The 8-point algorithm

16-385 Computer Vision (Kris Kitani)
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Fundamental Matrix Estimation

Given a set of matched image points

\[ \{ x_i, x'_i \} \]

Estimate the Fundamental Matrix \( F \)

What’s the relationship between \( F \) and \( x \)?
Assume you have $M$ point correspondences
\[
\{ \mathbf{x}_m, \mathbf{x}_m' \} \quad m = 1, \ldots, M
\]

Each correspondence should satisfy
\[
\mathbf{x}_m'^\top \mathbf{F} \mathbf{x}_m = 0
\]

How would you solve for the $3 \times 3 \mathbf{F}$ matrix?
Assume you have $M$ point correspondences

$$ \{ \mathbf{x}_m, \mathbf{x}'_m \} \quad m = 1, \ldots, M $$

Each correspondence should satisfy

$$ \mathbf{x}'_m \mathbf{F} \mathbf{x}_m = 0 $$

How would you solve for the $3 \times 3$ $\mathbf{F}$ matrix?
Assume you have $M$ point correspondences

$$\left\{ \mathbf{x}_m, \mathbf{x}'_m \right\} \quad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\mathbf{x}'_m \mathbf{F} \mathbf{x}_m = 0$$

How would you solve for the 3 x 3 $\mathbf{F}$ matrix?

$$\mathbf{S} \quad \mathbf{V}$$
Assume you have $M$ point correspondences

$$\{\mathbf{x}_m, \mathbf{x}'_m\} \quad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\mathbf{x}'_m^\top \mathbf{F} \mathbf{x}_m = 0$$

How would you solve for the $3 \times 3 \mathbf{F}$ matrix?

\[ \begin{array}{cccc}
S & V & D \\
\end{array} \]
Assume you have $M$ point correspondences 

$$\{x_m, x'_m\} \quad m = 1, \ldots, M$$

Each correspondence should satisfy

$$x'_m \trans F x_m = 0$$

How would you solve for the $3 \times 3$ $F$ matrix?

Set up a homogeneous linear system with 9 unknowns
How many equation do you get from one correspondence?
\[
\begin{bmatrix}
  x'_m & y'_m & 1
\end{bmatrix}
\begin{bmatrix}
  f_1 & f_2 & f_3 \\
  f_4 & f_5 & f_6 \\
  f_7 & f_8 & f_9
\end{bmatrix}
\begin{bmatrix}
  x_m \\
  y_m \\
  1
\end{bmatrix} = 0
\]

ONE correspondence gives you ONE equation

\[
x_m x'_m f_1 + x_m y'_m f_2 + x_m f_3 + y_m x'_m f_4 + y_m y'_m f_5 + y_m f_6 + x'_m f_7 + y'_m f_8 + f_9 = 0
\]
Set up a homogeneous linear system with 9 unknowns

\[
\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0
\]

\[
\begin{bmatrix} x_1 x'_1 & x_1 y'_1 & x_1 & y_1 x'_1 & y_1 y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_M x'_M & x_M y'_M & x_M & y_M x'_M & y_M y'_M & y_M & x'_M & y'_M & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{bmatrix} = 0
\]

How many equations do you need?
Each point pair (according to epipolar constraint) contributes only one scalar equation

$$x'_m^\top F x_m = 0$$

**Note:** This is different from the Homography estimation where each point pair contributes 2 equations.
Each point pair (according to epipolar constraint) contributes only one scalar equation

$$x'_m \mathbf{F} x_m = 0$$

**Note:** This is different from the Homography estimation where each point pair contributes 2 equations.

We need at least 8 points

Hence, the 8 point algorithm!
How do you solve a homogeneous linear system?

$$AX = 0$$
How do you solve a homogeneous linear system?

\[ \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = 0 \]

**Total Least Squares**

minimize  \[ ||Ax||^2 \]

subject to  \[ ||x||^2 = 1 \]
How do you solve a homogeneous linear system?

\[ AX = 0 \]

**Total Least Squares**

minimize \( \|Ax\|^2 \)

subject to \( \|x\|^2 = 1 \)

S
How do you solve a homogeneous linear system?

\[ \mathbf{A} \mathbf{X} = \mathbf{0} \]

**Total Least Squares**

minimize \[ \| \mathbf{A} \mathbf{x} \|^2 \]

subject to \[ \| \mathbf{x} \|^2 = 1 \]
How do you solve a homogeneous linear system?

\[ AX = 0 \]

**Total Least Squares**

minimize \( \|Ax\|^2 \)

subject to \( \|x\|^2 = 1 \)

**SVD!**
Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix $A$
2. Find the SVD of $A^T A$
3. Entries of $F$ are the elements of column of $V$ corresponding to the least singular value
4. (Enforce rank 2 constraint on $F$)
5. (Un-normalize $F$)
Example
epipolar lines
$$F = \begin{bmatrix}
-0.00310695 & -0.0025646 & 2.96584 \\
-0.028094 & -0.00771621 & 56.3813 \\
13.1905 & -29.2007 & -9999.79
\end{bmatrix}$$

$$x = \begin{bmatrix}
343.53 \\
221.70 \\
1.0
\end{bmatrix}$$

$$l' = Fx = \begin{bmatrix}
0.0295 \\
0.9996 \\
-265.1531
\end{bmatrix}$$
\[ l' = Fx \]

\[
= \begin{bmatrix}
0.0295 \\
0.9996 \\
-265.1531
\end{bmatrix}
\]
Where is the epipole?

How would you compute it?
The epipole is in the right null space of $\mathbf{F}$

$$\mathbf{F}e = 0$$

*How would you solve for the epipole?*

(hint: this is a homogeneous linear system)
The epipole is in the right null space of $F$.

How would you solve for the epipole?

(hint: this is a homogeneous linear system)
The epipole is in the right null space of $F$

$$Fe = 0$$

*How would you solve for the epipole?*

(hint: this is a homogeneous linear system)

$$SV$$
\[ \mathbf{F} \mathbf{e} = 0 \]

The epipole is in the right null space of \( \mathbf{F} \)

*How would you solve for the epipole?*

(hint: this is a homogeneous linear system)

**SVD!**
\[
\begin{align*}
\text{eigenvectors} \\
u &= \begin{bmatrix} -0.0013 & 0.2586 & -0.9660 \\ 0.0029 & -0.9660 & -0.2586 \\ 1.0000 & 0.0032 & -0.0005 \end{bmatrix} \\
\text{eigenvalue} \\
d &= \begin{bmatrix} -1.0000 & 0 & 0 \\ 0 & -0.0000 & 0 \\ 0 & 0 & -0.0000 \end{bmatrix}
\end{align*}
\]
$$\begin{align*}
> & \ [u,d] = \text{eigs}(F' \ast F) \\
& \text{eigenvectors} \\
& u = \\
& \begin{bmatrix}
-0.0013 & 0.2586 & -0.9660 \\
0.0029 & -0.9660 & -0.2586 \\
1.0000 & 0.0032 & -0.0005
\end{bmatrix} \\
& \text{eigenvalue} \\
& d = 1.0e8* \\
& \begin{bmatrix}
-1.0000 & 0 & 0 \\
0 & -0.0000 & 0 \\
0 & 0 & -0.0000
\end{bmatrix}
\end{align*}$$
\[
[u,d] = eigs(F' * F)
\]
eigenvectors
\[
u = \\
\begin{bmatrix}
-0.0013 & 0.2586 & -0.9660 \\
0.0029 & -0.9660 & -0.2586 \\
1.0000 & 0.0032 & -0.0005 \\
\end{bmatrix}
\]
eigenvalue
\[
d = 1.0e8* \\
\begin{bmatrix}
-1.0000 & 0 & 0 \\
0 & -0.0000 & 0 \\
0 & 0 & -0.0000 \\
\end{bmatrix}
\]

Eigenvector associated with smallest eigenvalue
\[
>> uu = u(:,3) \\
\begin{bmatrix}
-0.9660 & -0.2586 & -0.0005 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
-0.0013 & 0.2586 & -0.9660 \\
0.0029 & -0.9660 & -0.2586 \\
1.0000 & 0.0032 & -0.0005
\end{bmatrix}
\]

Eigenvector associated with smallest eigenvalue

Epipole projected to image coordinates
Epipole projected to image coordinates

\[
\gg uu / uu(3) \\
(1861.02 \quad 498.21 \quad 1.0)
\]