Game Theory II: Price of Anarchy

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BACK TO PRISON

• The only Nash equilibrium in Prisoner’s dilemma is bad; but how bad is it?

• **Objective function**: social cost = sum of costs

• NE is six times worse than the optimum

• We can make this arbitrarily bad

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<th>-1,-1</th>
<th>-9,0</th>
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<td>0,-9</td>
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ANARCHY AND STABILITY

• Fix a class of games, an objective function, and an equilibrium concept
• The price of anarchy (stability) is the worst-case ratio between the worst (best) objective function value of an equilibrium of the game, and that of the optimal solution
• In this lecture:
  ◦ Objective function = social cost
  ◦ Equilibrium concept = Nash equilibrium
EXAMPLE: COST SHARING

- $n$ players in weighted directed graph $G$
- Player $i$ wants to get from $s_i$ to $t_i$; strategy space is $s_i \rightarrow t_i$ paths
- Each edge $e$ has cost $c_e$
- Cost of edge is split between all players using edge
- Cost of player is sum of costs over edges on path
EXAMPLE: COST SHARING

• With $n$ players, the example on the right has a NE with social cost $n$
• Optimal social cost is 1
• It follows that the price of anarchy of cost sharing games is at least $n$
• It is easy to see that the price of anarchy of cost sharing games is at most $n$ — why?
EXAMPLE: COST SHARING

• Think of the 1 edges as cars, and the $k$ edge as mass transit
• Bad Nash equilibrium with cost $n$
• Good Nash equilibrium with cost $k$
• Now let’s modify the example...
EXAMPLE: COST SHARING

- OPT = $k + 1$
- Only equilibrium has cost $k \cdot H(n)$
- Therefore, the price of stability of cost sharing games is at least $\Omega(\log n)$
- We will show that the price of stability is $\Theta(\log n)$
POTENTIAL GAMES

- A game is an **exact potential game** if there exists a function $\Phi: \prod_{i=1}^{n} S_i \rightarrow \mathbb{R}$ such that for all $i \in N$, for all $s \in \prod_{i=1}^{n} S_i$, and for all $s'_i \in S_i$,
  \[
  \text{cost}_i(s'_i, s_{-i}) - \text{cost}_i(s) = \Phi(s'_i, s_{-i}) - \Phi(s)
  \]
- The existence of an exact potential function implies the existence of a pure Nash equilibrium — why?
POTENTIAL GAMES

• **Theorem:** the cost sharing game is an exact potential game

• **Proof:**
  
  - Let $n_e(s)$ be the number of players using $e$ under $s$
  - Define the potential function
    
    $$\Phi(s) = \sum_{e} \sum_{k=1}^{n_e(s)} \frac{c_e}{k}$$

  - If player changes paths, pays $\frac{c_e}{n_e(s)+1}$ for each new edge, gets $\frac{c_e}{n_e(s)}$ for each old edge, so $\Delta \text{cost}_i = \Delta \Phi$  ■
POTENTIAL GAMES

• **Theorem:** The cost of stability of cost sharing games is $O(\log n)$

• **Proof:**
  ◦ It holds that
    \[
    \text{cost}(s) \leq \Phi(s) \leq H(n) \cdot \text{cost}(s)
    \]
  ◦ Take a strategy profile $s$ that minimizes $\Phi$
  ◦ $s$ is an NE
  ◦ $\text{cost}(s) \leq \Phi(s) \leq \Phi(\text{OPT}) \leq H(n) \cdot \text{cost}(\text{OPT})$
COST SHARING SUMMARY

• **Upper bounds:** ∀cost sharing game,
  ◦ **PoA:** ∀NE \( s \),
    \[
    \text{cost}(s) \leq n \cdot \text{cost(OPT)}
    \]
  ◦ **PoS:** ∃NE \( s \) s.t.
    \[
    \text{cost}(s) \leq H(n) \cdot \text{cost(OPT)}
    \]

• **Lower bounds:** ∃cost sharing game s.t.
  ◦ **PoA:** ∃NE \( s \) s.t.
    \[
    \text{cost}(s) \geq n \cdot \text{cost(OPT)}
    \]
  ◦ **PoS:** ∀NE \( s \),
    \[
    \text{cost}(s) \geq H(n) \cdot \text{cost(OPT)}
    \]
NETWORK FORMATION GAMES

• Each player is a vertex $v$
• Strategy of $v$: set of undirected edges to build that touch $v$
• Strategy profile $s$ induces undirected graph $G(s)$
• Cost of building any edge is $\alpha$
• $\text{cost}_v(s) = \alpha n_v(s) + \sum_u d(u, v)$, where $n_v = \#\text{edges bought by } v$, $d$ is shortest path in $\#\text{edges}$
• $\text{cost}(s) = \sum_{u \neq v} d(u, v) + \alpha |E|$
EXAMPLE: NETWORK FORMATION

NE with $\alpha = 3$

Suboptimal

Optimal
EXAMPLE: NETWORK FORMATION

• **Lemma:** If $\alpha \geq 2$ then any star is optimal, and if $\alpha \leq 2$ then a complete graph is optimal

• **Proof:**
  ◦ Suppose $\alpha \leq 2$, and consider any graph that is not complete
  ◦ Adding an edge will decrease the sum of distances by at least 2, and costs only $\alpha$
  ◦ Suppose $\alpha \geq 2$ and the graph contains a star, so the diameter is at most 2; deleting a non-star edge increases the sum of distances by at most 2, and saves $\alpha$ ■
EXAMPLE: NETWORK FORMATION

Poll 1

For which values of $\alpha$ is any star a NE, and for which is any complete graph a NE?

1. $\alpha \geq 1$, $\alpha \leq 1$
2. $\alpha \geq 2$, $\alpha \leq 1$
3. $\alpha \geq 1$, none
4. $\alpha \geq 2$, none

• Theorem:

1. If $\alpha \geq 2$ or $\alpha \leq 1$, PoS = 1
2. For $1 < \alpha < 2$, PoS $\leq \frac{4}{3}$
PROOF OF THEOREM

• Part 1 is immediate from the lemma and poll
• For $1 < \alpha < 2$, the star is a NE, while OPT is a complete graph
• Worst case ratio when $\alpha \to 1$:
  \[
  \frac{2n(n - 1) - 2(n - 1) + (n - 1)}{n(n - 1) + n(n - 1)/2} = \frac{4n^2 - 6n + 2}{3n^2 - 3n} < \frac{4}{3}
  \]
EXAMPLE: NETWORK CREATION

• **Theorem [Fabrikant et al. 2003]**: The price of anarchy of network creation games is $O(\sqrt{\alpha})$

• **Lemma**: If $s$ is a Nash equilibrium that induces a graph of diameter $d$, then $\text{cost}(s) \leq O(d) \cdot \text{OPT}$
PROOF OF LEMMA

• \( \text{OPT} = \Omega(an + n^2) \)
  - Buying a connected graph costs at least \( (n - 1)\alpha \)
  - There are \( \Omega(n^2) \) distances

• Distance costs \( \leq dn^2 \Rightarrow \text{focus on edge costs} \)

• There are at most \( n - 1 \) cut edges \( \Rightarrow \text{focus on noncut edges} \)
**PROOF OF LEMMA**

- **Claim:** Let $e = (u, v)$ be a noncut edge, then the distance $d(u, v)$ with $e$ deleted $\leq 2d$
  - $V_e =$ set of nodes s.t. the shortest path from $u$ uses $e$
  - Figure shows shortest path avoiding $e$, $e' = (u', v')$ is the edge on the path entering $V_e$
  - $P_u$ is the shortest path from $u$ to $u' \Rightarrow |P_u| \leq d$
  - $|P_v| \leq d - 1$ as $P_v \cup \{e\}$ is shortest path from $u$ to $v'$ □

![Diagram showing shortest paths and nodes](image-url)
• **Claim:** There are $O(nd/\alpha)$ noncut edges paid for by any vertex $u$
  - Let $e = (u, v)$ be an edge paid for by $u$
  - By previous claim, deleting $e$ increases distances from $u$ by at most $2d|V_e|$
  - $G$ is an equilibrium $\Rightarrow \alpha \leq 2d|V_e| \Rightarrow |V_e| \geq \alpha/2d$
  - $n$ vertices overall $\Rightarrow$ can’t be more than $2nd/\alpha$ sets $V_e$
PROOF OF LEMMA

- $O(nd/\alpha)$ noncut edges per vertex
- $O(nd)$ total payment for these per vertex
- $O(n^2d)$ overall

∎
PROOF OF THEOREM

• By lemma, it is enough to show that the diameter at a NE $\leq 2\sqrt{\alpha}$
• Suppose $d(u, v) \geq 2k$ for some $k$
• By adding the edge $(u, v)$, $u$ pays $\alpha$ and improves distance to second half of the $u \rightarrow v$ shortest path by
  $$(2k - 1) + (2k - 3) + \cdots + 1 = k^2$$
• If
  $$\alpha < k^2 \leq \left(\frac{d(u, v)}{2}\right)^2 \Rightarrow d(u, v) > 2\sqrt{\alpha}$$
then it is beneficial to add edge — contradiction $\blacksquare$