2D Image Transforms

16-385 Computer Vision (Kris Kitani)

Carnegie Mellon University
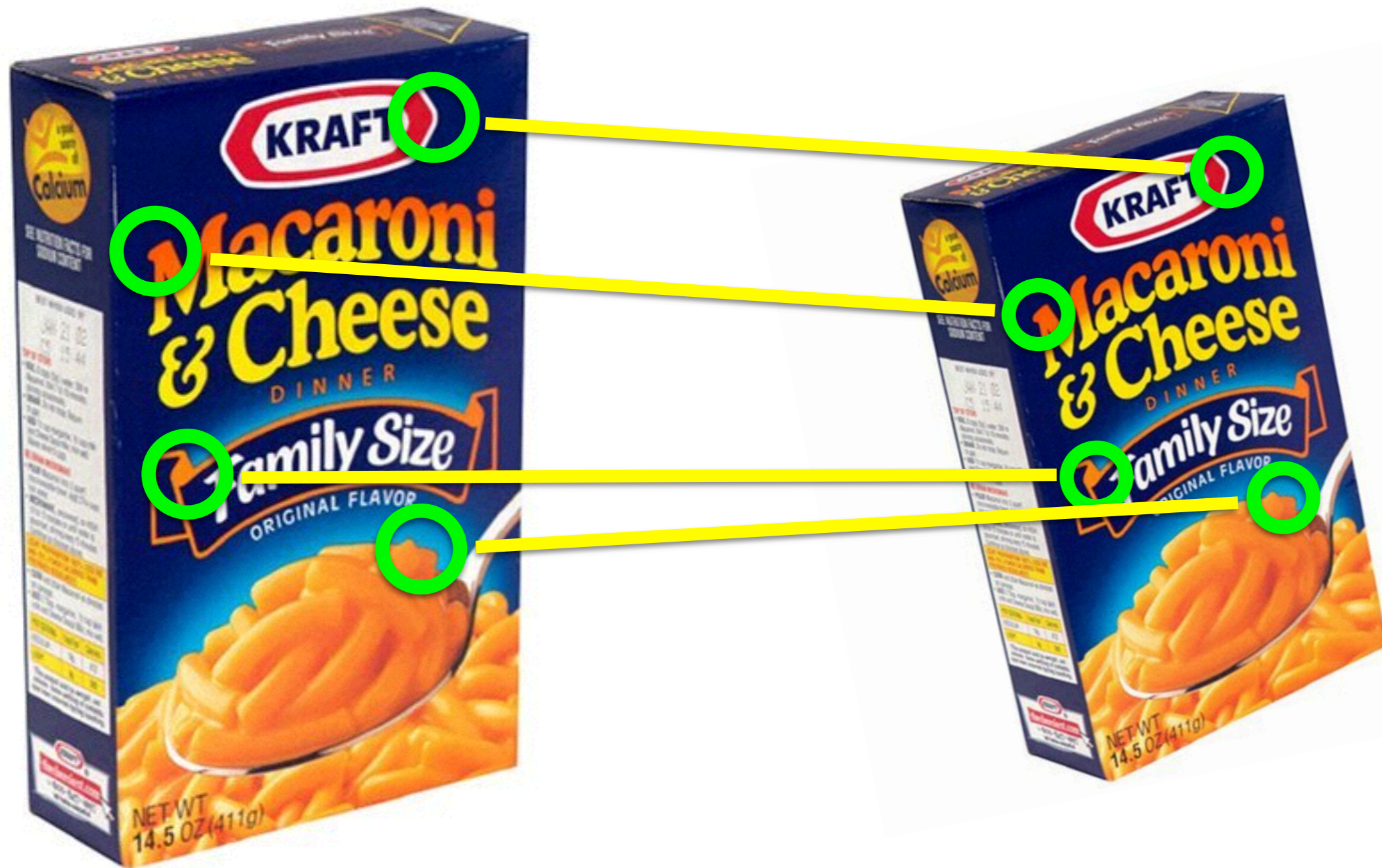
Extract features from an image …

what do we do next?

Feature matching

(object recognition, 3D reconstruction, augmented reality, image stitching)

How do you compute the transformation?

Given a set of matched feature points

$$\{ \boldsymbol{x}_i, \boldsymbol{x}_i' \}$$ ← set of point correspondences

point in one image   point in the other image

and a transformation

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

transformation function   parameters

Find the best estimate of

$$\boldsymbol{p}$$

What kind of transformation functions are there?

$$x' = f(x; p)$$

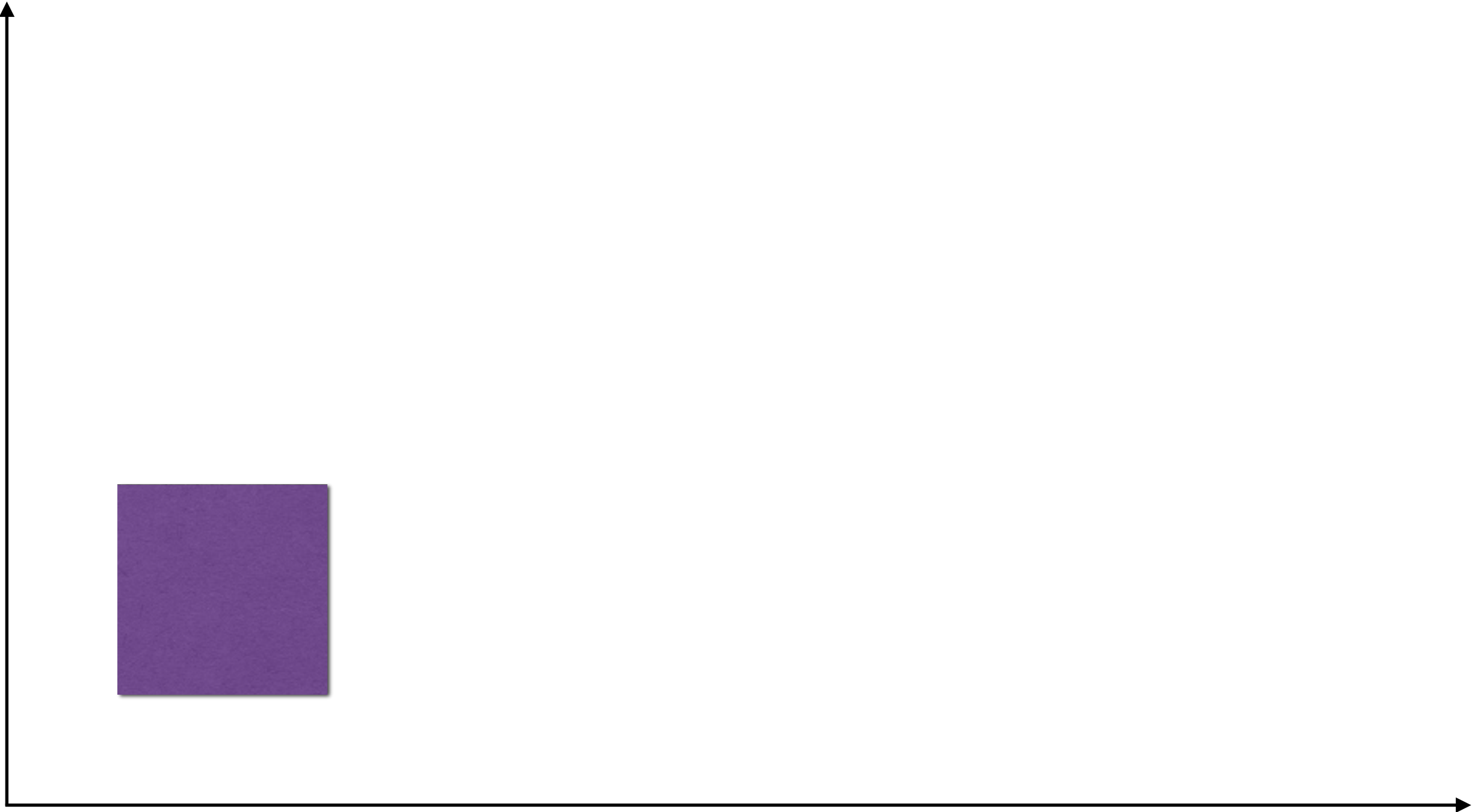# 2D Transformations

translation

rotation

aspect

affine

perspective

cylindrical

2D Planar Transformations

2D Planar Transformations

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

2D Planar Transformations

Scale

$$x' = ax$$

$$y' = by$$

Scale

- Each component multiplied by a scalar
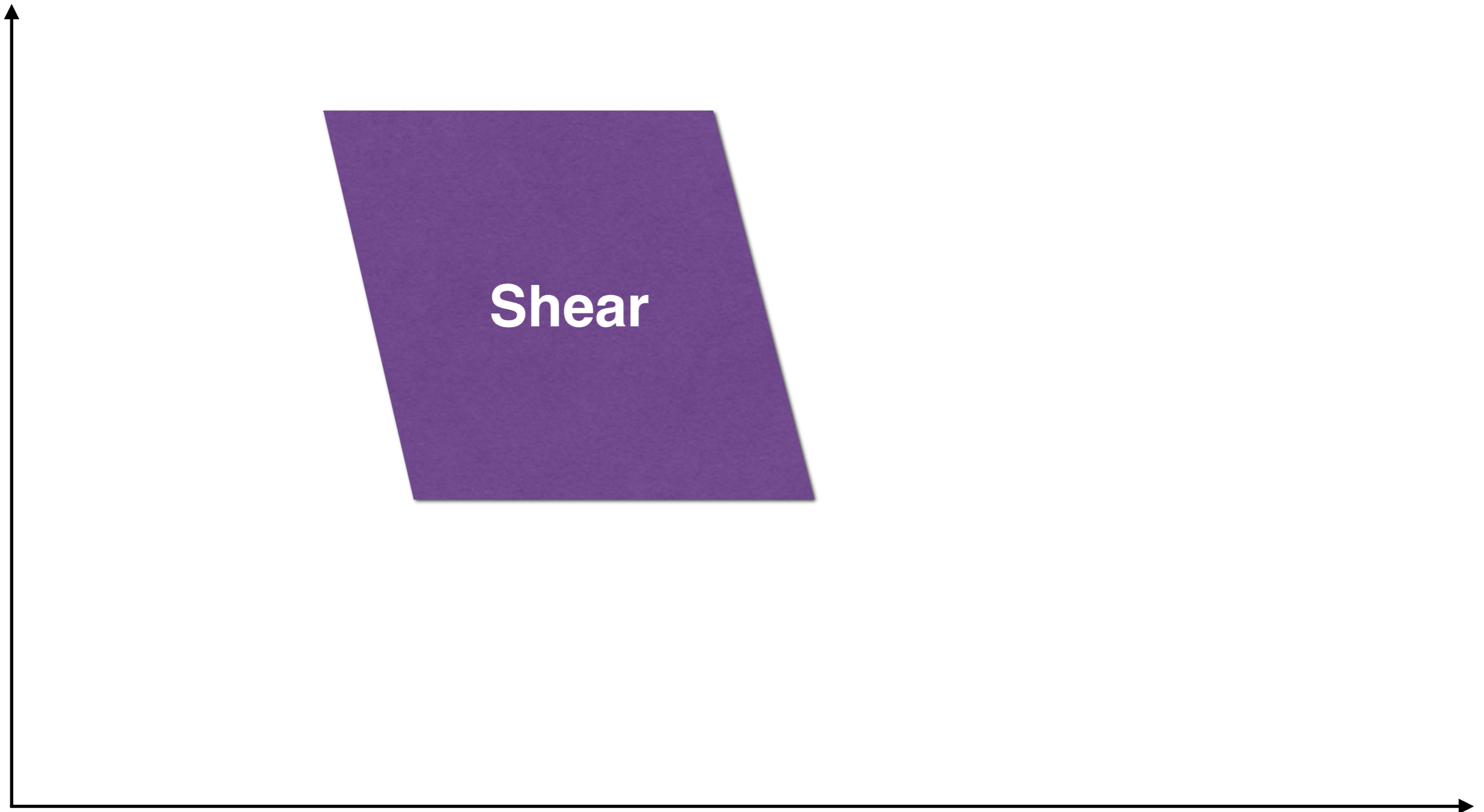- Uniform scaling - same scalar for each component

2D Planar Transformations

Scale

$$\left[\begin{array}{c} x' \\ y' \end{array} \right] = \left[\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right]$$

scaling matrix S

Scale

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

2D Planar Transformations

Shear

2D Planar Transformations

Shear

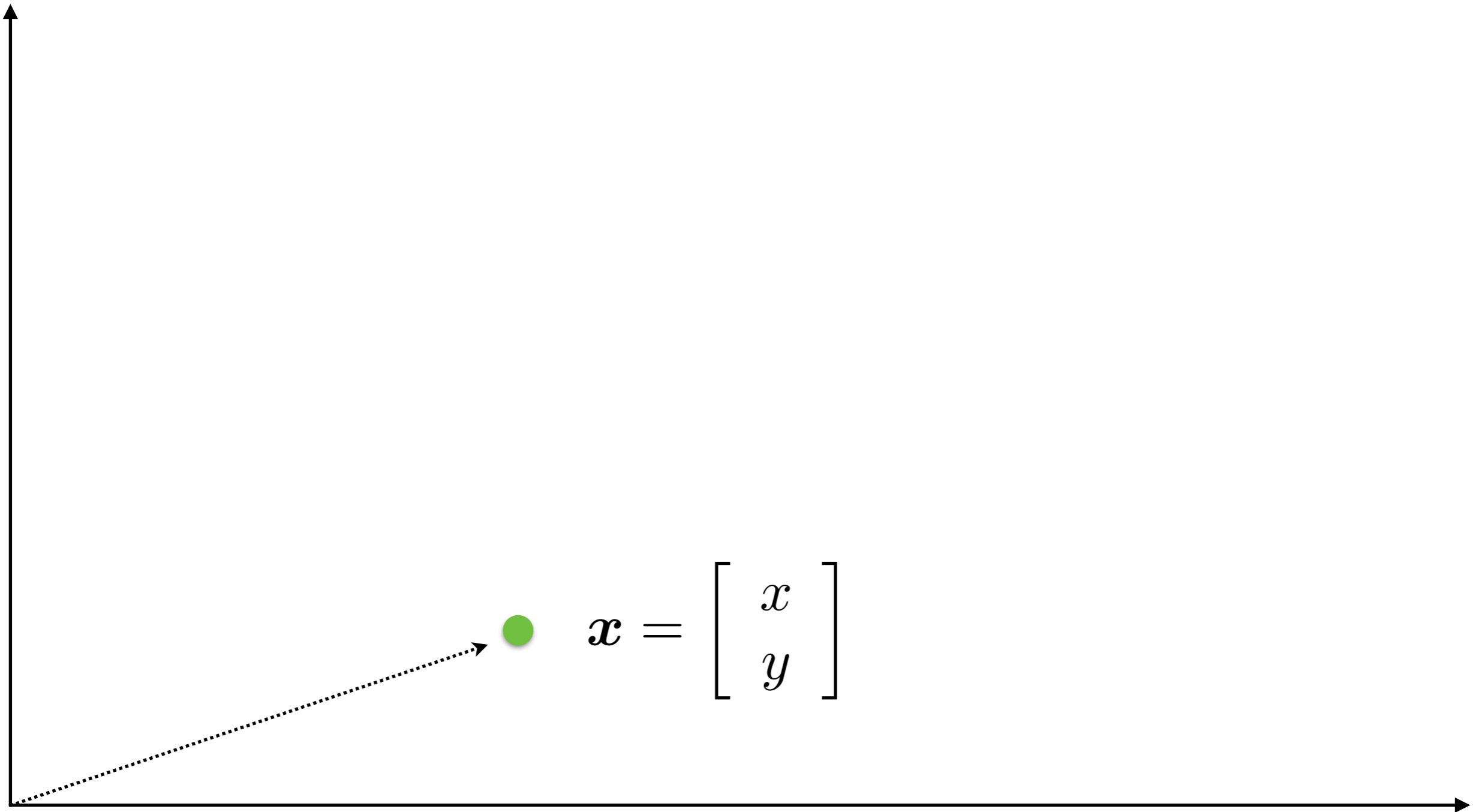$$x' = x + a \cdot y$$
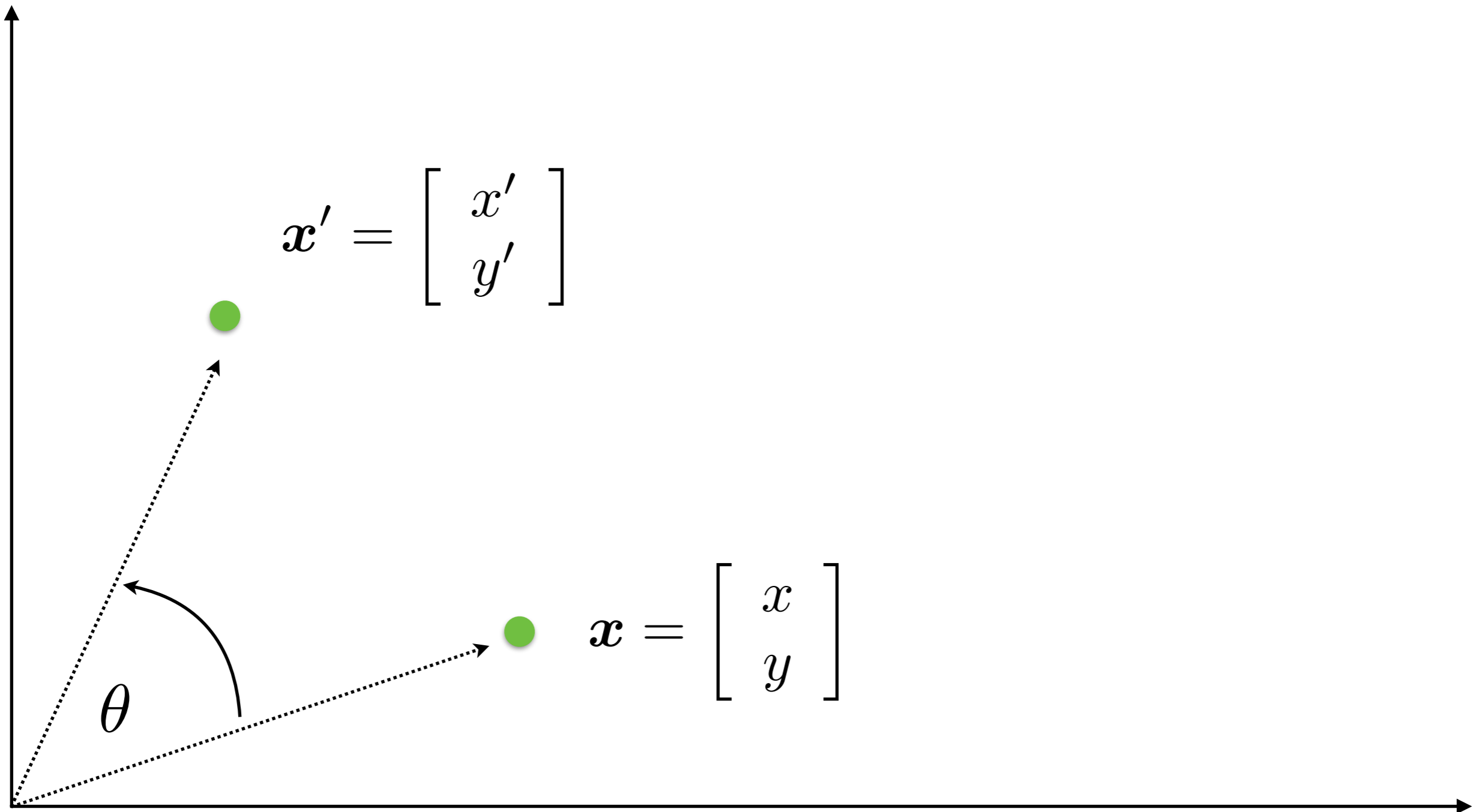$$y' = b \cdot x + y$$

2D Planar Transformations

Shear

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Planar Transformations
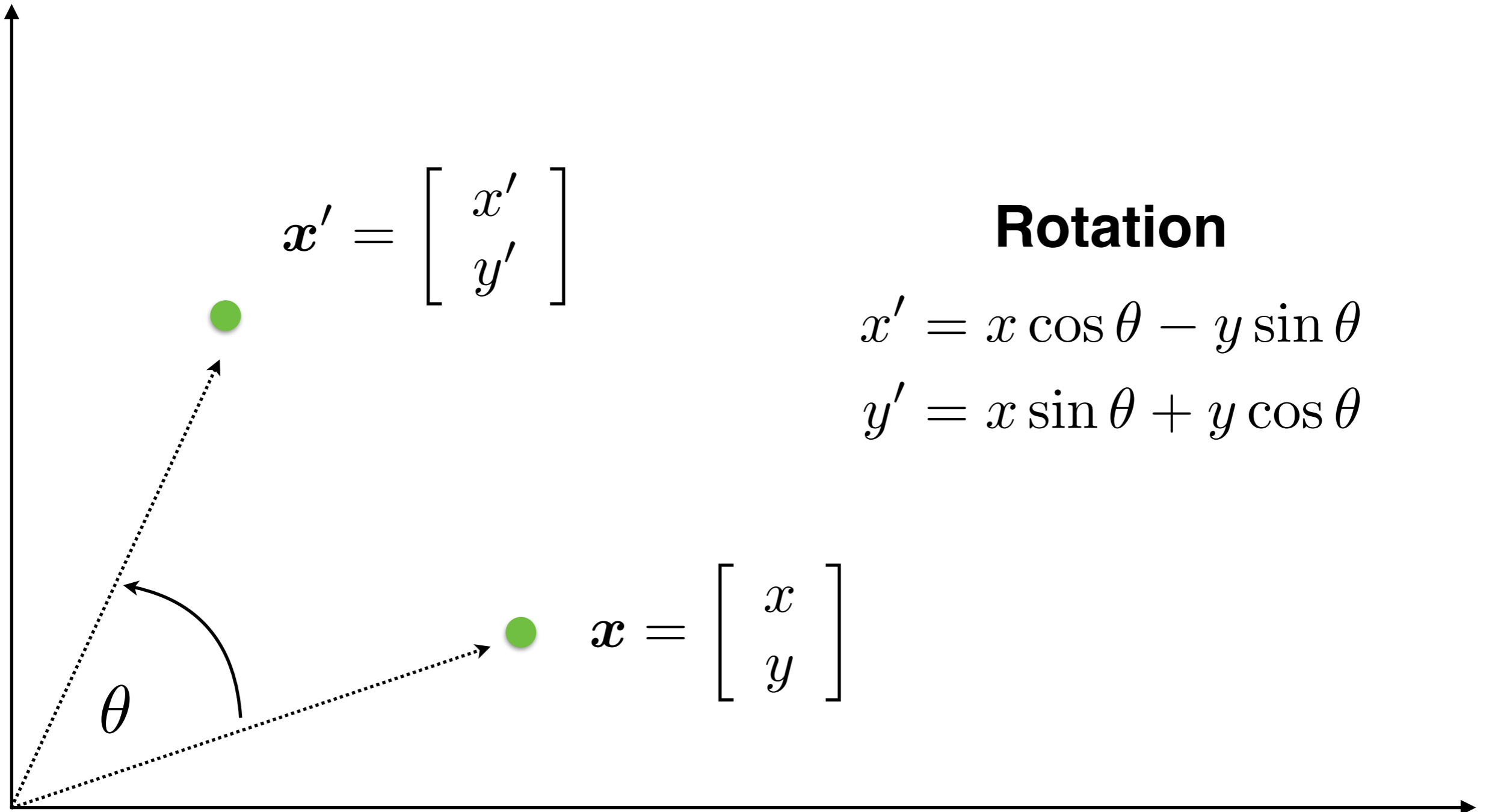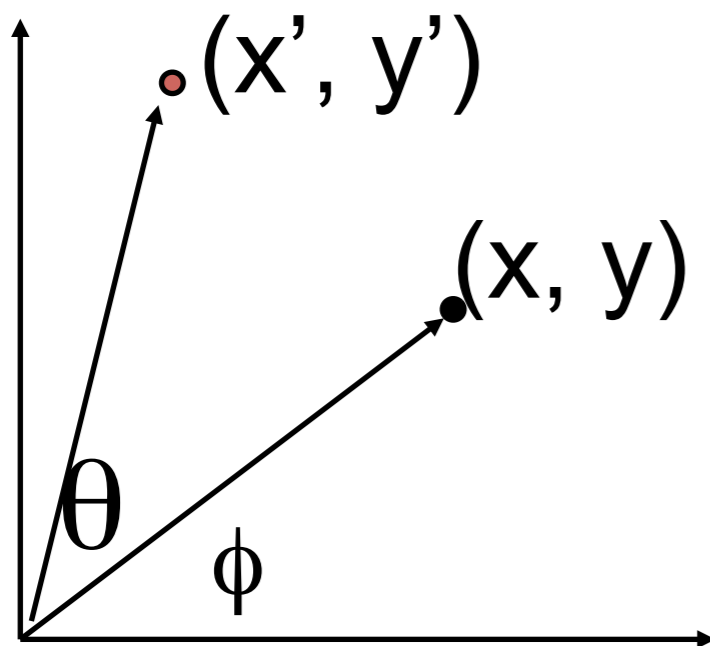
$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Planar Transformations

$$\boldsymbol{x}' = \left[\begin{array}{c} x' \\ y' \end{array} \right]$$

$$\boldsymbol{x} = \left[\begin{array}{c} x \\ y \end{array} \right]$$

θ

2D Planar Transformations

$$\boldsymbol{x}' = \left[\begin{array}{c} x' \\ y' \end{array} \right]$$

$$\boldsymbol{x} = \left[\begin{array}{c} x \\ y \end{array} \right]$$

θ

Rotation

$$x' = x \cos \theta - y \sin \theta$$
$$y' = x \sin \theta + y \cos \theta$$

Polar coordinates…
$x = r \cos(\phi)$
$y = r \sin(\phi)$
$x' = r \cos(\phi + \theta)$
$y' = r \sin(\phi + \theta)$

Trig Identity…
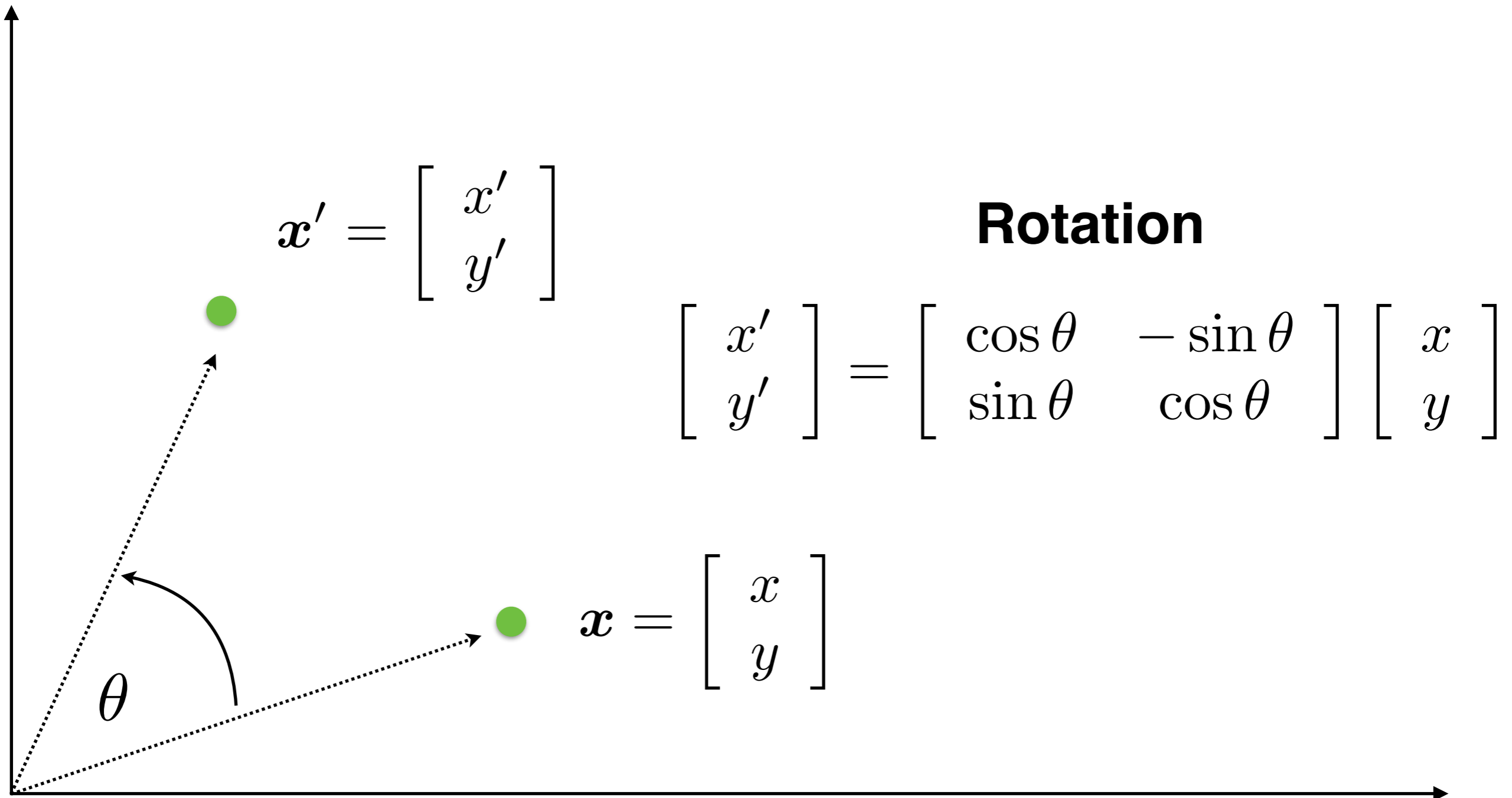$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$
$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute…
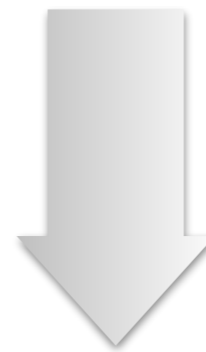$x' = x \cos(\theta) - y \sin(\theta)$
$y' = x \sin(\theta) + y \cos(\theta)$

2D Planar Transformations

$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

θ

2D **<u>linear</u>** transformation
(can be written in matrix form)

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$
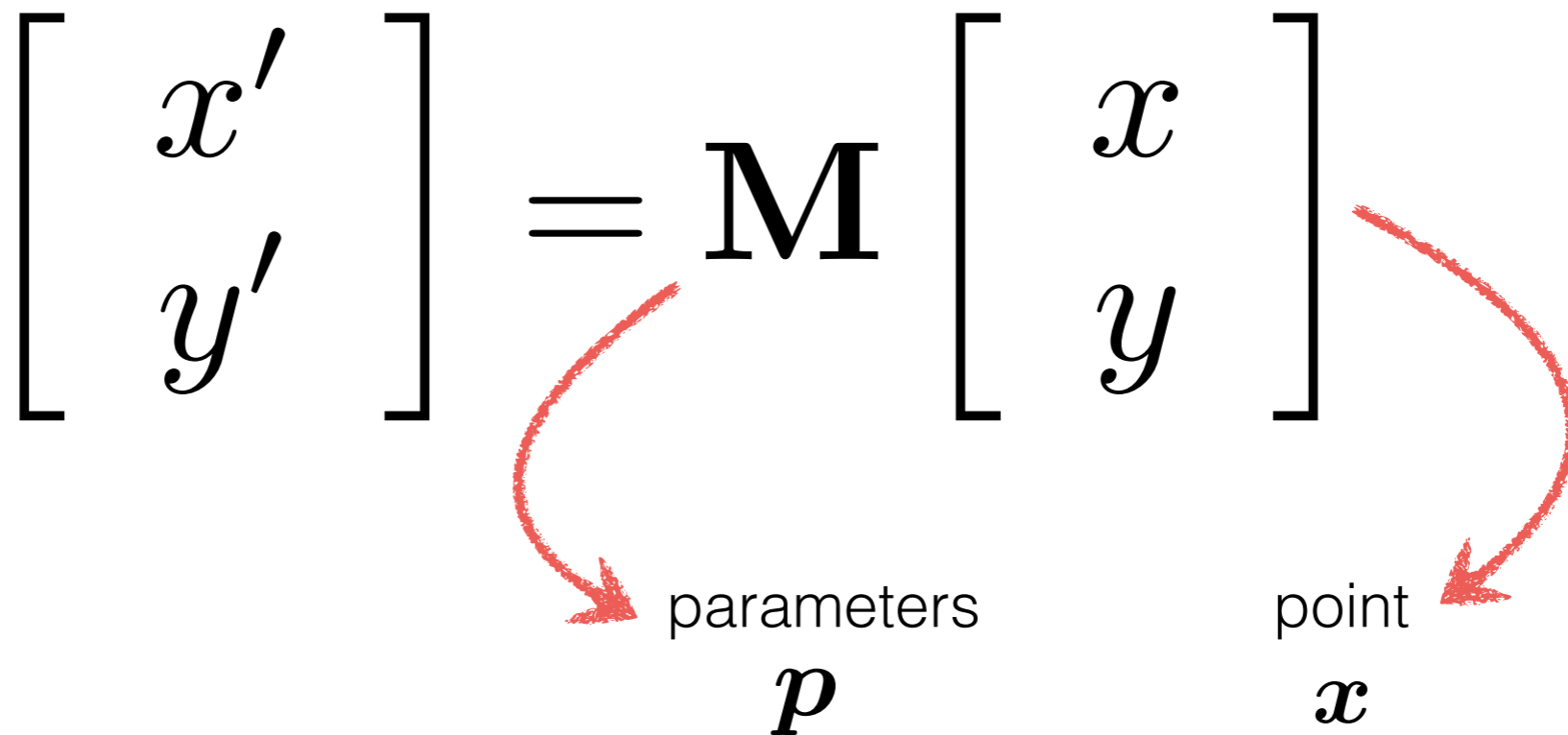
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

parameters
$\boldsymbol{p}$

point
$\boldsymbol{x}$

Scale

$$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Rotate

$$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Shear

$$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$

Flip across y

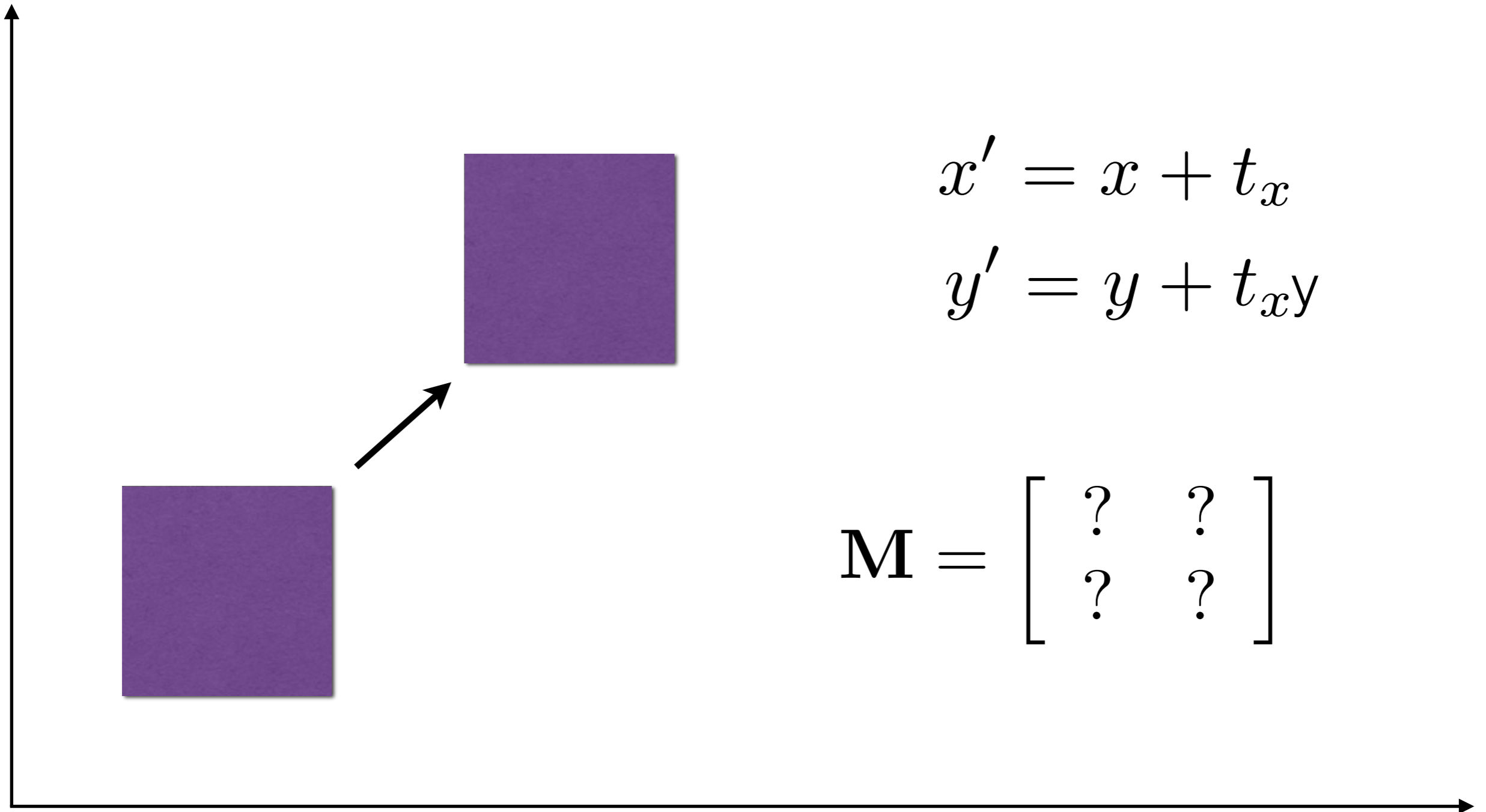$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Flip across origin

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Identity

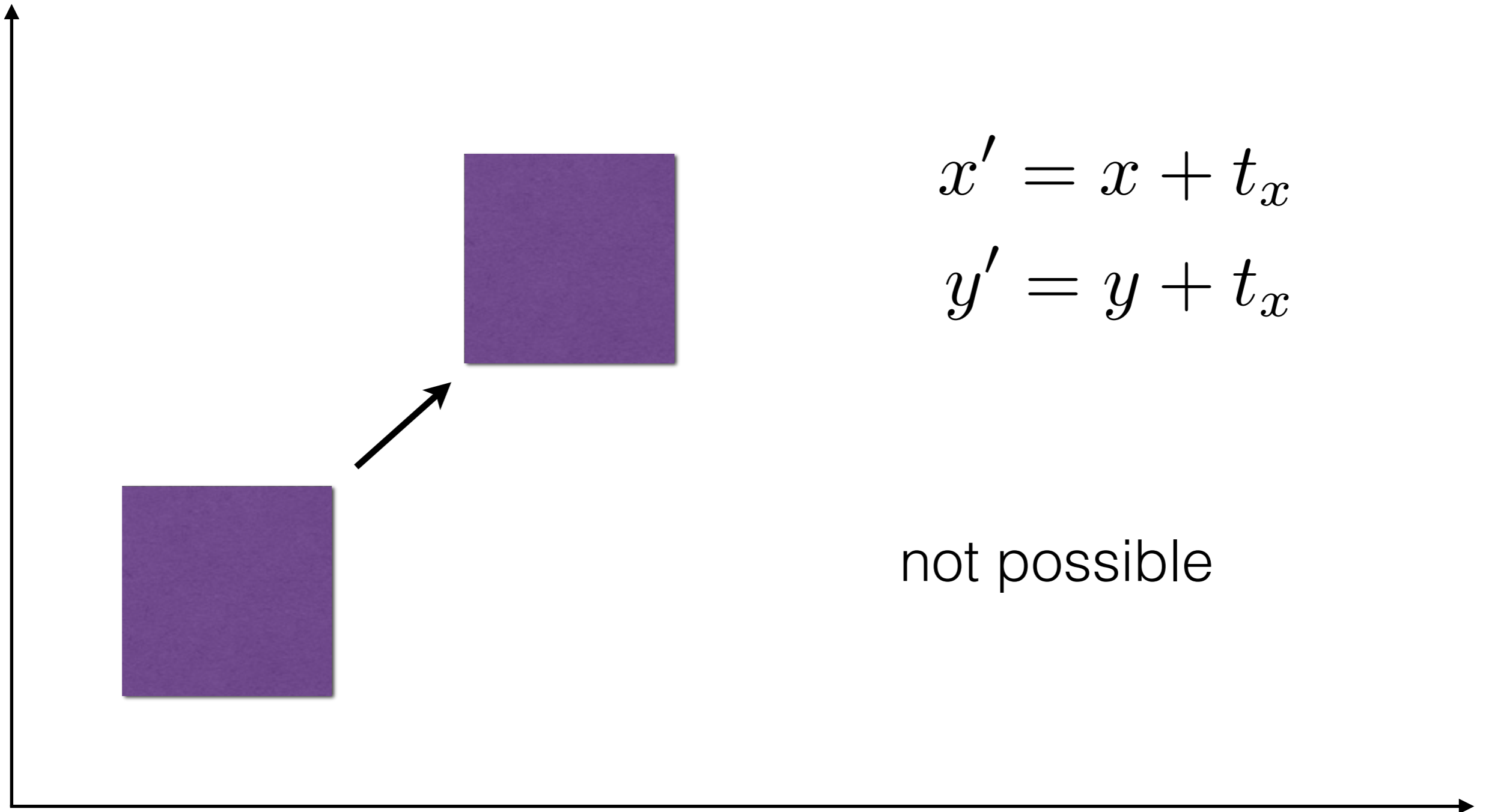$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

How do you represent translation with a 2 x 2 matrix?

$$x' = x + t_x$$
$$y' = y + t_x\mathsf{y}$$

$$\mathbf{M} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

How do you represent translation with a 2 x 2 matrix?

$$x' = x + t_x$$
$$y' = y + t_x$$

not possible

Q: How can we represent translation in matrix form?

$$x' = x + t_x$$

$$y' = y + t_y$$

Homogeneous Coordinates

add a one here

$$\begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

inhomogenous
coordinates

homogenous
coordinates
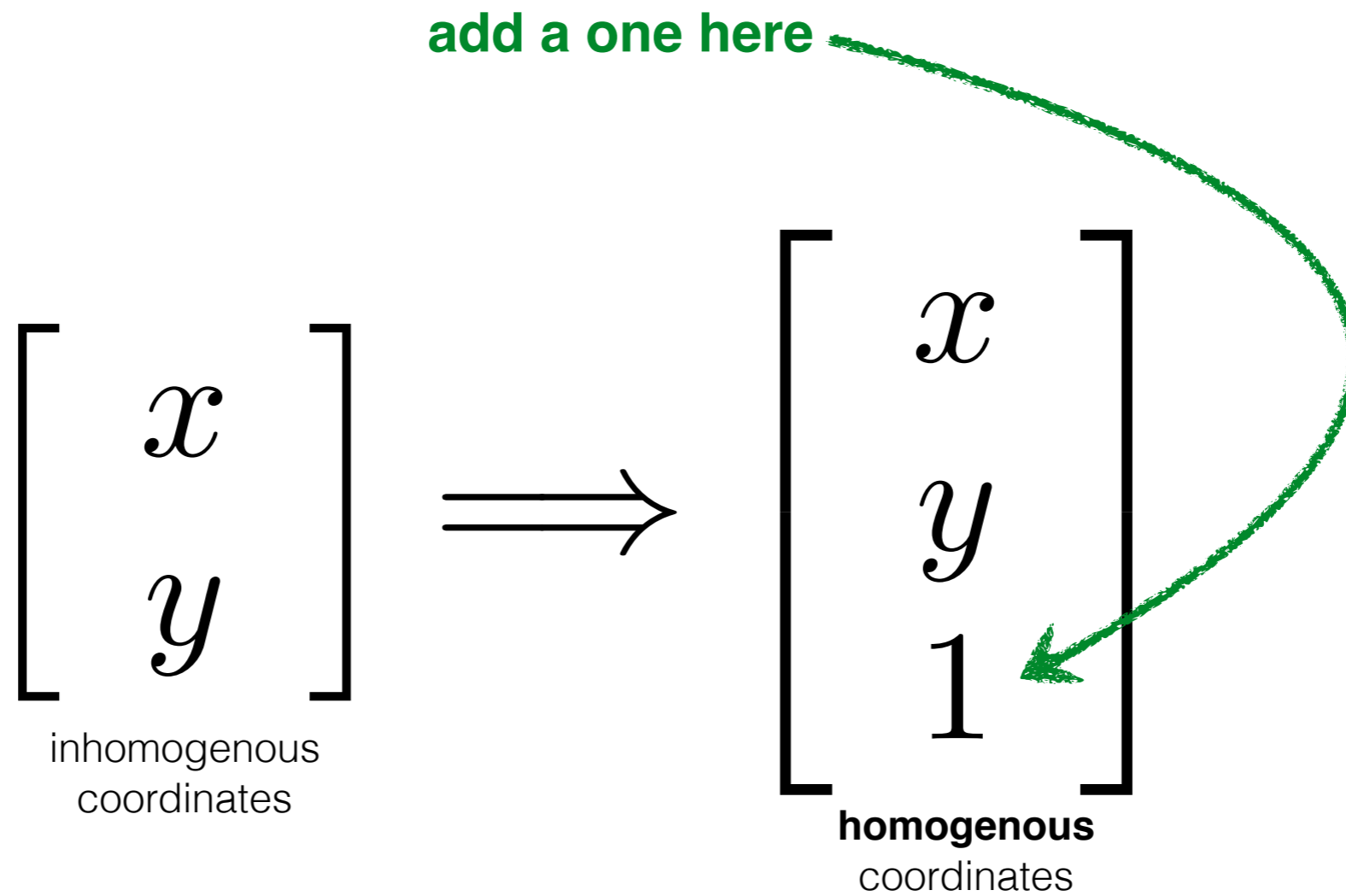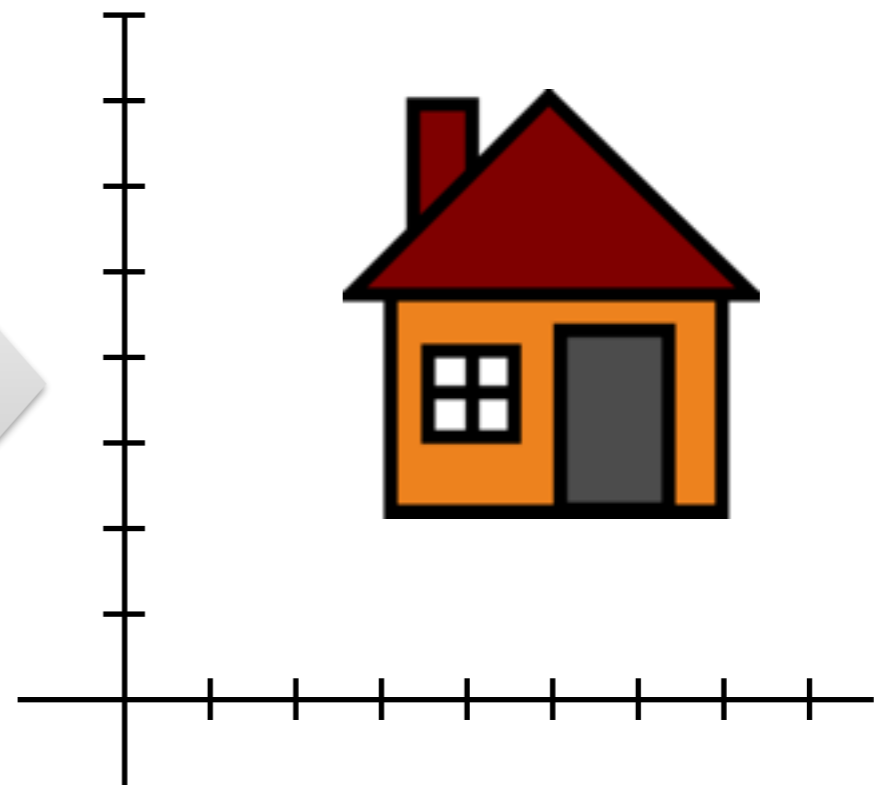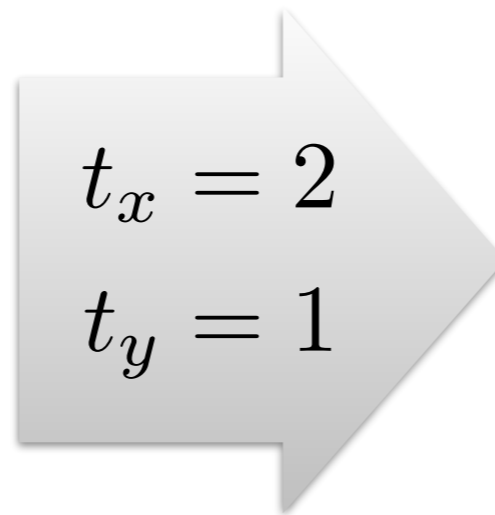
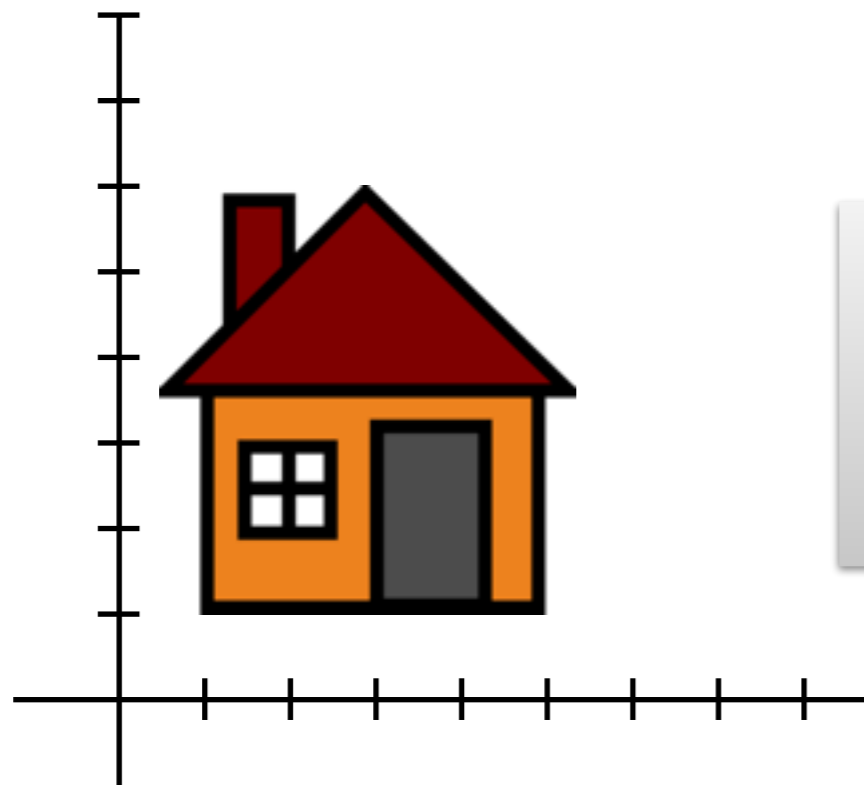Represent 2D point with a 3D vector

$$x' = x + t_x$$

$$y' = y + t_y$$

A: append 3rd element and append 3rd column & row

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous
Coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

$t_x = 2$
$t_y = 1$

A 2D point in an image can be represented as a 3D vector

$$\boldsymbol{x} = \left[\begin{array}{c} x \\ y \end{array} \right] \quad \Longleftrightarrow \quad \boldsymbol{X} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right]$$

where $\quad x = \dfrac{x_1}{x_3} \quad y = \dfrac{x_2}{x_3}$

Why?

Think of a point on the image plane in 3D

X is a projection of a point **P** on the image plane
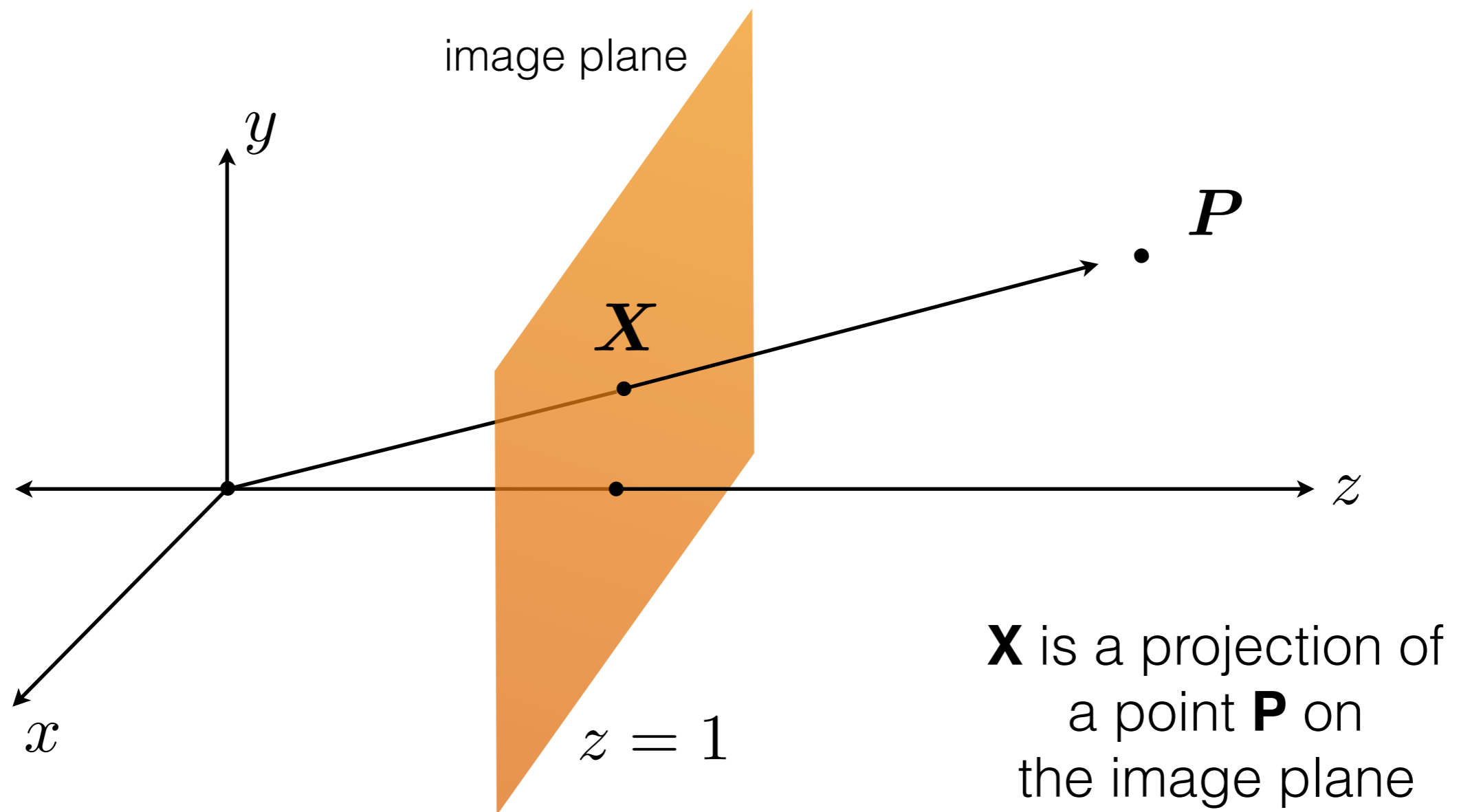
You can think of a conversion to homogenous coordinates as a conversion of a **point** to a **ray**

Conversion:

- 2D point ➙ homogeneous point

  $$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

  append 1 as 3rd coordinate

- homogeneous point ➙ 2D point

  $$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

  divide by 3rd coordinate

Special Properties

- Scale invariant

  $$\begin{bmatrix} x & y & w \end{bmatrix}^\top = \lambda \begin{bmatrix} x & y & w \end{bmatrix}^\top$$

- Point at infinity

  $$\begin{bmatrix} x & y & 0 \end{bmatrix}$$

- Undefined

  $$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s_x} & 0 & 0 \\ 0 & \boldsymbol{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Matrix Composition

Transformations can be combined by matrix multiplication

$$
\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}
$$

p' = T(t$_x$,t$_y$) R(Θ) S(s$_x$,s$_y$) p

Does the order of multiplication matter?

2D transformations
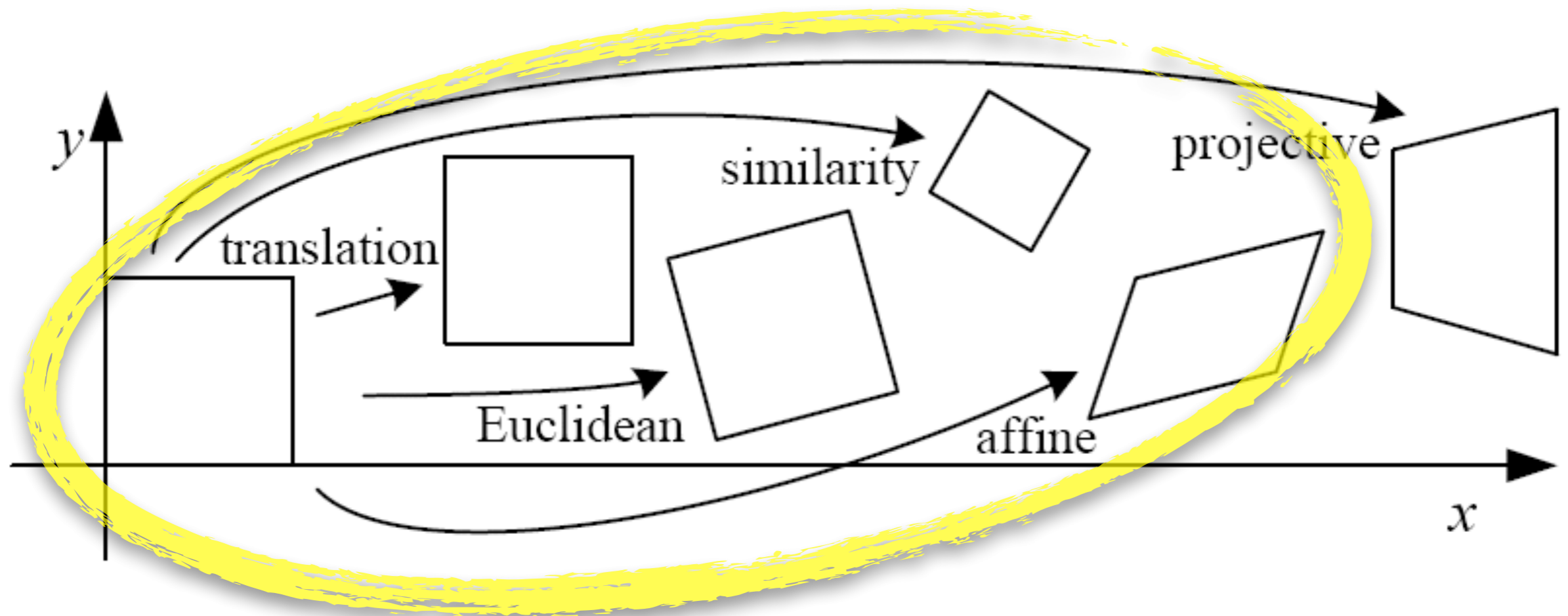
Figure 1: Basic set of 2D planar transformations

| Name	Matrix	# D.O.F.
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2 \times 3}$	2
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2 \times 3}$	3
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2 \times 3}$	4
affine	$\begin{bmatrix} A \end{bmatrix}_{2 \times 3}$	6
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3 \times 3}$	8

Affine Transformation

Affine transformations are combinations of

- •Linear transformations, and
- •Translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- •Origin does not necessarily map to origin
- •Lines map to lines
- •Parallel lines remain parallel
- •Ratios are preserved
- •Closed under composition (affine times affine is affine)
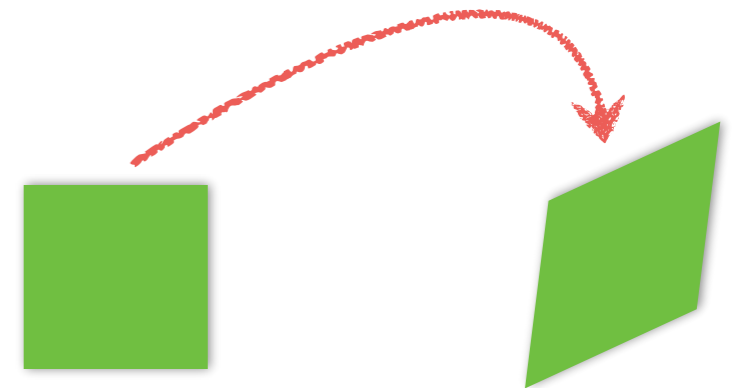
Will the last coordinate w ever change?

Coming soon…

Projective Transform

Projective transformations are combos of

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition
- Models change of basis
- Projective matrix is defined up to a scale (8 DOF)