Photometric stereo and shape from shading
Course announcements

- Homework 3 has been posted and is due on March 9th.
  - Any questions about the homework?
  - How many of you have looked at/started/finished homework 3?

- Office hours for Yannis’ this week: Wednesday 3-5 pm.

- Gaurav’s office hours now happen on Smith 200.

- Many more talks this week:
  1. Manolis Savva, "Human-centric Understanding of 3D Environments," Wednesday March 7, 2:00 PM, NSH 3305.

  2. David Fouhey, "Recovering a Functional and Three Dimensional Understanding of Images," Thursday March 8, 4:00 PM, NSH 3305.
Overview of today’s lecture

- Light sources.
- Shape from shading.
- Photometric stereo.
Most of these slides were adapted from:

- Srinivasa Narasimhan (16-385, Spring 2014).
- Todd Zickler (Harvard University).
- Steven Gortler (Harvard University).
Light sources
“Physics-based” computer vision (a.k.a “inverse optics”)

\[ I \rightarrow \text{shape, illumination, reflectance} \]
Lighting models: Plenoptic function

- Radiance as a function of position and direction
- Radiance as a function of position, direction, and time
- Spectral radiance as a function of position, direction, time and wavelength

\[ L(x, \omega, t, \lambda) \]

Fig. 1.3
The plenoptic function describes the information available to an observer at any point in space and time. Shown here are two schematic eyes— which one should consider to have punctate pupils— gathering pencils of light rays. A real observer cannot see the light rays coming from behind, but the plenoptic function does include these rays.

[Adelson and Bergen, 1991]
Lighting models: far-field approximation

- Assume that, over the observed region of interest, all sources of incoming flux are relatively far away

\[ L(x, \omega, t, \lambda) \rightarrow L(\omega, t, \lambda) \]
\[ L(x, \omega) \rightarrow L(\omega) \]

[Debevec, 1998]
Lighting models: far-field approximation

• Assume that, over the observed region of interest, all source of incoming flux are relatively far away

\[ L(x, \omega, t, \lambda) \rightarrow L(\omega, t, \lambda) \]
\[ L(x, \omega) \rightarrow L(\omega) \]

[Debevec, 1998]
Lighting models: far-field approximation

• Assume that, over the observed region of interest, all source of incoming flux are relatively far away

\[
\begin{align*}
L(x, \omega, t, \lambda) &\rightarrow L(\omega, t, \lambda) \\
L(x, \omega) &\rightarrow L(\omega)
\end{align*}
\]

radiance only depends on direction; not location

ignores close inter-reflections

[Debevec, 1998]
Application: augmented reality

$L(\omega)$
Application: augmented reality

[Debevec, 1998]
Application: augmented reality

(a) Background photograph

(b) Camera calibration grid and light probe

(g) Final result with differential rendering

[Debevec, 1998]
Application: augmented reality

http://gl.ict.usc.edu/LightStages/
Application: augmented reality

[Introducing ARKit](https://developer.apple.com/arkit/)

[ARCore Overview](https://developers.google.com/ar/)

ARCore is a platform for building augmented reality apps on Android. ARCore uses three key technologies to integrate virtual content with the real world as seen through your phone’s camera:

- **Motion tracking** allows the phone to understand and track its position relative to the world.
- **Environmental understanding** allows the phone to detect the size and location of flat horizontal surfaces like the ground or a coffee table.
- **Light estimation** allows the phone to estimate the environment’s current lighting conditions.
Lighting models: far-field approximation

- One can download far-field lighting environments that have been captured by others:
  [http://gl.ict.usc.edu/Data/HighResProbes/]

- A number of apps and software exist to help you capture your own environments using a light probe.

**TABLE OF LIGHT PROBES:**

<table>
<thead>
<tr>
<th>Location</th>
<th>Description</th>
<th>Interactive Preview</th>
<th>Download</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uffizi Gallery, Italy</td>
<td>Assembled from 18 14mm images taken using the Kodak DCS 520 camera</td>
<td>LDR panorama</td>
<td>HDR (1.4MB)</td>
</tr>
<tr>
<td>Grace Cathedral, San Francisco, California</td>
<td>Assembled from three 18mm fisheye images taken using the Canon EOS-1ds camera</td>
<td>LDR panorama</td>
<td>HDR (1.4MB)</td>
</tr>
<tr>
<td>Dining room of the Ennis Brown House, Los Angeles, California (website)</td>
<td>Assembled from six 18mm fisheye images taken using the Canon EOS-1ds camera</td>
<td>LDR panorama</td>
<td>HDR (1.4MB)</td>
</tr>
<tr>
<td>On a glacier in Banff National Forest, Canada</td>
<td>Assembled from three 18mm fisheye images taken using the Canon EOS-1ds camera</td>
<td>LDR panorama</td>
<td>HDR (1.4MB)</td>
</tr>
<tr>
<td>Piazzetta at dusk, Italy</td>
<td>Assembled from three 18mm fisheye images taken using the Canon 5D camera</td>
<td>LDR panorama</td>
<td>HDR (2.0MB)</td>
</tr>
<tr>
<td>Courtyard of the Doge's palace, Venice, Italy</td>
<td>Assembled from five 18mm fisheye images taken using the Canon 5D camera</td>
<td>LDR panorama</td>
<td>HDR (2.2MB)</td>
</tr>
</tbody>
</table>

*Figure 6. To produce the equal-area cylindrical projection of a spherical map, one projects each point on the surface of the sphere horizontally outward onto the cylinder, and then unwraps the cylinder to obtain a rectangular “panoramic” map.*
Application: inferring outdoor illumination

From a single image (left), we estimate the most likely sky appearance (middle) and insert a 3-D object (right). Illumination estimation was done entirely automatically.

[Lalonde et al., 2009]
A further simplification:
Low-frequency illumination

\[ L(\omega) = \sum_{i} a_i Y_i(\omega) \]

First nine basis functions are sufficient for re-creating Lambertian appearance

[Ramamoorthi and Hanrahan, 2001; Basri and Jacobs, 2003]
Low-frequency illumination

Fig. 2. On the left, a white sphere illuminated by three directional (distant point) sources of light. All the lights are parallel to the image plane, one source illuminates the sphere from above and the two others illuminate the sphere from diagonal directions. In the middle, a cross-section of the lighting function with three peaks corresponding to the three light sources. On the right, a cross-section indicating how the sphere reflects light. We will make precise the intuition that the material acts as a low-pass filtering, smoothing the light as it reflects it.

Figure 3. Plot of spherical harmonic terms in Lambertian BRDF filter.
Low-frequency illumination

\[ L(\omega) = \sum_i a_i Y_i(\omega) \]

\[ \vec{\ell} = (\ell_1, \ldots, \ell_9) \]

[Ramamoorthi and Hanrahan, 2001; Basri and Jacobs, 2003]
Application: Trivial rendering

Capture light probe

Rendering a (convex) diffuse object in this environment simply requires a lookup based on the surface normal at each pixel

Low-pass filter (truncate to first nine SHs)
White-out: Snow and Overcast Skies

CAN’ T perceive the shape of the snow covered terrain!

CAN perceive shape in regions lit by the street lamp!!

WHY?
Diffuse Reflection from Uniform Sky

Assume Lambertian Surface with Albedo = 1 (no absorption)

Assume Sky radiance is constant

Substituting in above Equation:

\[ L_{\text{surface}}(\theta_r, \phi_r) = \int_{-\pi}^{\pi} \int_{0}^{\pi/2} L_{\text{src}}(\theta_i, \phi_i) f(\theta_i, \phi_i; \theta_r, \phi_r) \cos \theta_i \sin \theta_i d\theta_i d\phi_i \]

- \( f(\theta_i, \phi_i; \theta_r, \phi_r) = \frac{1}{\pi} \)
- \( L_{\text{src}}(\theta_i, \phi_i) = L_{\text{sky}} \)
- Substituting in above Equation:

\[
L_{\text{surface}}(\theta_r, \phi_r) = L_{\text{sky}}
\]

Radiance of any patch is the same as Sky radiance !! (white-out condition)
Even simpler:
Directional lighting

- Assume that, over the observed region of interest, all source of incoming flux is from one direction

\[ L(x, \omega, t, \lambda) \rightarrow L(x, t, \lambda) \rightarrow s(t, \lambda) \delta(\omega = \omega_o(t)) \]
\[ L(x, \omega) \rightarrow L(\omega) \rightarrow s \delta(\omega = \omega_o) \]

- Convenient representation

\[ \vec{\ell} = (\ell_x, \ell_y, \ell_z) \]

“light direction” \[ \hat{\ell} = \frac{\vec{\ell}}{||\vec{\ell}||} \]

“light strength” \[ ||\vec{\ell}|| \]
Simple shading

ASSUMPTION 1: LAMBERTIAN

ASSUMPTION 2: DIRECTIONAL LIGHTING

\[ L_{\text{out}}(\hat{\omega}) = \int_{\Omega_{\text{in}}} f(\hat{\omega}_{\text{in}}, \hat{\omega}_{\text{out}}) L_{\text{in}}(\hat{\omega}_{\text{in}}) \cos \theta_{\text{in}} d\hat{\omega}_{\text{in}} \]

\[ I = a \hat{n} \cdot \hat{l} \]

[Prados, 2004]
“N-dot-l” shading

ASSUMPTION 1: LAMBERTIAN

\[
L^{\text{out}}(\hat{\omega}) = \int_{\Omega_{\text{in}}} f(\hat{\omega}_{\text{in}}, \hat{\omega}_{\text{out}}) L^{\text{in}}(\hat{\omega}_{\text{in}}) \cos \theta_{\text{in}} d\hat{\omega}_{\text{in}}
\]

\[
I = a \hat{n} \cdot \hat{l}
\]

ASSUMPTION 2: DIRECTIONAL LIGHTING

[Prados, 2004]
An ideal point light source

$$L(x, \omega) = \frac{s}{||x - x_o||^2} \delta \left( \omega = \frac{x - x_o}{||x - x_o||} \right)$$

Think of this as a spatially-varying directional source where
1. the direction is away from $x_o$
2. the strength is proportional to $1/(\text{distance})^2$
Summary of some useful lighting models

- plenoptic function (function on 5D domain)
- far-field illumination (function on 2D domain)
- low-frequency far-field illumination (nine numbers)
- directional lighting (three numbers = direction and strength)
- point source (four numbers = location and strength)
Shape from shading
Image Intensity and 3D Geometry

- Shading as a cue for shape reconstruction
- What is the relation between intensity and shape?
  - Reflectance Map
Application: Detecting composite photos

Fake photo

Real photo
"N-dot-l" shading

ASSUMPTION 1: LAMBERTIAN

\[ L^{\text{out}}(\hat{\omega}) = \int_{\Omega_{\text{in}}} f(\hat{\omega}_{\text{in}}, \hat{\omega}_{\text{out}}) L^{\text{in}}(\hat{\omega}_{\text{in}}) \cos \theta_{\text{in}} d\hat{\omega}_{\text{in}} \]

\[ I = a \hat{n} \uparrow \vec{l} \]

ASSUMPTION 2: DIRECTIONAL LIGHTING

[Prados, 2004]
Surface Normal

Equation of plane

\[ \frac{Ax}{C} + \frac{By}{C} + \frac{Cz}{C} + \frac{D}{C} = 0 \]

or

\[ \frac{A}{C} x + \frac{B}{C} y + z + \frac{D}{C} = 0 \]

Let

\[ \frac{z}{x} = \frac{A}{C} = p \]
\[ \frac{z}{y} = \frac{B}{C} = q \]

Surface normal

\[ N = \frac{A}{C}, \frac{B}{C}, 1 = (p, q, 1) \]
The $z = 1$ plane is called the Gradient Space ($pq$ plane).

- Every point on it corresponds to a particular surface orientation.

**Normal vector**

$$n = \frac{N}{|N|} = \frac{(p, q, 1)}{\sqrt{p^2 + q^2 + 1}}$$

**Source vector**

$$s = \frac{S}{|S|} = \frac{(p_s, q_s, 1)}{\sqrt{p_s^2 + q_s^2 + 1}}$$

$$\cos i = n \times s = \frac{(pp_s + qq_s + 1)}{\sqrt{p^2 + q^2 + 1}\sqrt{p_s^2 + q_s^2 + 1}}$$
Reflectance Map

- Relates image irradiance $I(x,y)$ to surface orientation $(p,q)$ for given source direction and surface reflectance.

Lambertian case:

- $k$: source brightness
- $c$: surface albedo (reflectance)
- $c$: constant (optical system)

Image irradiance:

$$I = -kc \cos i = -kc n \times s$$

Let $-kc = 1$ then $I = \cos i = n \times s$
Reflectance Map

- Lambertian case

\[ I = \cos \theta = n \times s = \frac{(pp_s + qq_s + 1)}{\sqrt{p_s^2 + q_s^2 + 1} \sqrt{p^2 + q^2 + 1}} = R(p, q) \]

Iso-brightness contour

Reflectance Map (Lambertian)

cone of constant

\[ i \]
Reflectance Map

- Lambertian case

Note: $R(p, q)$ is maximum when $(p, q) = (p_s, q_s)$
Shape from a Single Image?

- Given a single image of an object with known surface reflectance taken under a known light source, can we recover the shape of the object?
- Given $R(p,q)$ (($p_S,q_S$) and surface reflectance) can we determine $(p,q)$ uniquely for each image point?

\begin{answerbox}{NO}
\end{answerbox}
How Do We Do It?
• Humans have to make assumptions about illumination: 
bump (left) is perceived as hole (right) when upside down
Illumination direction is unknown. It is assumed to come from above
Does Shading Play a Central Role?

- Contour plays a more important role
- Variations in intensity are same on both shapes
- Upper region is perceived as cylindrical, illuminated from one side
- Lower region is perceived as sinusoidal, illuminated from above
- Note the ambiguities of the surface perceptions, depending on assumed illumination direction

2 possible illumination hypotheses
Examples of the classic bump/dent stimuli used to test lighting assumptions when judging shape from shading, with shading orientations (a) 0° and (b) 180° from the vertical.
Human Perception

• Our brain often perceives shape from shading.

• Mostly, it makes many assumptions to do so.

• For example:

  Light is coming from above (sun).

  Biased by occluding contours.
Stereographic Projection

\((p,q)\)-space (gradient space)

\((f,g)\)-space

Problem

\((p,q)\) can be infinite when \(= 90^\circ\)

\[
f = \frac{2p}{1 + \sqrt{1 + p^2 + q^2}} \quad g = \frac{2q}{1 + \sqrt{1 + p^2 + q^2}}
\]

Redefine reflectance map as \(R(f, g)\)
Occluding Boundaries

\textbf{n} \textbf{e}, \textbf{n} \textbf{v} \quad \textbf{n}=\textbf{e} \textbf{v} \quad \textbf{e} \text{and} \textbf{v} \text{ are known}

The \textbf{n} values on the occluding boundary can be used as the boundary condition for shape-from-shading
Image Irradiance Constraint

- Image irradiance should match the reflectance map

\[
e_i = \left( I(x, y) \cdot R(f, g) \right)^2 \text{d}x\text{d}y
\]

(image)

(minimize errors in image irradiance in the image)
Smoothness Constraint

- Used to constrain shape-from-shading
- Relates orientations $(f, g)$ of neighboring surface points

Minimize

$$e_s = \int_{\text{image}} \left( f_x^2 + f_y^2 \right) + \left( g_x^2 + g_y^2 \right) dxdy$$

$(f, g)$: surface orientation under stereographic projection

$$f_x = \frac{f}{x}, f_y = \frac{f}{y}, g_x = \frac{g}{x}, g_y = \frac{g}{y}$$

(penalize rapid changes in surface orientation $f$ and $g$ over the image)
Shape-from-Shading

- Find surface orientations \((f, g)\) at all image points that minimize

\[
e = e_s + e_i
\]

Minimize

\[
e = \left( f_x^2 + f_y^2 \right) + \left( g_x^2 + g_y^2 \right) + \left( I(x, y) \cdot R(f, g) \right)^2 \, dx \, dy
\]
Numerical Shape-from-Shading

- **Smoothness error** at image point \((i,j)\)

\[
s_{i,j} = \frac{1}{4} \left( \left( f_{i+1,j} - f_{i,j} \right)^2 + \left( f_{i,j+1} - f_{i,j} \right)^2 + \left( g_{i+1,j} - g_{i,j} \right)^2 + \left( g_{i,j+1} - g_{i,j} \right)^2 \right)
\]

Of course you can consider more neighbors (smoother results)

- **Image irradiance error** at image point \((i,j)\)

\[
r_{i,j} = \left( I_{i,j} - R(f_{i,j}, g_{i,j}) \right)^2
\]

Find \(\{f_{i,j}\}\) and \(\{g_{i,j}\}\) that minimize

\[
e = \sum_{i,j} \left( s_{i,j} + r_{i,j} \right)
\]

(Ikeuchi & Horn 89)
Results

by Ikeuchi and Horn
Results

Scanning Electron Microscope image (inverse intensity)

by Ikeuchi and Horn
More modern results

Resolution: 640 x 500; Re-rendering Error: 0.0075.

Resolution: 590 x 690; Re-rendering Error: 0.0083.
Single-lighting is ambiguous

ASSUMPTION 1: LAMBERTIAN

\[ L_{\text{out}}(\hat{\omega}) = \int_{\Omega_{\text{in}}} f(\hat{\omega}_{\text{in}}, \hat{\omega}_{\text{out}}) L_{\text{in}}(\hat{\omega}_{\text{in}}) \cos \theta_{\text{in}} d\hat{\omega}_{\text{in}} \]

\[ I = a\hat{n}^\top \vec{\ell} \]

ASSUMPTION 2: DIRECTIONAL LIGHTING

[Prados, 2004]
Photometric stereo
Lambertian photometric stereo

\[
\begin{align*}
I_1 &= a\hat{n}^\top \ell_1 \\
I_2 &= a\hat{n}^\top \ell_2 \\
\vdots
I_N &= a\hat{n}^\top \ell_N
\end{align*}
\]
Lambertian photometric stereo

\[
\begin{align*}
I_1 &= a\hat{n}^\top \ell_1 \\
I_2 &= a\hat{n}^\top \ell_2 \\
\vdots \\
I_N &= a\hat{n}^\top \ell_N
\end{align*}
\]

define “pseudo-normal” \[ \vec{b} \triangleq a\hat{n} \]

solve linear system for pseudo-normal

\[
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_N
\end{bmatrix}_{N \times 1}
= 
\begin{bmatrix}
\ell_1^\top \\
\ell_2^\top \\
\vdots \\
\ell_N^\top
\end{bmatrix}_{N \times 3}
\begin{bmatrix}
\vec{b}
\end{bmatrix}_{3 \times 1}
\]

once system is solved, \( b \) gives normal direction and albedo
Photometric Stereo

\[ (p_s^1, q_s^1) \]

\[ (p_s^2, q_s^2) \]

\[ (p_s^3, q_s^3) \]
Solving the Equations

\[
\begin{align*}
I_1 &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
I_2 &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
I_2 &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
s_1^T &= \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \\
s_2^T &= \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \\
s_3^T &= \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \\
S &= \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \\
\tilde{n} &= \begin{pmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{pmatrix} \\
\end{align*}
\]

\[
\tilde{n} = S^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

\[
\begin{align*}
n &= \frac{\tilde{n}}{|\tilde{n}|} = \frac{\tilde{n}}{|\tilde{n}|} \\
\end{align*}
\]
More than Three Light Sources

• Get better results by using more lights

\[
I_1 \quad s_1^T
\]
\[
\vdots = \vdots \quad n
\]
\[
I_N \quad s_N^T
\]

• Least squares solution:

\[
I = S \tilde{n}
\]

\[
S^T I = S^T S \tilde{n}
\]

\[
\tilde{n} = (S^T S)^{-1} S^T I
\]

• Solve for \( n \), \( n \) as before
Computing light source directions

• Trick: place a chrome sphere in the scene
  
  – the location of the highlight tells you the source direction
Specular Reflection - Recap

- For a perfect mirror, light is reflected about \( \mathbf{N} \)

\[
R_e = \begin{cases} 
R_i & \text{if } \mathbf{v} = \mathbf{r} \\
0 & \text{otherwise}
\end{cases}
\]

- We see a highlight when \( \mathbf{v} = \mathbf{r} \)
- Then \( \mathbf{s} \) is given as follows:

\[
\mathbf{s} = 2(\mathbf{n} \times \mathbf{r})\mathbf{n} \cdot \mathbf{r}
\]
Computing the Light Source Direction

Chrome sphere that has a highlight at position \( h \) in the image

\[ N \]

\[ rN \]

\[ C = (c_x, c_y, c_z) \]

\[ h = (h_x, h_y) \]

\[ c = (c_x, c_y) \]

• Can compute \( N \) by studying this figure
  
  Hints:
  
  • use this equation: \( \| H - C \| = r \)
  
  • can measure \( c, h, \) and \( r \) in the image
Limitations

• Big problems
  – Doesn’t work for shiny things, semi-translucent things
  – Shadows, inter-reflections

• Smaller problems
  – Camera and lights have to be distant
  – Calibration requirements
    • measure light source directions, intensities
    • camera response function
Trick for Handling Shadows

• Weight each equation by the pixel brightness:

\[ I_i (I_i) = I_i (n \times s_i) \]

• Gives weighted least-squares matrix equation:

\[
\begin{align*}
I_1^2 & \quad I_1 s_1^T \\
\vdots & \quad \vdots \quad n \\
I_N^2 & \quad I_N s_N^T
\end{align*}
\]

• Solve for \( n \) as before
Depth from normals

- Solving the linear system per-pixel gives us an estimated surface normal for each pixel.

- How can we compute depth from normals?
  - Normals are like the “derivative” of the true depth.
Normal Integration

- Integrating a set of derivatives is easy in 1D
  - (similar to Euler’s method from diff. eq. class)

- Could just integrate normals in each column / row separately
- Instead, we formulate as a linear system and solve for depths that *best agree with the surface normals*
Depth from normals

Get a similar equation for \( \mathbf{V}_2 \)

- Each normal gives us two linear constraints on \( z \)
- compute \( z \) values by solving a matrix equation

\[
\begin{align*}
\mathbf{V}_1 &= (x + 1, y, z_{x+1,y}) - (x, y, z_{xy}) \\
&= (1, 0, z_{x+1,y} - z_{xy}) \\
0 &= \mathbf{N} \cdot \mathbf{V}_1 \\
&= (n_x, n_y, n_z) \cdot (1, 0, z_{x+1,y} - z_{xy}) \\
&= n_x + n_z(z_{x+1,y} - z_{xy})
\end{align*}
\]
Results

1. Estimate light source directions
2. Compute surface normals
3. Compute albedo values
4. Estimate depth from surface normals
5. Relight the object (with original texture and uniform albedo)
Results: Lambertian Sphere

Input Images

Needles are projections of surface normals on image plane

Estimated Surface Normals

Estimated Albedo
Lambertain Mask
Results – Albedo and Surface Normal
Results – Shape of Mask
Results: Lambertian Toy
Original Images
Results - Shape

- Shallow reconstruction (effect of interreflections)
- Accurate reconstruction (after removing interreflections)
What if the light directions are unknown?
What if the light directions are unknown?

\[
\begin{align*}
I_1 &= a\hat{n}^\top \ell_1 \\
I_2 &= a\hat{n}^\top \ell_2 \\
\vdots \\
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define "pseudo-normal" \( \vec{b} \triangleq a\hat{n} \)

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\end{align*}
\]

define “pseudo-normal” \( \vec{b} \triangleq a \hat{n} \)

solve linear system for pseudo-normal at each image pixel

\[
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_N
\end{bmatrix}_{N \times M} =
\begin{bmatrix}
\ell_1^\top \\
\ell_2^\top \\
\vdots \\
\ell_N^\top
\end{bmatrix}_{N \times 3} \begin{bmatrix}
B
\end{bmatrix}_{3 \times M}
\]
What if the light directions are unknown?

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\vdots \\
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\end{bmatrix}_{N \times 3}
\begin{bmatrix}
B
\end{bmatrix}_{3 \times M}
\]

How do we solve this system without knowing light matrix L?
Factorizing the measurement matrix

\[ \text{Measurements} = \text{Lights} \times \text{Pseudonormals} \]

What are the dimensions?
Factorizing the measurement matrix

- Singular value decomposition:

\[ D = UWV^T \]

To reduce to rank 3, we just need to set all the singular values to 0 except for the first 3.

This decomposition minimizes \(|I-LB|^2\)
Are the results unique?
Bas-relief ambiguity

\[ I = L B = (L Q^{-1})(Q B) \]
What assumptions have we made for all this?
What assumptions have we made for all this?

• Lambertian BRDF

• Directional lighting

• No interreflections or scattering
Shape independent of BRDF via reciprocity: "Helmholtz Stereopsis"

\[ I = f(\text{shape, illumination, reflectance}) \]

\[ f^{-1} = \]
What assumptions have we made for all this?

• Lambertian BRDF
• Directional lighting
• No interreflections or scattering
References

Basic reading:
• Szeliski, Section 2.2, 12.1.
• Gortler, Chapter 21.