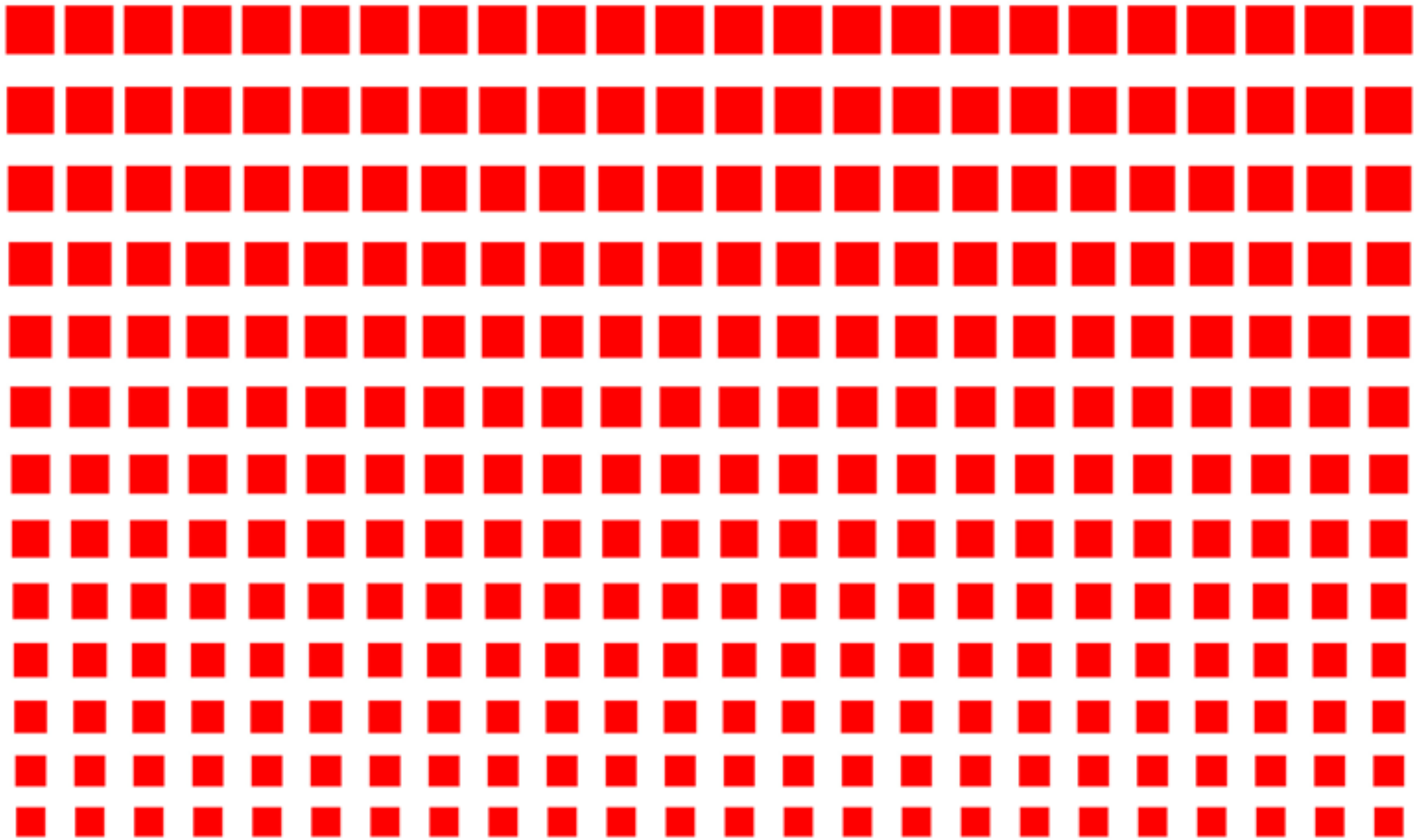


# 2D Alignment: Linear Least Squares

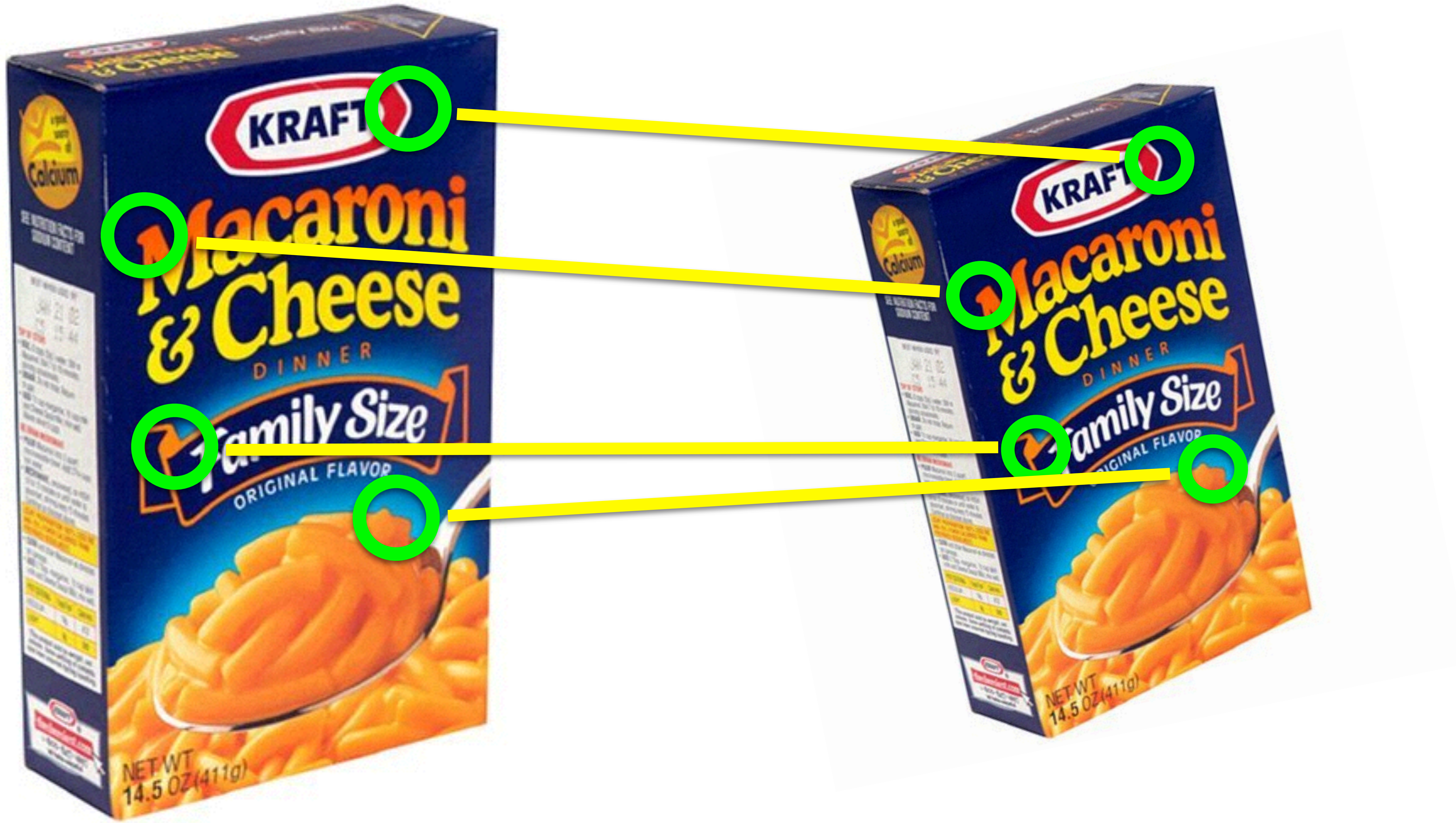
16-385 Computer Vision (Kris Kitani)
**Carnegie Mellon University**

# Extract features from an image …


what do we do next?

# Feature matching

(object recognition, 3D reconstruction, augmented reality, image stitching)




*How do we estimate the transformation?*

Given a set of matched feature points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

point in one image     point in the other image

and a transformation

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

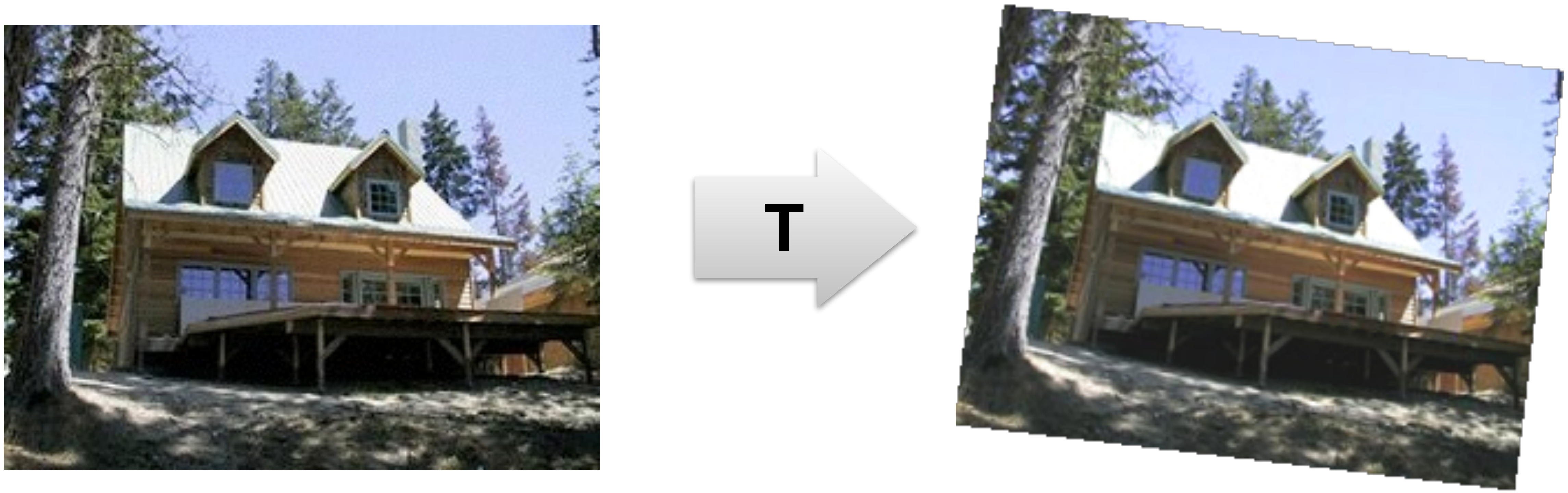
transformation function     parameters

Find the best estimate of

$$\boldsymbol{p}$$

# Model fitting

# Recover the transformation



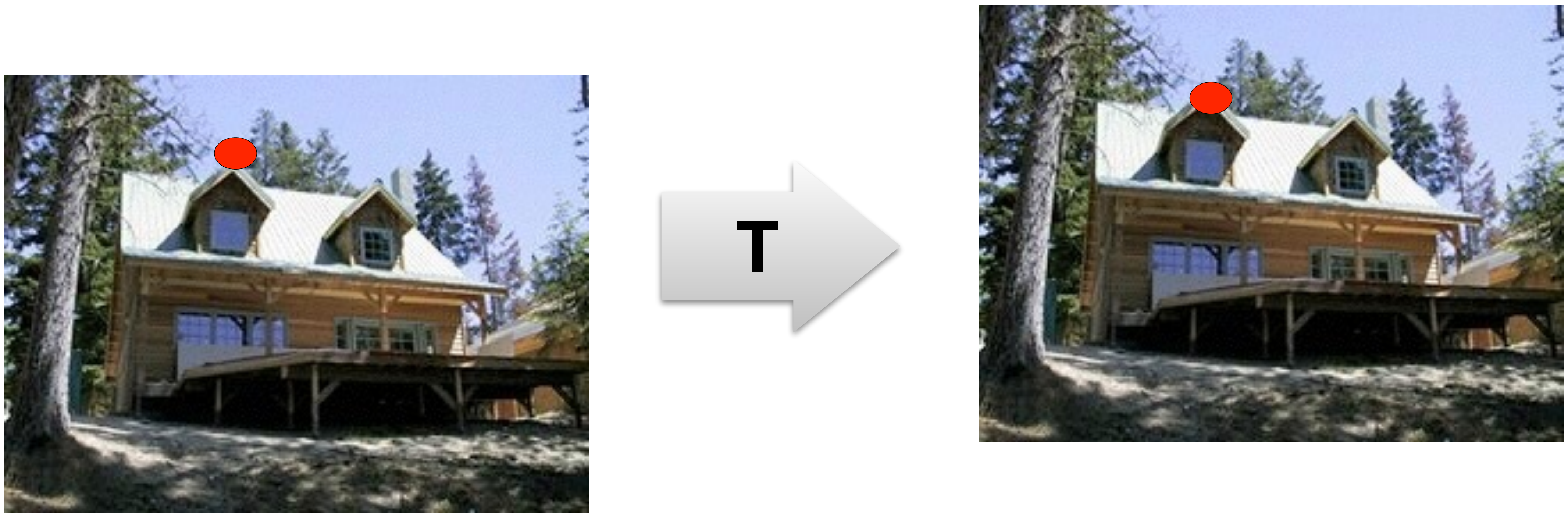

f(x,y) g(x,y)

*Given f and g, how would you recover the transform T?*

*(user will provide correspondences)*

*How many do we need?*

# Translation

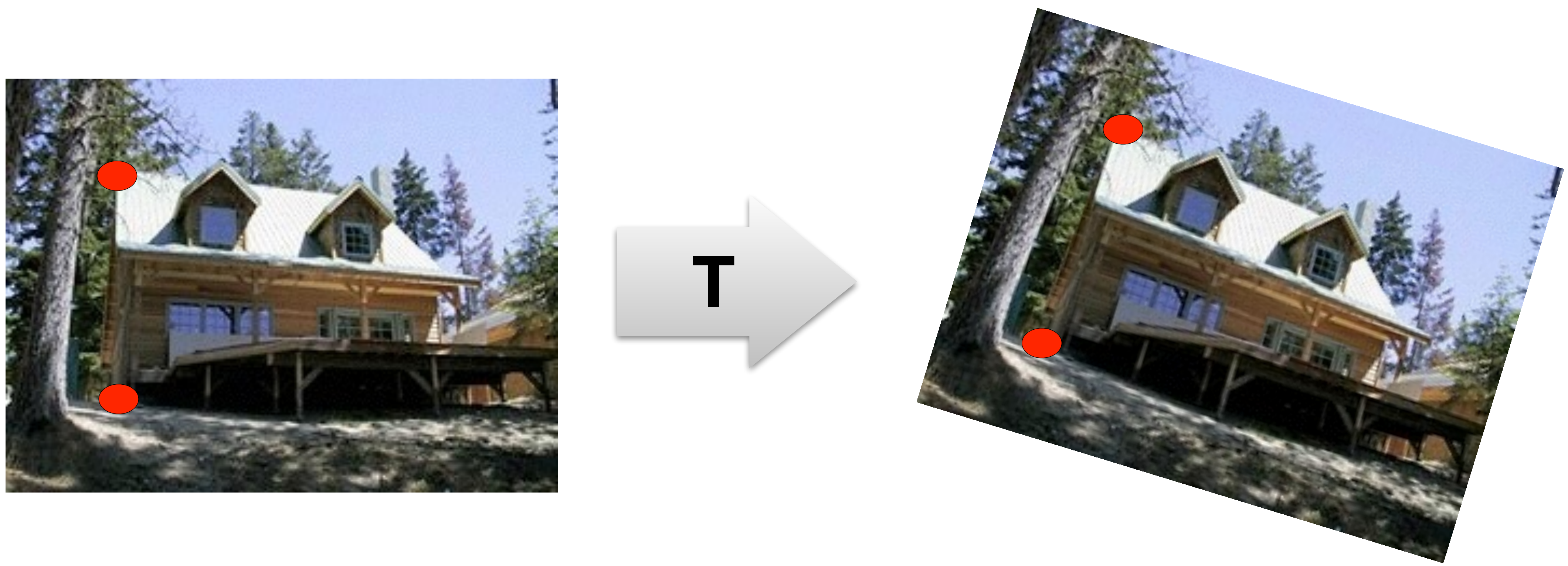



f(x,y) g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed?*
- *What is the transformation matrix?*

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & p'_x - p_x \\ 0 & 1 & p'_y - p_y \\ 0 & 0 & 1 \end{bmatrix}$$

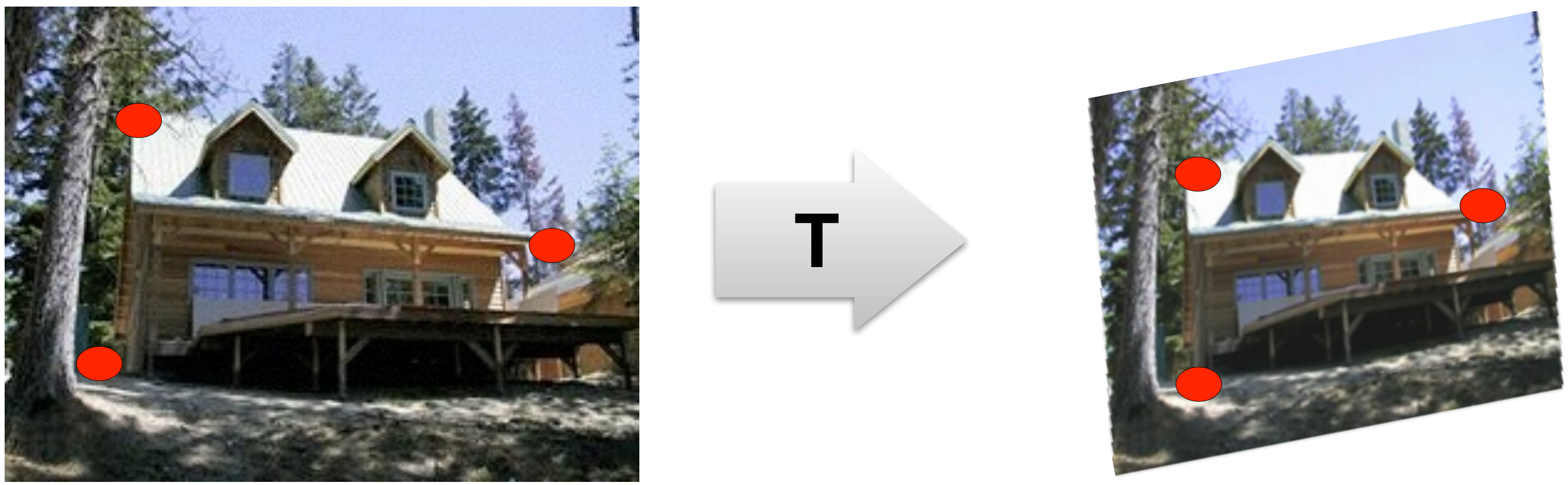
# Euclidean




f(x,y) g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed for translation+rotation?*
- *What is the transformation matrix?*

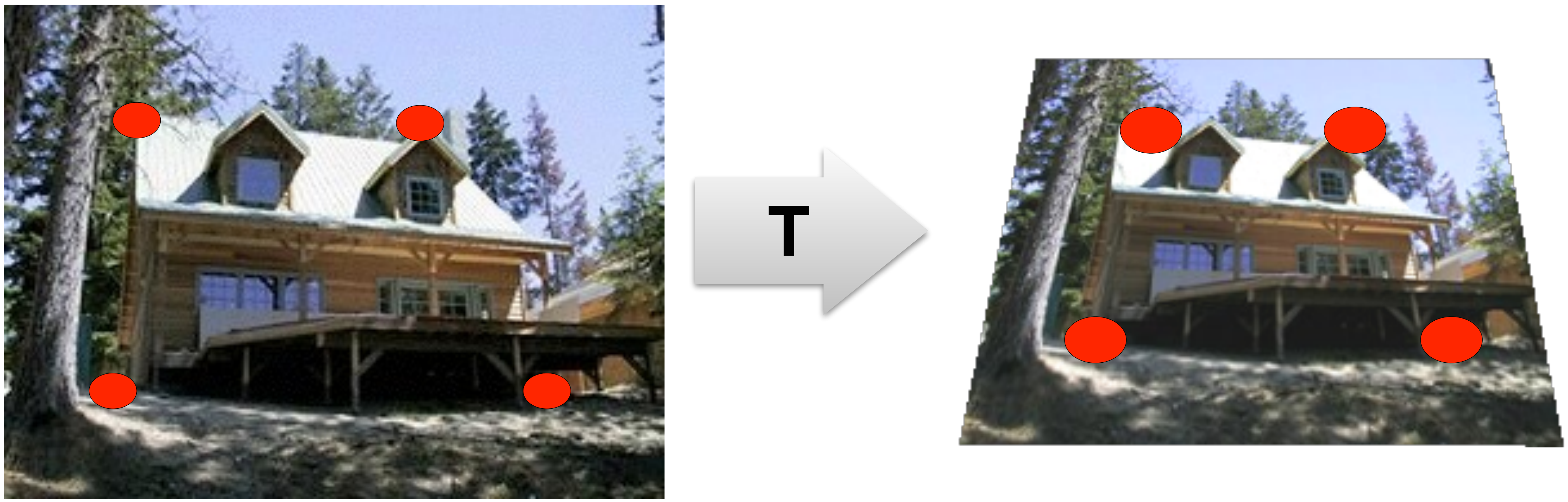
# Affine




f(x,y) g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed for affine?*
- *What is the transformation matrix?*

# Projective



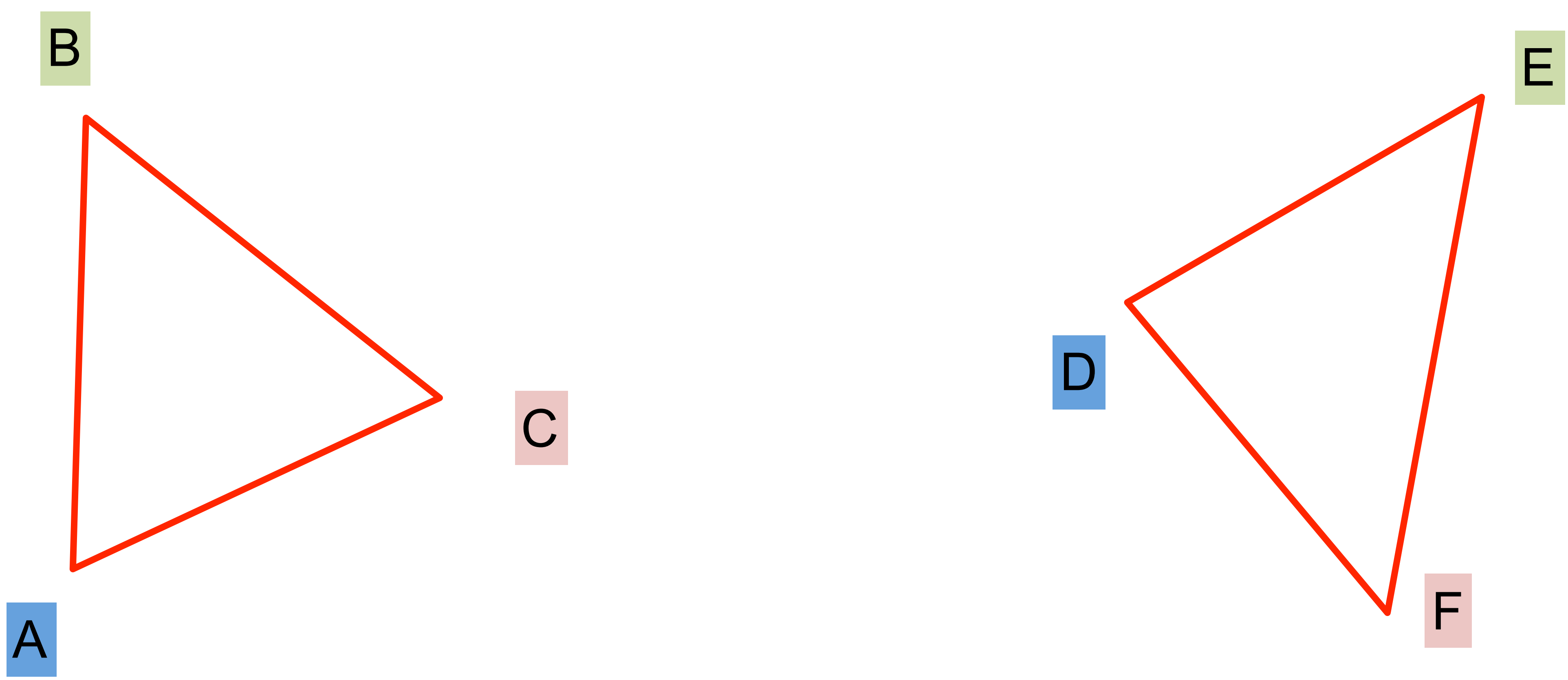

f(x,y) g(x,y)

- *How many Degrees of Freedom?*
- *How many correspondences needed for projective?*
- *What is the transformation matrix?*

Suppose we have two triangles: ABC and DEF.

What transformation will map A to D, B to E, and C to F?

How can we get the parameters?



Estimate transformation parameters using

# Linear least squares

Given a set of matched feature points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

point in one image     point in the other image

and a transformation

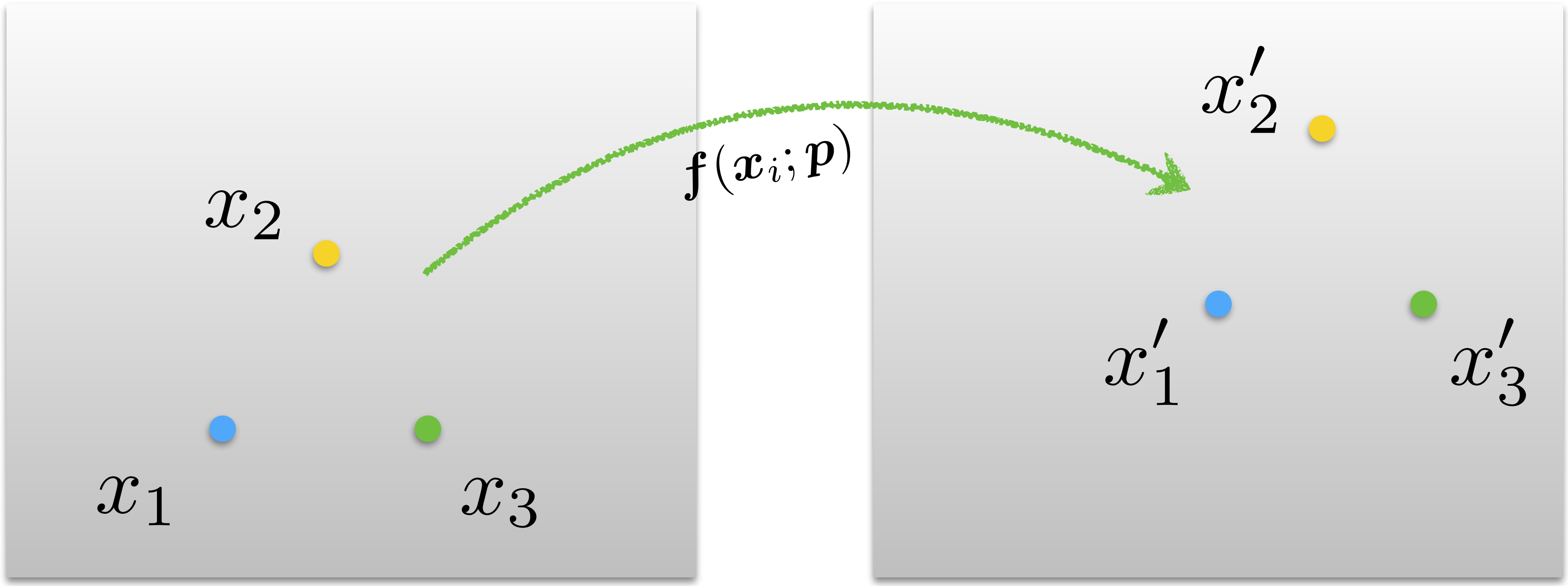
$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

transformation function     parameters

Find the best estimate of

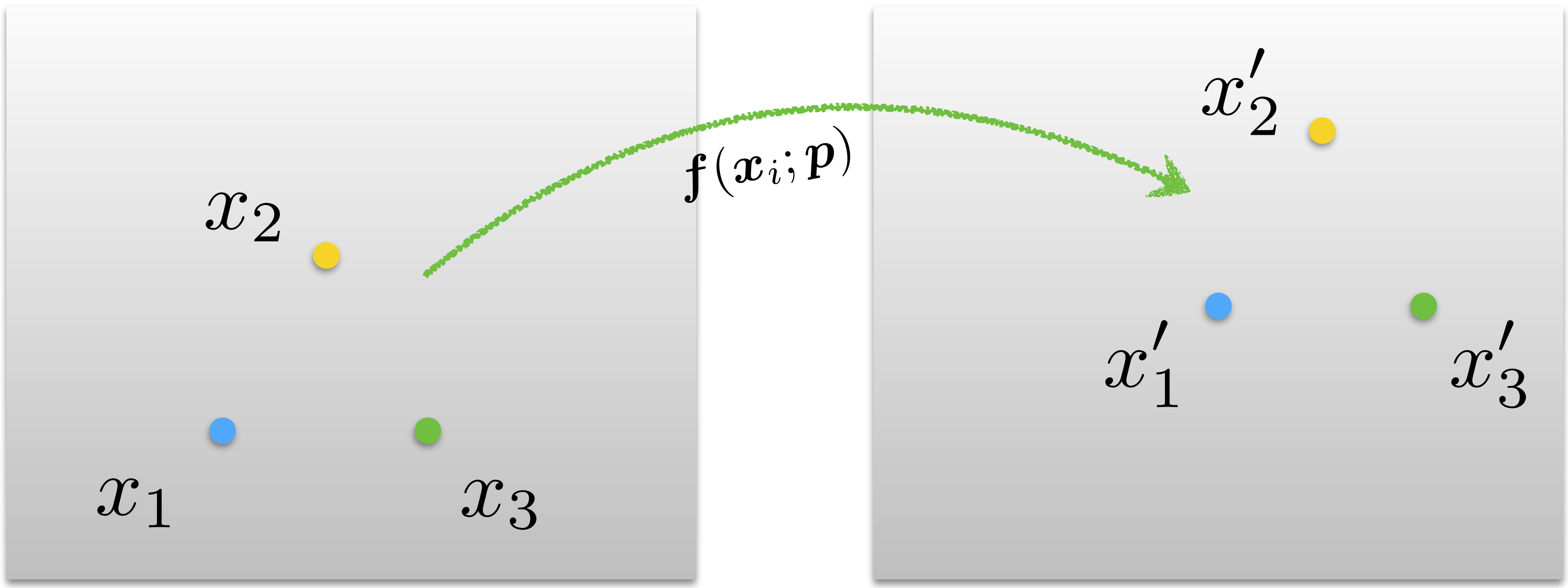
$$\boldsymbol{p}$$


Given point correspondences



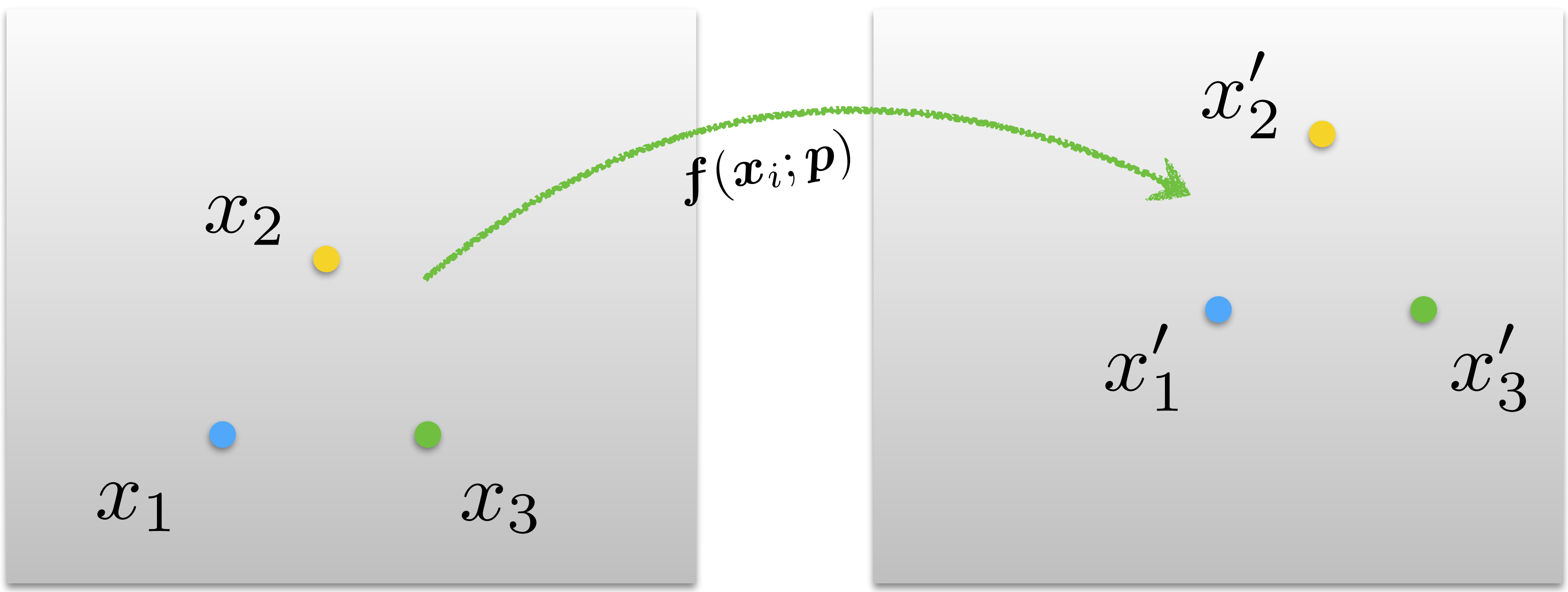

Given point correspondences

*How can you solve for the transformation?*




**Least Squares Error**

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

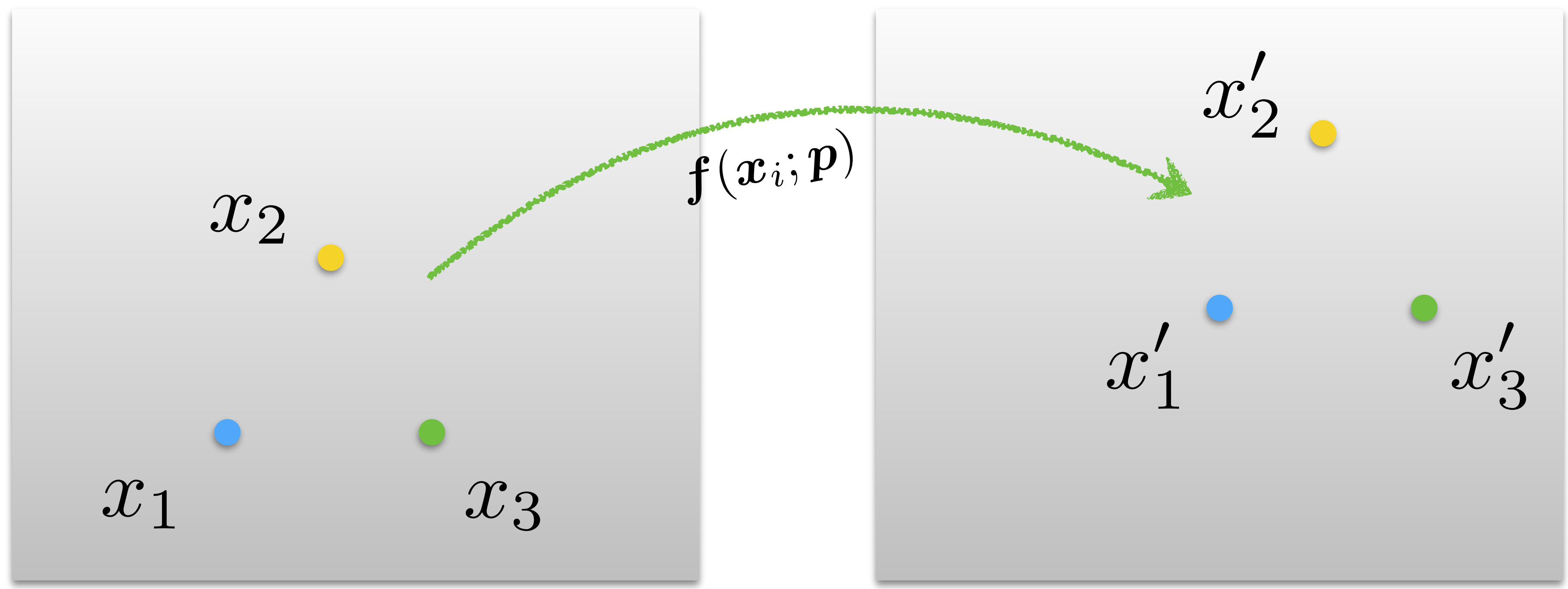



**Least Squares Error**

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i;\boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

*What is this?*

*What is this?*

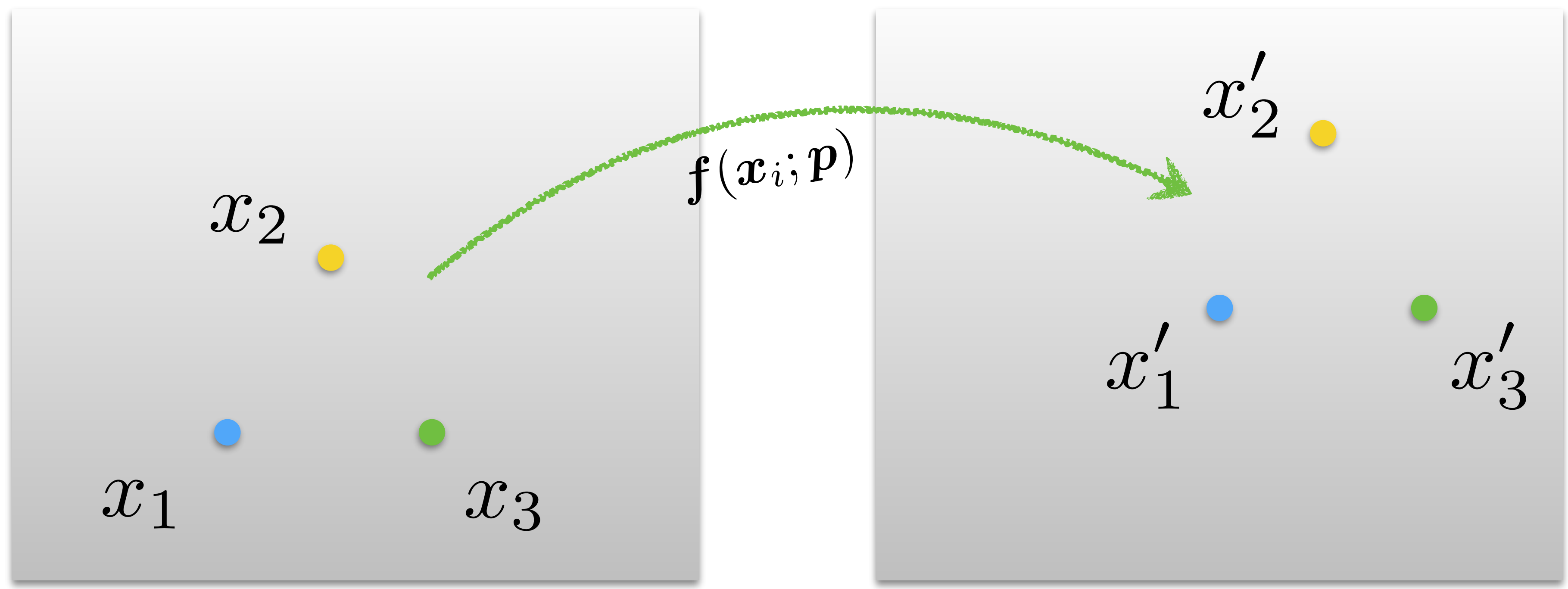



**Least Squares Error**

What is this?

$$E_{\mathrm{LS}} = \sum_i \| \boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i \|^2$$

What is this? What is this?




$$||\boldsymbol{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

**Least Squares Error**

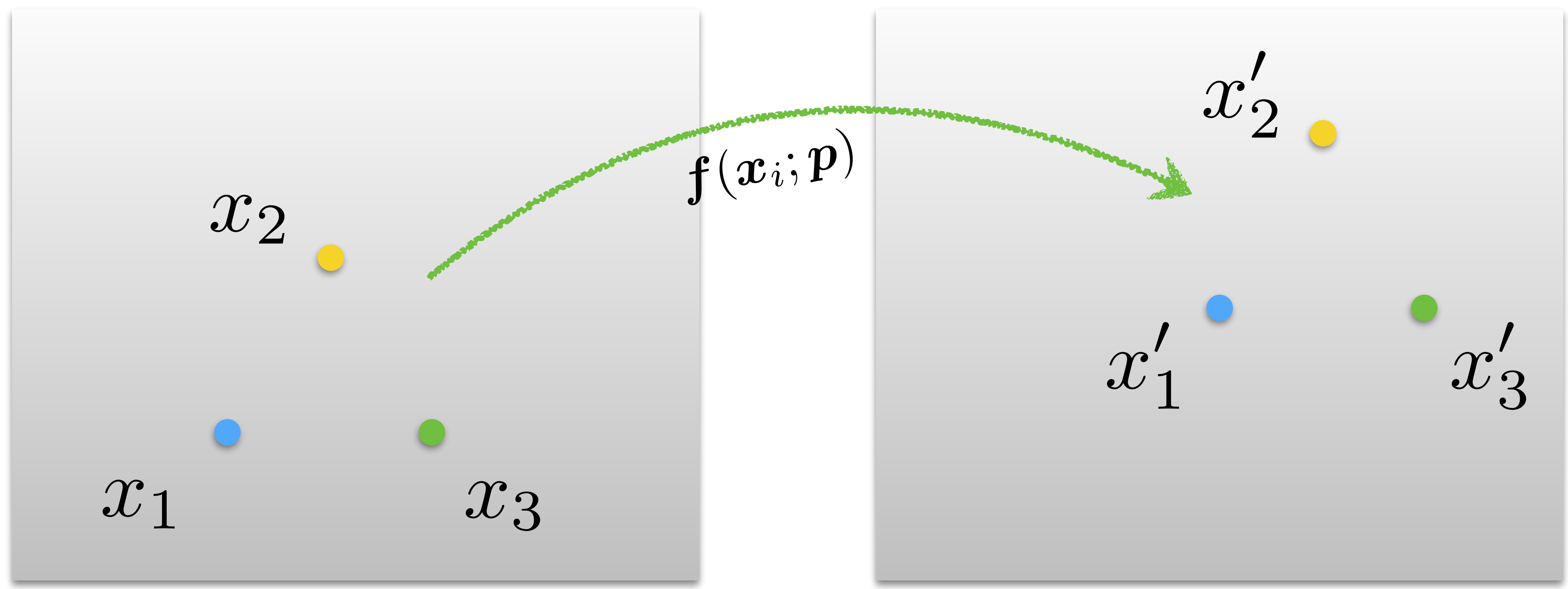
Euclidean (L2) norm

$$E_{\mathrm{LS}} = \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i;\boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

squared!

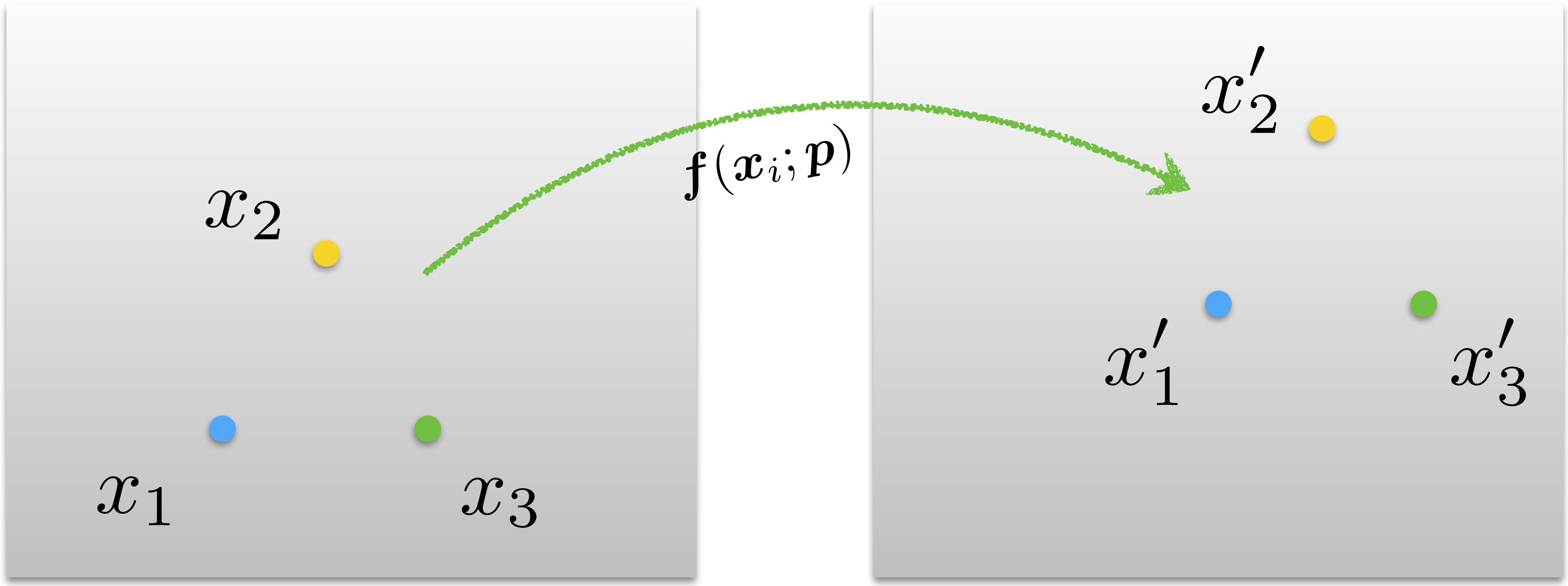
predicted location

measured location




## Least Squares Error

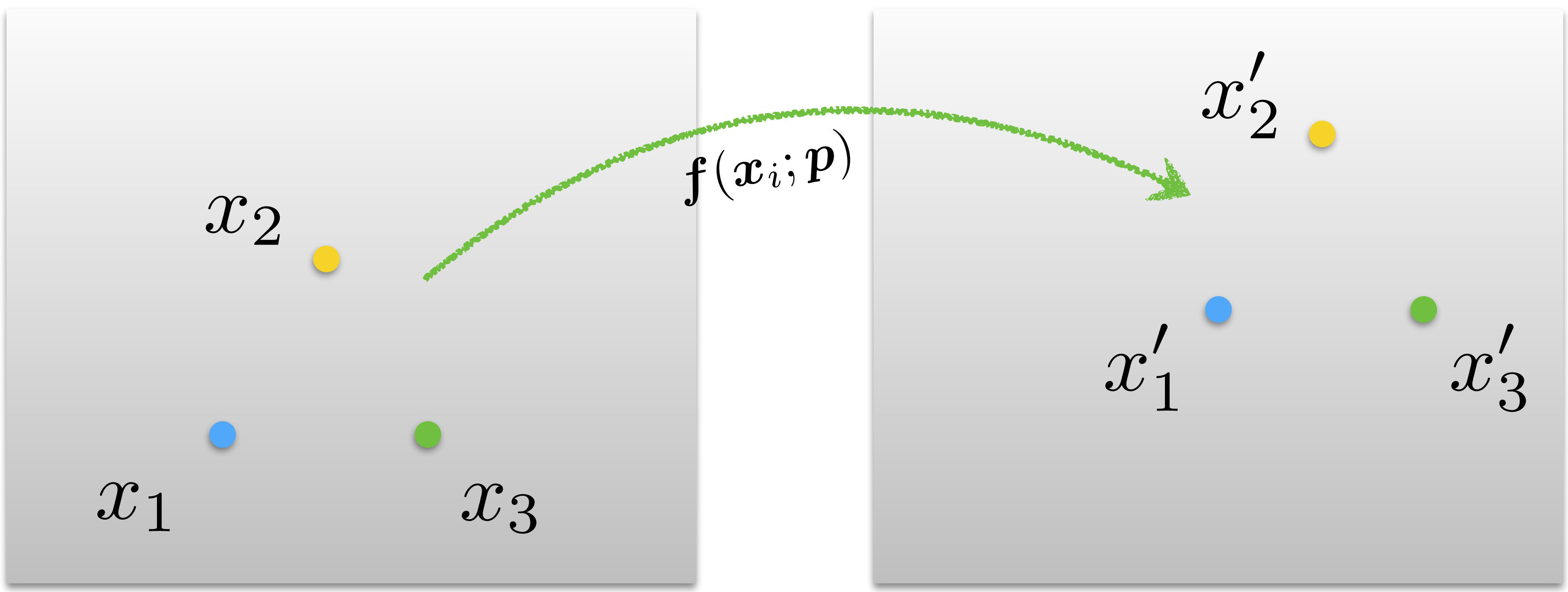
$$E_{\mathrm{LS}} = \sum_i \underbrace{\|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i\|^2}_{\text{Residual (projection error)}}$$




## Least Squares Error

$$E_{\mathrm{LS}} = \sum_i \| \boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i \|^2$$

*What is the free variable?*
*What do we want to optimize?*




Find parameters that minimize squared error

$$\hat{\boldsymbol{p}} = \arg\min_{\boldsymbol{p}} \sum_i \|\boldsymbol{f}(\boldsymbol{x}_i; \boldsymbol{p}) - \boldsymbol{x}'_i\|^2$$

# General form of linear least squares

(**Warning:** change of notation. x is a vector of parameters!)

$$E_{\mathrm{LLS}} = \sum_i |\boldsymbol{a}_i \boldsymbol{x} - \boldsymbol{b}_i|^2$$

$$= \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|^2 \qquad \text{(matrix form)}$$

This function is quadratic.

*How do you find the root of a quadratic?*

# General form of linear least squares

(**Warning:** change of notation. x is a vector of parameters!)

$$E_{\mathrm{LLS}} = \sum_i |\boldsymbol{a}_i \boldsymbol{x} - \boldsymbol{b}_i|^2$$

$$= \|\mathbf{A}\boldsymbol{x} - \boldsymbol{b}\|^2$$

(matrix form)

## Minimize the error:

Expand

$$E_{\mathrm{LLS}} = \boldsymbol{x}^\top(\mathbf{A}^\top\mathbf{A})\boldsymbol{x} - 2\boldsymbol{x}^\top(\mathbf{A}^\top\boldsymbol{b}) + \|\boldsymbol{b}\|^2$$

Take derivative, set to zero

$$(\mathbf{A}^\top\mathbf{A})\boldsymbol{x} = \mathbf{A}^\top\boldsymbol{b}$$

(normal equation)

Solve for x

$$\boxed{\boldsymbol{x} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\boldsymbol{b}}$$

For the Affine transformation $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x};\boldsymbol{p})$

$$\boldsymbol{x}' = \mathbf{M}\boldsymbol{x} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

**Vectorize** transformation parameters

$$\underbrace{\begin{bmatrix} x' \\ y' \\ x' \\ y' \\ \vdots \\ x' \\ y' \end{bmatrix}}_{\boldsymbol{b}} = \underbrace{\begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \\ x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \\ & & & \vdots & & \\ x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix}}_{\boldsymbol{x}}$$

Notation in general form

**Linear**

least squares

estimation

only works

when the

transform function

is

**?**

Linear

least squares

estimation

only works

when the

transform function

is

**linear!**

Also

doesn't

deal well

with

**outliers**