Fair Division IV: Rent Division

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THE WHINING PHILOSOPHERS PROBLEM
SPERNER’S LEMMA

• Triangle $T$ partitioned into elementary triangles

• Label vertices by $\{1, 2, 3\}$ using Sperner labeling:
  ◦ Main vertices are different
  ◦ Label of vertex on an edge $(i, j)$ of $T$ is $i$ or $j$

• Lemma: Any Sperner labeling contains at least one fully labeled elementary triangle
PROOF OF LEMMA

• Doors are 12 edges
• Rooms are elementary triangles
• #doors on the boundary of $T$ is odd
• Every room has $\leq 2$ doors; one door iff the room is 123
PROOF OF LEMMA

• Start at door on boundary and walk through it
• Room is fully labeled or it has another door...
• No room visited twice
• Eventually walk into fully labeled room or back to boundary
• But #doors on boundary is odd □
THE MODEL

• Assume there are three players A, B, C
• Goal is to assign the rooms and divide the rent in a way that is envy free: each player wants a different room at the given prices
• Sum of prices for three rooms is 1
• Theorem [Su 99]: An envy-free solution always exists under some assumptions
PROOF OF THEOREM
PROOF OF THEOREM

• “Triangulate” and assign “ownership” of each vertex to each of A, B, and C, in a way that each elementary triangle is an ABC triangle
PROOF OF THEOREM

• Ask the owner of each vertex to tell us which room he prefers
• This gives a new labeling by 1, 2, 3
• Assume that a person wants a free room if one is offered to him
PROOF OF THEOREM

• Choice of rooms on edges is constrained by free room assumption
PROOF OF THEOREM

• Sperner’s lemma (variant): such a labeling must have a 123 triangle
PROOF OF THEOREM

• Such a triangle is nothing but an approximately EF solution!
• By making the triangulation finer, we can approach envy-freeness
• Under additional closedness assumption, leads to existence of an EF solution ■
DISCUSSION

• It is possible to derive an algorithm from the proof

• Same techniques generalize to more housemates

• Same proof (with the original Sperner’s Lemma) shows existence of EF cake division!
QUASI-LINEAR UTILITIES

- Suppose each player $i \in N$ has value $v_{ir}$ for room $r$
- $\sum_r v_{ir} = R$, where $R$ is the total rent
- The utility of player $i$ for getting room $r$ at price $p_r$ is $v_{ir} - p_r$
- A solution consists of an assignment $\pi$ and a price vector $p$, where $p_r$ is the price of room $r$
- Solution $(\pi, p)$ is envy free if
  $$\forall i, j \in N, v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\pi(j)} - p_{\pi(j)}$$
- Theorem [Svensson 1983]: An envy-free solution always exists under quasi-linearity
Total rent: $10

Room 1

Room 2

Room 3
PROPERTIES OF EF SOLUTIONS

• Allocation \( \pi \) is **welfare-maximizing** if

\[
\pi \in \arg\max_{\sigma} \sum_{i \in N} v_{i\sigma(i)}
\]

• **Lemma 1:** If \( (\pi, p) \) is an EF solution, then \( \pi \) is a welfare-maximizing assignment

• **Lemma 2:** If \( (\pi, p) \) is an EF solution and \( \sigma \) is a welfare-maximizing assignment, then \( (\sigma, p) \) is an EF solution, and for all \( i \),

\[
v_{i\pi(i)} - p_{\pi(i)} = v_{i\sigma(i)} - p_{\sigma(i)}
\]
PROOF OF LEMMA 1

- Let \((\pi, p)\) be an EF solution, and let \(\sigma\) be another assignment.
- Due to EF, for all \(i\),
  \[ v_{i\pi(i)} - p_{\pi(i)} \geq v_{i\sigma(i)} - p_{\sigma(i)} \]
- Summing over all \(i\),
  \[ \sum_{i \in N} v_{i\pi(i)} - \sum_{i \in N} p_{\pi(i)} \geq \sum_{i \in N} v_{i\sigma(i)} - \sum_{i \in N} p_{\sigma(i)} \]
- We get the desired inequality because prices sum up to \(R\) \(\blacksquare\)
POLYNOMIAL-TIME ALGORITHM

• Consider the algorithm that finds a welfare-maximizing assignment $\pi$, and then finds prices $p$ that satisfy the EF constraint

• Theorem [Gal et al. 2017]: The algorithm always returns an EF solution, and can be implemented in polynomial time

• Proof:
  ◦ We know that an EF solution $(\sigma, p)$ exists, by Lemma 2 $(\pi, p)$ is EF, so we would be able to find prices satisfying the EF constraint
  ◦ The first part is max weight matching, the second part is a linear program □
Total rent: $3
OPTIMAL EF SOLUTIONS

- **Straw Man Solution**
  - Max sum of utilities
  - Subject to envy freeness

- **Maximin Solution**
  - Max min utility
  - Subject to envy freeness

- **Equitable solution**
  - Min max difference in utilities
  - Subject to envy freeness
• Theorem [Gal et al. 2017]: The maximin and equitable solutions can be computed in polynomial time

• Theorem [Alkan et al. 1991]: The maximin solution is unique

Suppose that the values are
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1/2 & 1/2 \\
1/3 & 1/3 & 1/3
\end{pmatrix}
\]

What is the min utility under the maximin solution?

• $2/6 = 1/3$
• $2/8 = 1/4$
• $2/7$
• $2/9$
OPTIMAL EF SOLUTION

• Theorem [Gal et al. 2017]: The maximin solution is equitable, but not vice versa.

• Rent division instance from Spliddit where the equitable solution is not maximin:

\[
\begin{pmatrix}
2227 & 708 & 0 \\
258 & 1378 & 1299 \\
1000 & 1000 & 935
\end{pmatrix}
\]

• Maximin solution gives room $i$ to player $i$, with prices and utilities:

\[
\left(1813 \frac{1}{3}, 600 \frac{1}{3}, 521 \frac{1}{3}\right), \left(413 \frac{2}{3}, 777 \frac{2}{3}, 413 \frac{2}{3}\right)
\]

• The max difference in utilities is 364.

• The following prices and utilities have the same max difference, but lower minimum utility:

\[
\left(1570 \frac{2}{3}, 721 \frac{2}{3}, 642 \frac{2}{3}\right), \left(656 \frac{1}{3}, 656 \frac{1}{3}, 292 \frac{1}{3}\right)
\]
CAVEAT: STRATEGYPROOFNESS

- Lemma 1 tells us that any EF solution is welfare maximizing
- Therefore, any EF solution is Pareto efficient
- But there is no rent division algorithm that is both EF and Pareto efficient [Green and Laffont 1979]
- However, strategic behavior is largely a nonissue in practice in the rent division domain
CAVEAT: NEGATIVE RENT

• Envy-freeness may require negative rent, as the following example shows:

\[
\begin{pmatrix}
36 & 34 & 30 & 0 \\
31 & 36 & 33 & 0 \\
34 & 30 & 36 & 0 \\
32 & 33 & 35 & 0
\end{pmatrix}
\]

• Whatever player \( i \) gets room 4 must pay 0, and the prices of the other rooms must be exactly his values to prevent envy.

• Easy to verify that \( i \) can’t be any of the players.
WHICH MODEL IS BETTER?

• Advantages of quasi-linear utilities:
  ◦ Preference elicitation is easy: Each player reports a single number in one shot
  ◦ Can choose among EF solutions

• Disadvantage of quasi-linear utilities: does not correctly model real-world situations
  ◦ I want the room but I really can’t spend more than $500 on rent
INTERFACES

NY TIMES (rental harmony)

Spliddit (quasi-linear utilities)
http://www.spliddit.org/apps/rent