Truth Justice Algors

Mechanism Design: Recent Advances

Teachers: Ariel Procaccia and Alex Psomas (this time)
SO FAR

- Revelation Principle
- Single parameter environments
  - Second price auctions
  - Myerson’s lemma
  - Myerson’s optimal auction
  - Cremer-McLean auction for correlated buyers
  - Prophet inequalities
  - Bulow-Klemperer
- Multiparameter environments
  - The VCG mechanism
  - Challenges
  - Revenue optimal auctions are strange
TODAY

• Computing the optimal auction
  ◦ Reduced forms

• Simple vs Optimal mechanisms
  ◦ $SRev$ and $BRev$ are not good approximations
  ◦ $\max\{SRev, BRev\}$ is
  ◦ Langrangian duality

• Dynamic mechanisms
CAN WE COMPUTE STUFF FOR MANY BIDDERS?

• Assume that buyers are additive over items.
• DSIC: Too many constraints to even write down!
• Standard approach: BIC (Bayesian Incentive Compatible)
  ◦ “If everyone is telling the truth, bidding my true values is the optimal strategy”

\[
\sum_{v_{-i} \sim V_{-i}} \Pr[V_{-i} = v_{-i}] \left( \sum_{j} v_{ij} x_{ij} (\vec{v}) - p_i(\vec{v}) \right) \\
\geq \sum_{v_{-i} \sim V_{-i}} \Pr[V_{-i} = v_{-i}] \left( \sum_{j} v_{ij} x_{ij} (\vec{v}') - p_i(\vec{v}') \right)
\]
CAN WE COMPUTE STUFF FOR MANY BIDDERS?

\[ \sum_{v_{-i} \sim V_{-i}} \Pr[V_{-i} = v_{-i}] (\sum_j v_{ij}x_{ij}(\tilde{v}) - p_i(\tilde{v})) \]

\[ \geq \sum_{v_{-i} \sim V_{-i}} \Pr[V_{-i} = v_{-i}] (\sum_j v_{ij}x_{ij}(\tilde{v}') - p_i(\tilde{v}')) \]

• \( n \) bidders, \( m \) items, \( |V_i| \) = support of \( V_i \)

Poll

How many variables?

1. \( \Theta(nm \prod_i |V_i|) \)
2. \( \Theta(n^m \Sigma_i |V_i|) \)
3. \( \Theta(\Sigma_i |V_i|) \)
4. Beats me
CAN WE COMPUTE STUFF FOR MANY BIDDERS?

• Reduced form

\[ \pi_{ij}(\vec{v}_i) = \Pr[\text{item } j \text{ goes to } i \text{ if she reports } \vec{v}_i] \]

  ◦ “Interim allocation rule”

• BIC:

\[
\sum_j v_{ij} \pi_{ij}(\vec{v}_i) - p_i(\vec{v}_i) \geq \sum_j v_{ij} \pi_{ij}(\vec{v}_i') - p_i(\vec{v}_i')
\]

• Down to \( \Theta(nm \cdot \max_i |V_i|) \) variables and constraints!

• New problem: How do we know that there is an auction that corresponds to a given reduced form?
REDUCED FORMS

• One item, two bidders: \( V_1 = U\{A, B, C\}, V_2 = U\{D, E, F\} \)

• Question: Is the following r.f. feasible?

\[
\begin{align*}
\pi_{11}(A) &= 1 & \pi_{21}(D) &= \frac{2}{3} \\
\pi_{11}(B) &= \frac{1}{2} & \pi_{11}(E) &= \frac{5}{9} \\
\pi_{11}(C) &= 0 & \pi_{21}(F) &= 0
\end{align*}
\]

• \((A, D / E / F) \rightarrow 1\) wins \((\pi_{11}(A) = 1)\)

• \((B / C, D) \rightarrow 2\) wins \((\pi_{21}(D) = \frac{2}{3})\)

• \((B, F) \rightarrow 1\) wins \((\frac{2}{3}\) out of \(1/2, 1/6\) to go)

• \((C, E) \rightarrow 2\) wins \((\frac{1}{3}\) out of \(5/9, 2/9\) to go)

• \((B, E) \rightarrow \) ...

  • \(B\) needs to win with probability \(1/2\)
  • \(E\) needs to win with probability \(2/3\)
• Can we check if a reduced form is feasible quickly?
• Border’s theorem: The following a necessary and sufficient condition of a reduced form to be feasible. For every item \( j \) and every \( S_1 \subseteq V_1, \ldots, S_n \subseteq V_n \)

\[
\sum_{i \in [n]} \sum_{\tilde{v}_i \in S_i} \Pr[\tilde{v}_i] \pi_i(\tilde{v}_i) \leq 1 - \prod_{i \in [n]} (1 - \sum_{\tilde{v}_i \in S_i} \Pr[\tilde{v}_i])
\]

• LHS = Probability that winner has value in \( S_i \)
• RHS = Probability that there is someone with value in \( S_i \)
REDUCED FORMS

• For every item \( j \) and every \( S_1 \subseteq V_1, \ldots, S_n \subseteq V_n \)

\[
\sum_{i \in [n]} \sum_{\tilde{v}_i \in S_i} \Pr[\tilde{v}_i] \pi_i(\tilde{v}_i) \leq 1 - \prod_{i \in [n]} \left(1 - \sum_{\tilde{v}_i \in S_i} \Pr[\tilde{v}_i]\right)
\]

• That’s \( 2^{\sum_i |V_i|} \) conditions!

• [CDW’12]: We can check feasibility in time almost linear in \( \sum_i |V_i| \)
  ◦ Key result in solving the succinct LP.
For the remaining we focus on the case of a single additive buyer with $m$ independent items
CHARACTERIZATIONS OF THE OPTIMAL MECHANISM

• When is the revenue maximizing auction “nice”, even for a single buyer?
• For example, when is it optimal to post a price for the grand-bundle?
  ◦ Grand-bundle = all the items as a single bundle
• There are necessary and sufficient conditions! [DDT 15]
• Unfortunately, these conditions are not very intuitive
  ◦ Measure theory conditions
• Very interesting outcomes though:
  ◦ For every number of items $m$, there exists a $c$, such that the optimal mechanism for $m$ i.i.d. $U[c, c + 1]$ items is a grand-bundling mechanism
  ◦ On the other hand, for every $c$, there exists a number $m_0$, such that for all $m > m_0$, the grand-bundle mechanism is not optimal for $m$ i.i.d. $U[c, c + 1]$ items!
SIMPLE AND APPROXIMATELY OPTIMAL MECHANISMS

• Is selling only the grand bundle a good (constant) approximation to the optimal mechanism?

• No!
  ◦ Not even a good approximation to $SRev$
**Example:**

- $v_i \in \{0, M^i\}$, where $M$ is a large number
- $\Pr[v_i = M^i] = 1/M^i$
- $\text{Rev}(D_i) = 1$
  - So, $S\text{Rev} = m$
- $B\text{Rev} \leq \max_k M^k \cdot \Pr[\sum_j v_j \geq M^k]$
  - $\Pr[\sum_j v_j \geq M^k] \leq \sum_{j \geq k} \Pr[v_j = M^j] = M^{-j}$
  - $\sum_{j \geq k} M^{-j} = M^{1-k}/(M - 1)$
- $B\text{Rev} \leq 1 + 1/(M - 1)$
SIMPLE AND APPROXIMATELY OPTIMAL MECHANISMS

• Is selling each item separately a good (constant) approximation to the optimal mechanism?

• No!
  ◦ Example a bit too complicated...
  ◦ $m$ i.i.d. items from a “equal revenue” distribution: $F(x) = 1 - 1/x$
SIMPLE AND APPROXIMATELY OPTIMAL MECHANISMS

• What about the best of $SRev$ and $BRev$?
• Theorem [BILW 14]:

\[
\max\{SRev, BRev\} \geq \frac{1}{6} Rev
\]

• Some definitions
  ◦ $m =$ number of items
  ◦ $V_j$ random variable for the value of item $j$
  ◦ $f_j(v_j) = \Pr[V_j = v_j]$
  ◦ $R_j = \{\mathbf{v} : v_j \geq v_k, \forall k \in [m]\}$
    • Set of profiles where $j$ is the favorite item
PROOF SKETCH

• Two parts:

1. $\text{Rev} \leq \text{Benchmark}$
2. $\text{Benchmark} \leq 6\max\{S\text{Rev}, B\text{Rev}\}$

• Today: Part 1
A DETOUR: LAGRANGIAN DUALITY

• Optimization

\[
\begin{align*}
\text{max } x_1 + 3x_2 + 5x_3 \\
\text{Subject to } \\
x_2 + x_3 &\leq 10 \\
x_1 &\leq 2 \\
\ldots
\end{align*}
\]

• Lagrangian function

\[\mathcal{L}(x, \lambda) = x_1 + 3x_2 + 5x_3 + \lambda(10 - x_2 - x_3)\]
A DETOUR: LAGRANGIAN DUALITY

• Lagrangian function
  \[ \mathcal{L}(x, \lambda) = x_1 + 3x_2 + 5x_3 + \lambda(10 - x_2 - x_3) \]

• Let OPT be the optimal solution to the optimization problem

• Game:
  ◦ We pick \( \lambda \geq 0 \)
  ◦ Adversary picks \( x_1, \ldots \) that satisfy all the constraints \textbf{except} the one we “Lagrangified” in order to maximize \( \mathcal{L}(\vec{x}, \lambda) \)

• Theorem: \( \forall \lambda \geq 0, \text{OPT} \leq \max_{\vec{x}} \mathcal{L}(\vec{x}, \lambda) \)
A DETOUR: LAGRANGIAN DUALITY

• Lagrangian function
  \[ \mathcal{L}(x, \lambda) = x_1 + 3x_2 + 5x_3 + \lambda(10 - x_2 - x_3) \]

• Intuition:
  ◦ If \( \lambda = 0 \), then it’s as if we dropped that constraint
  ◦ If \( \lambda = \infty \), if we violate the Lagrangified constraint we pay an infinite penalty. But, if we strictly satisfy it we get a bonus
A DETOUR: LAGRANGIAN DUALITY

• Why would this be useful?
• Sometimes you know how to solve a problem if you “remove” a constraint
  ◦ Canonical example: Find the shortest path between $s$ and $t$, that also uses at most $k$ edges
  • Lagrangify the “at most $k$ edges” constraint.
For now, single buyer

Objective:

\[
\max \sum_{v \in V} p(v) \cdot \Pr[\text{value} = v]
\]

Constraints:

- IC: \( \forall v, v' \in V: vx(v) - p(v) \geq vx(v') - p(v') \)
- IR: \( \forall v \in V: vx(v) - p(v) \geq 0 \)
- Feasibility: \( \forall v \in V: 1 \geq x(v) \geq 0 \)
REVENUE

\[
\max \sum_{v \in V} p(v) \cdot f(v)
\]

\[
\forall v \in V, v' \in V \cup \{\perp\}: vx(v) - p(v) \geq vx(v') - p(v')
\]

\[
\forall v \in V: 1 \geq x(v) \geq 0
\]

• Lagrangify the IC+IR constraint!

\[
\mathcal{L} = \sum_{v \in V} f(v)p(v) + \sum_{v \in V} \sum_{v' \in V \cup \{\perp\}} \lambda(v, v') \cdot (vx(v) - p(v) - vx(v') + p(v'))
\]
REVENUE

• Re-arrange:

\[ L = \sum_{v \in V} x(v) \left( \sum_{v' \in V \cup \{\perp\}} v\lambda(v, v') - \sum_{v' \in V} v'\lambda(v', v) \right) \]

\[ + \sum_{v \in V} p(v)( f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\perp\}} \lambda(v,v')) \]

• Game:

  ◦ We pick \( \lambda(v, v') \geq 0 \) for all \( v, v' \)
  ◦ Adversary maximizes \( L \) subject to \( x(v) \in [0,1] \)

• Goal: make \( L^* \) as small as possible
REVENUE

\[ L = \sum_{v \in V} x(v) \left( \sum_{v' \in V \cup \{\perp\}} v \lambda(v, v') - \sum_{v' \in V} v' \lambda(v', v) \right) \]
\[ + \sum_{v \in V} p(v) \left( f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\perp\}} \lambda(v, v') \right) \]

- Observation: no constraints on \( p(v) \)
- Therefore:
  \[ f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\perp\}} \lambda(v, v') = 0 \]
- Otherwise, \( L^* = \infty \)
REVENUE

\[ f(v) + \sum_{v' \in V} \lambda(v', v) - \sum_{v' \in V \cup \{\bot\}} \lambda(v, v') = 0 \]

\( \lambda \)'s form a flow!!
• Simplify:
\[
\mathcal{L} = \sum_{v \in V} x(v) (vf(v) + \sum_{v' \in V} v\lambda(v', v) - \sum_{v' \in V} v'\lambda(v', v))
\]
\[
= \sum_{v \in V} f(v)x(v) \left( v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v) \right)
\]
• Game:
  ◦ We pick a flow \( \lambda \)
  ◦ Adversary tries to maximize \( \mathcal{L}(\lambda) \)
• Adversary will pointwise maximize
\[
\Phi(v) = v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v)
\]
EXAMPLE

• $D = U\{1,2,3,4,5\}$

$$
\Phi(v) = v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v)
$$

$$
\lambda(v, \bot) = f(v)
$$

$$
\Phi(v) = v
$$
EXAMPLE

- \( D = U\{1, 2, 3, 4, 5\} \)

\[
\Phi(v) = v - \frac{1}{f(v)} \sum_{v' \in V} \lambda(v', v)(v' - v)
\]

- \( \Phi(5) = 5 \)
- \( \Phi(4) = 4 - \frac{1}{\frac{1}{5}} \cdot \frac{1}{5} \cdot (5 - 4) = 3 \)
- \( \Phi(3) = 3 - \frac{1}{\frac{1}{5}} \cdot \frac{2}{5} \cdot (4 - 3) = 1 \)
- \( \Phi(2) = 2 - \frac{1}{\frac{1}{5}} \cdot \frac{3}{5} \cdot (3 - 2) = -1 \)
- \( \Phi(1) = 1 - \frac{1}{\frac{1}{5}} \cdot \frac{4}{5} \cdot (2 - 1) = -3 \)

Upper Bound = \( \frac{5 + 3 + 1}{5} = \frac{9}{5} \)

What’s OPT?
PROOF SKETCH

- Same idea for many items
- Have to find a good “flow”
PROOF SKETCH

• Lemma 1: $Rev$ is at most
  \[ \sum_{\vec{v}} \sum_{j} f(\vec{v}) \cdot x_j(\vec{v}) \cdot \phi_j(v_j) \cdot \mathbb{I}\{\vec{v} \in R_j\} \text{ (SINGLE)} \]
  \[ + \sum_{\vec{v}} \sum_{j} f(\vec{v}) \cdot x_j(\vec{v}) \cdot v_j \cdot \mathbb{I}\{\vec{v} \notin R_j\} \text{ (NONFAV)} \]

• Intuition:
  - SINGLE = Favorite item contributes its virtual value
  - NONFAV = Every other item contributes its value

• Theorem [BILW 14]: $\max\{SRev, BRev\} \geq \frac{1}{6} Rev$
  - Similar results exist for many buyers, even beyond additive valuation functions
DYNAMIC MECHANISMS

• Slight twist to the model
• Two items: one today, one tomorrow

Game:
• $D_1, D_2$ are public knowledge
• Buyer learns $v_1 \sim D_1$, submits $b_1$
• Item 1 and payments according to $x_1(b_1), p_1(b_1)$
• Buyer learns $v_2 \sim D_2$, submits $b_2$
• Item 2 and payments according to $x_2(b_1, b_2), p_2(b_1, b_2)$
DYNAMIC MECHANISMS

- When submitting $b_1$ buyer has to take into account how this will affect the (expected) utility she’ll get from item 2
- $D_1$ and $D_2$ could be correlated
- For now assume independence
- Independent?
  - Shouldn’t Myerson + Myerson be optimal?
  - Even if not optimal, it’s definitely a good approximation!
DYNAMIC MECHANISMS

\[ \nu_1 = 2^i \text{ with probability } 2^{-i}, \ i = 1 \ldots n \]
\[ \nu_2 = 2^i \text{ with probability } 2^{-i}, \ i = 1 \ldots 2^n \]
\hspace{1cm} \bullet \text{ With the remaining probability they’re equal to zero}
DYNAMIC MECHANISMS

• Myerson + Myerson = constant
• Consider the following auction
  ◦ $x_1(b_1) = 1, p_1(b_1) = b_1$
  ◦ $x_2(b_1, b_2) = b_1/E[D_2], p_2(b_1, b_2) = 0$
  ◦ So first day you pay your bid $b_1$
  ◦ Second you get it for free w.p. $b_1/E[D_2]$
• $E[\text{utility of reporting } b_1]$
  ◦ ut. from day 1 = $v_1 - b_1$
  ◦ $E[\text{ut. from day 2}] = \sum_{v_2} \Pr[v_2] v_2 \cdot \frac{b_1}{E[D_2]} = b_1$
• So, $E[\text{ut. of } b_1] = v_1$!
• $Rev = E[v_1] = n$
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