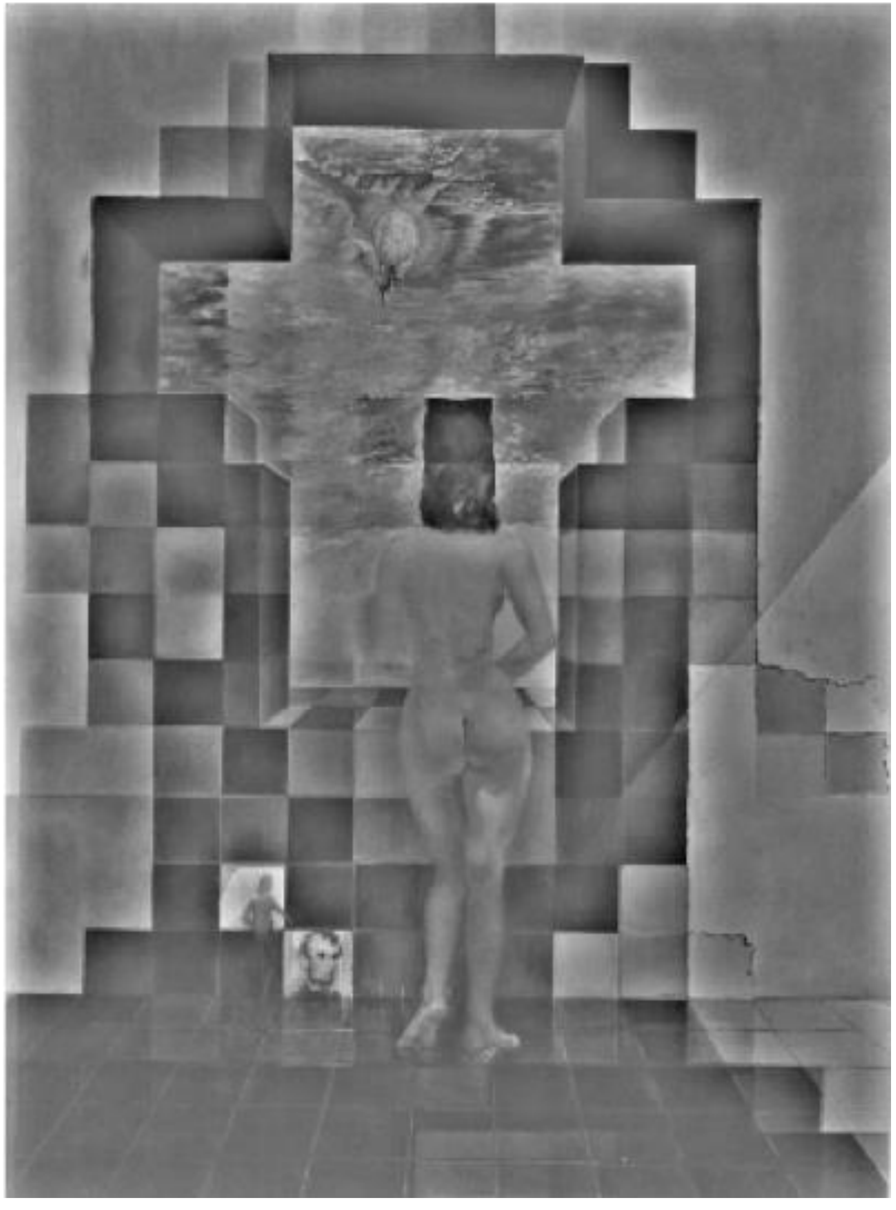
Salvador Dali, 1976

Frequency-domain filtering in human vision



Low-pass filtered version

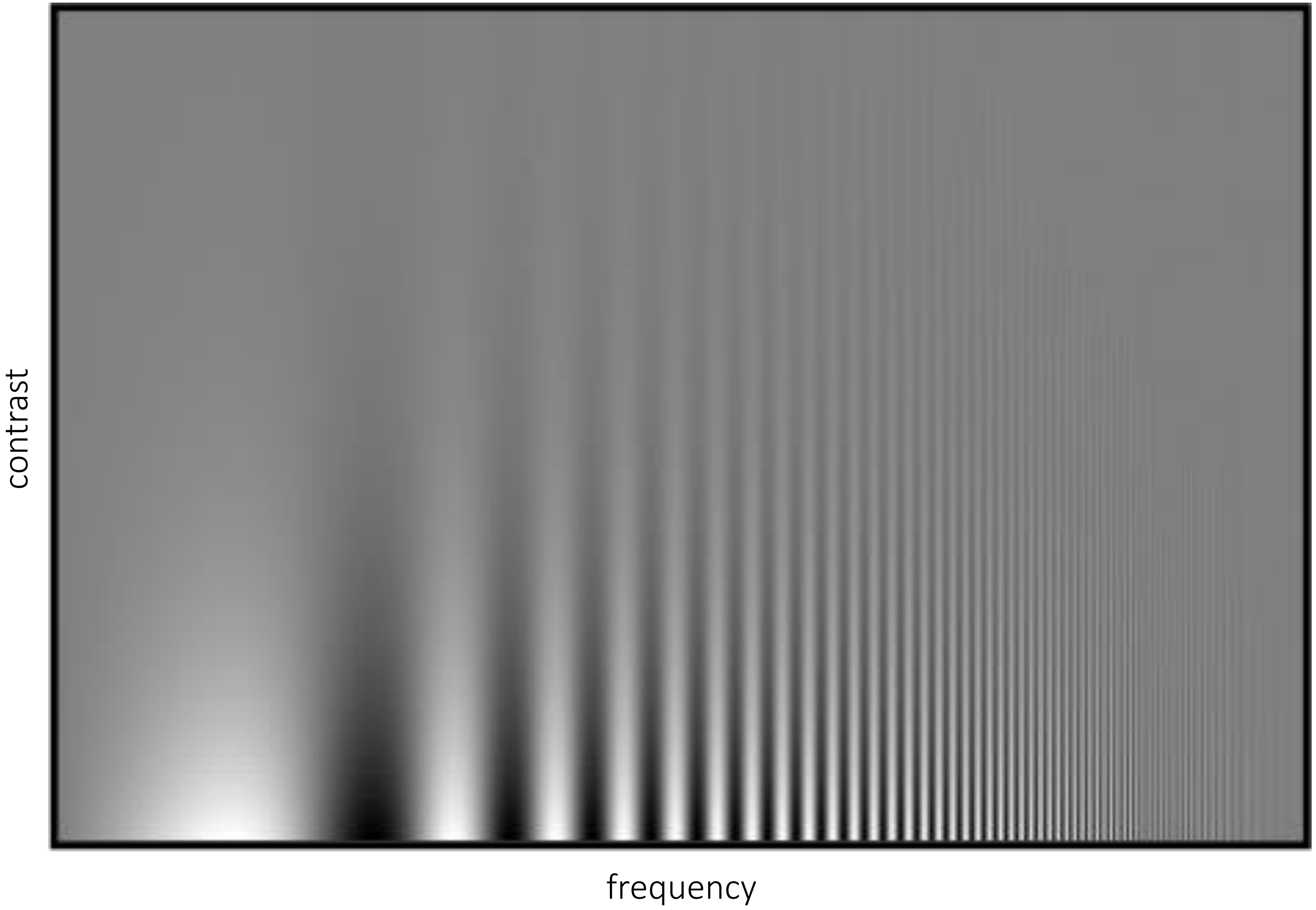
Frequency-domain filtering in human vision



High-pass filtered version

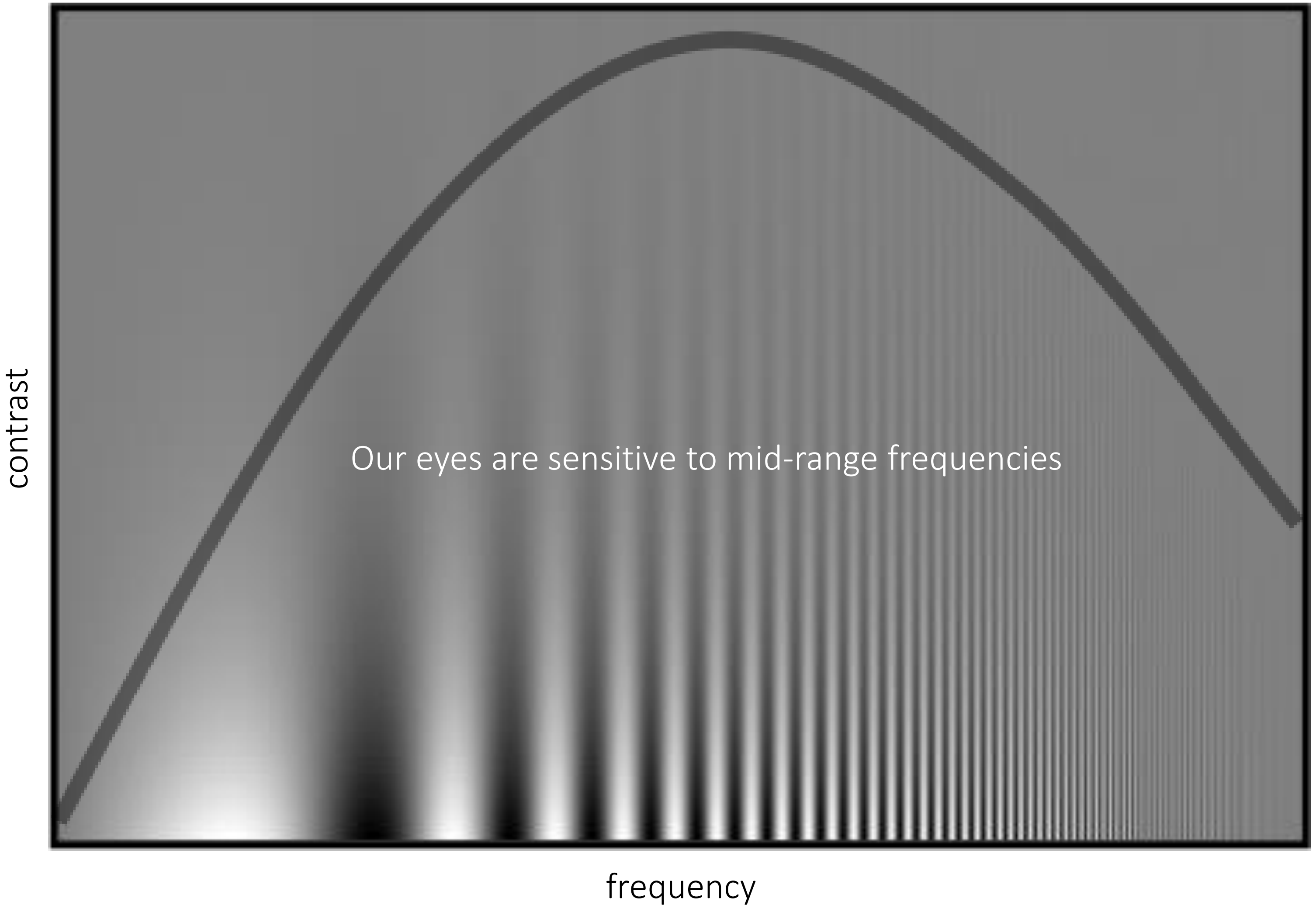
Variable frequency sensitivity

Experiment: Where do you see the stripes?



Variable frequency sensitivity

Campbell-Robson contrast sensitivity curve



- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid frequencies dominate perception

References

Basic reading:

- Szeliski textbook, Sections 3.4.

Additional reading:

- Hubel and Wiesel, “Receptive fields, binocular interaction and functional architecture in the cat's visual cortex,” The Journal of Physiology 1962

  a foundational paper describing information processing in the visual system, including the different types of filtering it performs; Hubel and Wiesel won the Nobel Prize in Medicine in 1981 for the discoveries described in this paper