Game Theory I: Basic Concepts

Teachers: Ariel Procaccia (this time) and Alex Psomas
NORMAL-FORM GAME

• A game in normal form consists of:
  ◦ Set of players $N = \{1, \ldots, n\}$
  ◦ Strategy set $S$
  ◦ For each $i \in N$, utility function $u_i : S^n \to \mathbb{R}$: if each $j \in N$ plays the strategy $s_j \in S$, the utility of player $i$ is $u_i(s_1, \ldots, s_n)$
THE PRISONER’S DILEMMA

• Two men are charged with a crime

• They are told that:
  ◦ If one rats out and the other does not, the rat will be freed, other jailed for nine years
  ◦ If both rat out, both will be jailed for six years

• They also know that if neither rats out, both will be jailed for one year
### THE PRISONER’S DILEMMA

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<thead>
<tr>
<th></th>
<th>Cooperate</th>
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<tr>
<td>Cooperate</td>
<td>-1,-1</td>
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<td>0,-9</td>
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What would you do?
ON TV

http://youtu.be/S0qjK3TWZE8
THE PROFESSOR'S DILEMMA

<table>
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<tr>
<th>Professor</th>
<th>Class</th>
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<tbody>
<tr>
<td>Make effort</td>
<td>Listen, 10^6, 10^6, Sleep, -10, 0</td>
</tr>
<tr>
<td>Slack off</td>
<td>0, -10, 0, 0</td>
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Dominant strategies?
NASH EQUILIBRIUM

• In a Nash equilibrium, no player wants to unilaterally deviate

• Each player’s strategy is a best response to strategies of others

• Formally, a Nash equilibrium is a vector of strategies $s = (s_1, \ldots, s_n) \in S^n$ such that for all $i \in N$, $s'_i \in S$, $u_i(s) \geq u_i(s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$
THE PROFESSOR’S DILEMMA

Professor
- Make effort
  - 10^6, 10^6
  - -10, 0
- Slack off
  - 0, -10
  - 0, 0

Class
- Listen
- Sleep

Nash equilibria?
# ROCK-PAPER-SCISSORS

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**Nash equilibria?**
MIXED STRATEGIES

• A mixed strategy is a probability distribution over (pure) strategies.

• The mixed strategy of player \( i \in N \) is \( x_i \), where

\[
x_i(s_i) = \Pr[i \text{ plays } s_i]
\]

• The utility of player \( i \in N \) is

\[
u_i(x_1, \ldots, x_n) = \sum_{(s_1, \ldots, s_n) \in S^n} u_i(s_1, \ldots, s_n) \cdot \prod_{j=1}^{n} x_j(s_j)
\]
EXERCISE: MIXED NE

• **Exercise:** player 1 plays \( \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \), player 2 plays \( \left( 0, \frac{1}{2}, \frac{1}{2} \right) \). What is \( u_1 \)?

• **Exercise:** Both players play \( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \). What is \( u_1 \)?

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EXERCISE: MIXED NE

Poll 1

Which is a NE?

1. \((\left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right))\)

2. \((\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right))\)

3. \((\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right))\)

4. \((\left(\frac{2}{3}, 0\right), \left(\frac{2}{3}, 0, \frac{1}{3}\right))\)

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NASH’S THEOREM

• Theorem [Nash, 1950]: In any (finite) game there exists at least one (possibly mixed) Nash equilibrium

• What about computing a Nash equilibrium? Stay tuned...
DOES NE MAKE SENSE?

- Two players, strategies are \{2, \ldots, 100\}
- If both choose the same number, that is what they get
- If one chooses \( s \), the other \( t \), and \( s < t \), the former player gets \( s + 2 \), and the latter gets \( s - 2 \)
- **Poll 2**: What would you choose?
CORRELATED EQUILIBRIUM

• Let $N = \{1, 2\}$ for simplicity
• A mediator chooses a pair of strategies $(s_1, s_2)$ according to a distribution $p$ over $S^2$
• Reveals $s_1$ to player 1 and $s_2$ to player 2
• When player 1 gets $s_1 \in S$, he knows the distribution over strategies of 2 is

$$\Pr[s_2 | s_1] = \frac{\Pr[s_1 \land s_2]}{\Pr[s_1]} = \frac{p(s_1, s_2)}{\Pr[s_1]}$$
CORRELATED EQUILIBRIUM

- Player 1 is best responding if for all $s_1' \in S$
  \[ \sum_{s_2 \in S} \Pr[s_2 | s_1] u_1(s_1, s_2) \geq \sum_{s_2 \in S} \Pr[s_2 | s_1] u_1(s_1', s_2) \]

- Equivalently,
  \[ \sum_{s_2 \in S} p(s_1, s_2) u_1(s_1, s_2) \geq \sum_{s_2 \in S} p(s_1, s_2) u_1(s_1', s_2) \]

- $p$ is a **correlated equilibrium (CE)** if both players are best responding

- Every Nash equilibrium is a correlated equilibrium, but not vice versa
GAME OF CHICKEN

http://youtu.be/u7hZ9jKrwo
GAME OF CHICKEN

• **Social welfare** is the sum of utilities

• Pure NE: (C,D) and (D,C), social welfare = 5

• Mixed NE: both \((1/2, 1/2)\), social welfare = 4

• Optimal social welfare = 6
GAME OF CHICKEN

• Correlated equilibrium:
  ◦ (D,D): 0
  ◦ (D,C): \frac{1}{3}
  ◦ (C,D): \frac{1}{3}
  ◦ (C,C): \frac{1}{3}

• Social welfare of CE = \frac{16}{3}
IMPLEMENTATION OF CE

• Instead of a mediator, use a hat!

• Balls in hat are labeled with “chicken” or “dare”, each blindfolded player takes a ball

Poll 3

Which balls implement the distribution of the previous slide?

1. 1 chicken, 1 dare  
2. 1 chicken, 2 dare  
3. 2 chicken, 1 dare  
4. 2 chicken, 2 dare
CE AS LP

- Can compute CE via linear programming in polynomial time!

\[
\text{find } \forall s_1, s_2 \in S, p(s_1, s_2) \\
\text{s.t. } \forall s_1, s'_1 \in S, \sum_{s_2 \in S} p(s_1, s_2) u_1(s_1, s_2) \geq \sum_{s_2 \in S} p(s_1, s_2) u_1(s'_1, s_2) \\
\forall s_2, s'_2 \in S, \sum_{s_1 \in S} p(s_1, s_2) u_2(s_1, s_2) \geq \sum_{s_1 \in S} p(s_1, s_2) u_2(s_1, s'_2) \\
\sum_{s_1, s_2 \in S} p(s_1, s_2) = 1 \\
\forall s_1, s_2 \in S, p(s_1, s_2) \in [0,1]
\]