16-385 Computer Vision, Spring 2020

Take-home Quiz 11

Due Date: Sunday May 3, 2020 23:59

Question 1

In the first part of this quiz, you will derive the Lucas-Kanade (or forward-additive) image alignment algorithm. Consider first a *warp function* $\mathbf{W}(\mathbf{x};\mathbf{p})$ that maps coordinate vectors $\mathbf{x} \in \mathbb{R}^2$ to other coordinate vectors in $\mathbb{R}^2$, with the mapping depending on a set of parameters $\mathbf{p} \in \mathbb{R}^N$. Given an image $I(\mathbf{x})$ and a template $T(\mathbf{x})$, we want to find the parameters $\mathbf{p}$ such that the warp $\mathbf{W}(\mathbf{x};\mathbf{p})$ best aligns the image with the template in terms of sum-of-squared-differences (SSD) error. That is, we want to find the parameters $\mathbf{p}$ that minimize the loss function:

$$\min_{\mathbf{p}} \sum_{\mathbf{x}} \left[I\left(\mathbf{W}\left(\mathbf{x};\mathbf{p}\right)\right) - T\left(\mathbf{x}\right)\right]^2 . \tag{1}$$

The Lucas-Kanade alignment algorithm minimizes Equation (1) using the Gauss-Newton algorithm. To this end, given some initial set of parameters $\mathbf{p}^0$, they are updated iteratively as:

$$\mathbf{p}^{t+1} = \mathbf{p}^t + \Delta\mathbf{p}^t, \tag{2}$$

for $t = 0, \dots, T$, where the number of iterations T can be selected based on any of the common convergence criteria. Then, the Gauss-Newton algorithm corresponds to selecting a specific form for the update vector $\Delta\mathbf{p}^t$, which you will derive below step-by-step.

1. Use the first-order Taylor expansion to linearize the composite function $I\left(\mathbf{W}\left(\mathbf{x};\mathbf{p}\right)\right)$ with respect to $\mathbf{p}$ around the value $\mathbf{p}^t$. Write out the expression for this Taylor expansion.

2. Combine the Taylor expansion expression with Equation (2), to obtain an approximation for $I\left(\mathbf{W}\left(\mathbf{x};\mathbf{p}^t + \Delta\mathbf{p}^t\right)\right)$.

3. Show that, using this approximation, the optimization problem of Equation (1) can be rewritten in the form:

$$\min_{\Delta\mathbf{p}^t} \left\|\mathbf{A}\Delta\mathbf{p}^t - \mathbf{b}\right\|^2 , \tag{3}$$

   for some matrix $\mathbf{A}$ and vector $\mathbf{b}$.

4. Show how to solve the optimization problem of Equation (3) for the parameter update $\Delta\mathbf{p}^t$, and write out an expression for this solution.

5. Finally, explain how this expression for $\Delta\mathbf{p}^t$ can be evaluated, using image convolutions, warps, element-wise operations, and matrix-vector operations. You can either explain this in words or provide pseudocode, but make sure to explain each step clearly.

Question 2

In the second part of this quiz, you will follow a similar sequence of steps to derive the Shum-Szeliski (or forward-compositional) image alignment algorithm. In this case, we are directly optimizing the warp function itself, rather than a parameterization of it. That is, given some initial warping function $\mathbf{W}^0$, it is updated iteratively through composition as:

$$\mathbf{W}^{t+1}(\mathbf{x}) = \mathbf{W}^t\left(\mathbf{W}\left(\mathbf{x}, \Delta\mathbf{p}^t\right)\right), \tag{4}$$

where the parameters $\mathbf{p}^t$ are selected by solving the optimization problem:

$$\min_{\Delta\mathbf{p}^t} \sum_{\mathbf{x}} \left[I\left(\mathbf{W}^t\left(\mathbf{W}\left(\mathbf{x}, \Delta\mathbf{p}^t\right)\right)\right) - T(\mathbf{x})\right]^2. \tag{5}$$

This update is performed for $t = 0, \ldots, T$, where the number of iterations T can be selected based on any of the common convergence criteria. Unfortunately, minimizing the non-linear Equation (8) exactly is difficult. Therefore, in practice, the parameter vector $\Delta\mathbf{p}^t$ is computed using an approximate solution of Equation (8), which you will derive below step-by-step. (The steps in this derivation are similar to the Lucas-Kanade derivation, but different in subtle-yet-important ways.)

1. Use the first-order Taylor expansion to linearize the composite function $I\left(\mathbf{W}^t\left(\mathbf{W}\left(\mathbf{x}, \Delta\mathbf{p}\right)\right)\right)$ with respect to $\Delta\mathbf{p}$ around the value $\mathbf{0}$ (this is a vector of all zeros). Write out the expression for this Taylor expansion.

2. Show that, by using this Taylor expansion expression, the optimization problem of Equation (1) can be rewritten in the form:

   $$\min_{\Delta\mathbf{p}^t} \left\|\mathbf{A}\Delta\mathbf{p}^t - \mathbf{b}\right\|^2, \tag{6}$$

   for some matrix $\mathbf{A}$ and vector $\mathbf{b}$.

3. Show how to solve the optimization problem of Equation (9) for the parameter vector $\Delta\mathbf{p}^t$, and write out an expression for this solution.

4. Explain how this expression for $\Delta\mathbf{p}^t$ can be evaluated, using image convolutions, warps, element-wise operations, and matrix-vector operations. You can either explain this in words or provide pseudocode, but make sure to explain each step clearly.

5. Finally, discuss in detail how the Lucas-Kanade and Shum-Szeliski algorithms are different from each other.

Question 3

In the third and las part of this quiz, you will perform yet another derivation along the same lines, to derive the Baker-Matthews (or Matthews-Baker, or inverse-compositional) image alignment algorithm. As in the previous algorithm, we are directly optimizing the warp function itself, rather than a parameterization of it. However, in this case, given some initial warping function $\mathbf{W}^0$, it is updated iteratively through composition as:

$$\mathbf{W}^{t+1}(\mathbf{x}) = \mathbf{W}^t\left(\mathbf{W}^{-1}\left(\mathbf{x}, \Delta\mathbf{p}^t\right)\right), \tag{7}$$

where the parameters $\mathbf{p}^t$ are selected by solving the optimization problem:

$$\min_{\Delta\mathbf{p}^t} \sum_{\mathbf{x}} \left[I\left(\mathbf{W}^t(\mathbf{x})\right) - T\left(\mathbf{W}\left(\mathbf{x}, \Delta\mathbf{p}^t\right)\right)\right]^2. \tag{8}$$

This update is performed for $t = 0, \ldots, T$, where the number of iterations T can be selected based on any of the common convergence criteria. As usual, minimizing the non-linear Equation (8) exactly is difficult, and we use a linearization procedure to obtain an expression for $\Delta\mathbf{p}^t$, which you will derive below step-by-step. (Yet again, the steps in this derivation are similar to the derivation of Shum-Szeliski, but different in subtle-yet-important ways.)

1. Use the first-order Taylor expansion to linearize the composite function $T(\mathbf{W}(\mathbf{x}, \Delta\mathbf{p}))$ with respect to $\Delta\mathbf{p}$ around the value $\mathbf{0}$. Write out the expression for this Taylor expansion.

2. Show that, by using this Taylor expansion expression, the optimization problem of Equation (1) can be rewritten in the form:

$$\min_{\Delta\mathbf{p}^t} \left\|\mathbf{A}\Delta\mathbf{p}^t - \mathbf{b}\right\|^2, \tag{9}$$

   for some matrix $\mathbf{A}$ and vector $\mathbf{b}$.

3. Show how to solve the optimization problem of Equation (9) for the parameter vector $\Delta\mathbf{p}^t$, and write out an expression for this solution.

4. Explain how this expression for $\Delta\mathbf{p}^t$ can be evaluated, using image convolutions, warps, element-wise operations, and matrix-vector operations. You can either explain this in words or provide pseudocode, but make sure to explain each step clearly.

5. Finally, discuss in detail how the Shum-Szeliski and Baker-Matthews algorithms are different from each other.

Instructions

1. **Integrity and collaboration:** Students are encouraged to work in groups but each student must submit their own work. If you work as a group, include the names of your collaborators in your write up. Plagiarism is strongly prohibited and may lead to failure of this course.

2. **Questions:** If you have any questions, please look at Piazza first. Other students may have encountered the same problem, and it may be solved already. If not, post your question on the discussion board. Teaching staff will respond as soon as possible.

3. **Write-up:** Your write-up should be typese in LaTeXand should consist of your answers to the theory questions. Please note that we **DO NOT** accept handwritten scans for your write-up in quizzes.

4. **Submission:** Your submission for this assignment should be a PDF file, `<andrew-id.pdf>`, composed of your write-up. **Please do not submit ZIP files.**