Social Networks I: Coordination Games

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BACKGROUND

- Spread of ideas and new behaviors through a population

- Examples:
  - Political movements
  - Adoption of technological innovations
  - Success of new product

- Process starts with early adopters and spreads through the social network
NETWORKED COORDINATION GAMES

• Simple model for the diffusion of ideas and innovations
• Social network is undirected graph $G = (V, E)$
• Choice between old behavior $A$ and new behavior $B$
• Parametrized by $q \in (0,1)$
NETWORKED COORDINATION GAMES

• Rewards for $u$ and $v$ when $(u, v) \in E$:
  - If both choose $A$, they receive $q$
  - If both choose $B$, they receive $1 - q$
  - Otherwise both receive 0

• Overall payoff to $v = \text{sum of payoffs}$

• Denote $d_v = \text{degree of } v$, $d_v^X = \#\text{neighbors playing } X$

• Payoff to $v$ from choosing $A$ is $qd_v^A$; reward from choosing $B$ is $(1 - q)d_v^B$

• $v$ adopts $B$ if $d_v^B \geq qd_v \Rightarrow q$ is a threshold
CASCADING BEHAVIOR

• Each node simultaneously updates its behavior in time steps \( t = 1, 2, \ldots \).
• Nodes in \( S \) initially adopt \( B \).
• \( h_q(S) = \) set of nodes adopting \( B \) after one round.
• \( h^k_q(S) = \) after \( k \) rounds of updates.
• **Question:** When does a small set of nodes convert to the entire population?
• \( V \) is countably infinite and each \( d_v \) is finite

• \( v \) is converted by \( S \) if \( \exists k \) s.t. \( v \in h^k_q(S) \)

• \( S \) is contagious if every node is converted

• Easier to be contagious when \( q \) is small

• Contagion threshold of \( G = \max q \) s.t. \( \exists \) finite contagious set
EXAMPLE

\[ q = \frac{1}{2} \]

Poll 1: What is the contagion threshold of \( G \)?
Poll 2: What is the contagion threshold of $G$?
PROGRESSIVE PROCESSES

• **Nonprogressive** process: Nodes can switch from $A$ to $B$ or $B$ to $A$

• **Progressive** process: Nodes can only switch from $A$ to $B$

• As before, a node $\nu$ switches to $B$ if a $q$ fraction of its neighbors $N(\nu)$ follow $B$

• $\bar{h}_q(S) =$ set of nodes adopting $B$ in progressive process; define $\bar{h}_q^k(S)$ as before
PROGRESSIVE PROCESSES

• With progressive processes intuitively the contagion threshold should be at least as high

• **Theorem [Morris, 2000]:** For any graph $G$, $\exists$ finite contagious set wrt $h_q \iff \exists$ finite contagious set wrt $\bar{h}_q$

• I.e., the contagion threshold is identical under both models
PROOF OF THEOREM

• Lemma: \( \overline{h}_q^k (X) = h_q \left( \overline{h}_q^{k-1} (X) \right) \cup X \)

• Proof:
  
  \( \overline{h}_q^k (X) = (\overline{h}_q^k (X) \setminus \overline{h}_q^{k-1} (X)) \cup (\overline{h}_q^{k-1} (X) \setminus X) \cup X \)

  \( \overline{h}_q^k (X) \setminus \overline{h}_q^{k-1} (X) = h_q \left( \overline{h}_q^{k-1} (X) \right) \setminus \overline{h}_q^{k-1} (X) \)

  For every \( v \in \overline{h}_q^{k-1} \setminus X \), \( v \in h_q \left( \overline{h}_q^{k-1} (X) \right) \), because \( v \) has at least as many \( B \) neighbors as when it converted

  Clearly \( X \subseteq h_q \left( \overline{h}_q^{k-1} (X) \right) \cup X \) ■
PROOF OF THEOREM

• Enough to show: given a set $S$ that is contagious wrt $\overline{h}_q$, there is a set $T$ that is contagious wrt $h_q$

• Let $\ell$ s.t. $S \cup N(S) \subseteq \overline{h}_q^\ell(S)$; this is our $T$

• For $k > \ell$, $\overline{h}_q^k(S) = h_q(\overline{h}_q^{k-1}(S)) \cup S$ by the lemma

• Since $N(S) \subseteq \overline{h}_q^{k-1}(S)$, $S \subseteq h_q(\overline{h}_q^{k-1}(S))$, and hence $\overline{h}_q^k(S) = h_q(\overline{h}_q^{k-1}(S))$

• By induction, all $k > \ell$,

$$\overline{h}_q^k(S) = h_q^{k-\ell}(\overline{h}_q^\ell(S)) = h_q^{k-\ell}(T) \blacksquare$$
CONTAGION THRESHOLD $\leq 1/2$

- Saw a graph with contagion threshold $1/2$
- Does there exist a graph with contagion threshold $> 1/2$?
- The previous theorem allows us to focus on the progressive case
- **Theorem [Morris, 2000]**: For any graph $G$, the contagion threshold $\leq 1/2$
PROOF OF THEOREM

• Let $q > 1/2$, finite $S$

• Denote $S_j = \overline{h}_q^j(S)$

• $\delta(X) =$ set of edges with exactly one end in $X$

• If $S_{j-1} \neq S_j$ then $|\delta(S_j)| < |\delta(S_{j-1})|$
  ◦ For each $v \in S_j \setminus S_{j-1}$, its edges into $S_{j-1}$ are in $\delta(S_{j-1}) \setminus \delta(S_j)$, and its edges into $V \setminus S_j$ are in $\delta(S_j) \setminus \delta(S_{j-1})$
  ◦ More of the former than the latter because $v$ converted and $q > 1/2$

• $\delta(S)$ is finite and $\delta(S_j) \geq 0$ for all $j$
MORE GENERAL MODELS

- Directed graphs to model asymmetric influence
- Redefine $N(v) = \{u \in V: (u, v) \in E\}$
- Assume progressive contagion
- Node is active if it adopts $B$; activated if switches from $A$ to $B$
LINEAR THRESHOLD MODEL

• Nonnegative weight $w_{uv}$ for each edge $(u, v) \in E$; $w_{uv} = 0$ otherwise
• Assume $\forall v \in V, \sum_u w_{uv} \leq 1$
• Each $v \in V$ has threshold $\theta_v$
• $v$ becomes active if

$$\sum_{\text{active } u} w_{uv} \geq \theta_v$$
GENERAL THRESHOLD MODEL

• Linear model assumes additive influences
  ◦ Switch if two co-workers and three family members switch?

• $\nu$ has a monotonic function $g_{\nu}(\cdot)$ defined on subsets $X \subseteq N(\nu)$

• $\nu$ becomes activated if the activated subset $X \subseteq N(\nu)$ satisfies $g_{\nu}(X) \geq \theta_{\nu}$
THE CASCADE MODEL

• When $\exists (u, v) \in E$ s.t. $u$ is active and $v$ is not, $u$ has one chance to activate $v$

• $v$ has an **incremental function** $p_v(u, X) = \text{probability that } u \text{ activates } v \text{ when } X \text{ have tried and failed}$

• Special cases:
  - Diminishing returns: $p_v(u, X) \geq p_v(u, Y)$ when $X \subseteq Y$
  - Independent cascade: $p_v(u, X) = p_{uv}$