Mechanism Design III:
Simple single item auctions

Teachers: Ariel Procaccia and Alex Psomas (this time)
SO FAR

• Revelation Principle
• Single parameter environments
  ◦ Second price auctions
  ◦ Myerson’s lemma
  ◦ Myerson’s optimal auction
CORRECTION IN THE DEFINITION OF MHR

• \( \phi(v) = v - \frac{1 - F(v)}{f(v)} \)

• \( D \) is MHR if \( \frac{1 - F(v)}{f(v)} \) is monotone non-increasing.
TODAY

• Cremer-McLean for correlated buyers
• Prophet Inequalities
• Bulow-Klemperer
BEYOND INDEPENDENCE

• Myerson: Optimal auction for independent bidders.
• What if the bidders’ values are correlated?
  ◦ Very realistic!
• We’ll see a 2 agent instance of a result of Cremer and McLean [1998]
  ◦ They show how to extract the full social welfare under very mild conditions on the correlation
How much revenue does a second price auction make (in expectation)?

1. 8/6
2. 10/6
3. 12/6
4. 14/6

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What’s the maximum possible revenue an auction can make?

1. 8/6
2. 10/6
3. 12/6
4. 14/6
**CREMERS-MCLEAN**

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- $P_{i,j} = \Pr[v_2 = j \mid v_1 = i]$

\[
P = \begin{pmatrix}
1/2 & 1/4 & 1/4 \\
1/4 & 1/2 & 1/4 \\
1/4 & 1/4 & 1/2
\end{pmatrix}
\]

- $E[\text{utility of } v_1 = 1 \text{ from } SP] = 0$
- $E[\text{utility of } v_1 = 2 \text{ from } SP] = 1/4 \cdot 1 = 1/4$
- $E[\text{utility of } v_1 = 3 \text{ from } SP] = 1/4 \cdot 2 + 1/4 \cdot 1 = 3/4$
Observation: $P$ has full rank

Therefore, $P \cdot (x_1, x_2, x_3)^T = (0, \frac{1}{4}, \frac{3}{4})^T$ has a solution:

- $x_1 = -1, x_2 = 0, x_3 = 2$

_The magic part_

Consider the following bet $B_1$ for player 1:

- I pay you 1 if $v_2 = 1$
- Nothing happens if $v_2 = 2$
- You pay me 2 if $v_2 = 3$
Consider the following bet $B_1$ for player 1: (a) I pay you 1 if $v_2 = 1$, (b) Nothing happens if $v_2 = 2$, (c) You pay me 2 if $v_2 = 3$

What’s the expected value for taking this bet if $v_1 = 1$?
- $1/2 \cdot 1 + 1/4 \cdot 0 + 1/4 \cdot (-2) = 0$

What if $v_1 = 2$? $-1/4$

What if $v_1 = 3$? $-3/4$

Similar bet $B_2$ for player 2

Auction: Player $i$ is offered bet $B_i$. After the bet we’ll run a second price auction

- $E[utility \ of \ v_1 = 1] = E[utility \ of \ B_1] + E[utility \ from \ SP] = 0$
- $E[ut. \ of \ v_1 = 2] = -1/4 + 1/4 = 0$
- $E[ut. \ of \ v_1 = 3] = -3/4 + 3/4 = 0$
• Since buyers always have zero utility, and the item is always sold, the seller must be extracting all of the social welfare

• Expected revenue = 14/6

• With just happened???

• That’s a pretty weird auction!

• This “prediction” is very unlikely to be observed in practice.
MYERSON IS WEIRD

- \( n = 2. D_1 = U[0,1], D_2 = U[0,100] \)
- \( \phi_1(v_1) = 2v_1 - 1, \phi_2(v_2) = 2v_2 - 100 \)
- Optimal auction
  - When \( v_1 \leq 1/2 \) and \( v_2 \geq 50 \): Sell to 2 for 50
  - When \( v_1 > 1/2 \) and \( v_2 < 50 \): Sell to 1 for \( 1/2 \)
  - When \( 0 < 2v_1 - 1 < 2v_2 - 100 \): Sell to 2 for \( (99 + 2v_1)/2 \) (slightly over 50)
  - When \( 0 < 2v_2 - 100 < 2v_1 - 1 \): Sell to 1 for \( (2v_2 - 99)/2 \) (slightly over \( 1/2 \))

- Wth is this???
- Impossible to explain, unless you go through all of Myerson’s calculations!
OPTIMAL AUCTIONS ARE WEIRD

• Weirdness inevitable if you want optimality
• Weirdness inevitable if you’re 100% confident in the model
• Take away: Optimality requires complexity
• In the remainder: ask for simplicity and settle for approximately optimal auctions.
CRITIQUE #1: TOO COMPLEX

A (cool) detour: Prophet inequalities!
PROPHET INEQUALITY

• $n$ treasure boxes.
• Treasure in box $i$ is distributed according to known distribution $D_i$
• In stage $i$ you open box $i$ and see the treasure (realization of the random variable) $x_i$
• After seeing $x_i$ you either take it, or discard it forever and move on to stage $i + 1$
• What should you do?
• Our goal will be to compete against a prophet who knows the realizations of the $D_i$s
PROPHET INEQUALITY

\[
D_1 = U[0,60] \quad D_1 = \text{Exp}[1/60] \quad D_1 = N[1,1] \quad D_1 = U[0,100]
\]

\[
x_1 = 54 \quad x_2 = 52 \quad x_3 = 1 \quad x_4 = 61
\]

Our value is 52, Prophet gets 61
PROPHET INEQUALITY

• Optimal policy: Solve it backwards!
  ◦ If we get to the last box, we should clearly take $x_n$
  ◦ For the second to last, we should take $x_{n-1}$ if it’s larger than $E[x_n]$
  ◦ We should take $x_{n-2}$ only if it’s larger than the expected value of the optimal policy starting at $n - 1$, i.e.
    
    $$
    \Pr[x_{n-1} > E[x_n]] \cdot E[x_{n-1}|x_{n-1} > E[x_n]] + \Pr[x_{n-1} \leq E[x_n]] \cdot E[x_n]
    $$
    
    ◦ And so on...

• Ok, that’s pretty complicated...

• Any simpler policies?
  ◦ Focus on policies that set a single threshold $t$ and accept $x_i$ if it’s above $t$, otherwise reject
  ◦ How good are those?
Theorem: There exists a single threshold \( t^* \) such that the policy that accepts \( x_i \) when \( x_i \geq t^* \) gives expected reward at least \( \frac{1}{2} E \left[ \max_i x_i \right] \), i.e. at least half of what the prophet makes (in expectation).
PROPHET INEQUALITY

Proof

• \( Z^+ = \max\{z, 0\} \)
• Given a “threshold policy” with threshold \( t \), let \( q(t) = \Pr[\text{policy accepts no prize}] \)
• Large \( t \): large \( q(t) \), but big rewards
• Small \( t \): small \( q(t) \), but small rewards
• \( E[\text{reward}] \geq q(t) \cdot 0 + (1 - q(t)) \cdot t \)
• A little too pessimistic...
• When \( x_i \geq t \) we’ll count \( x_i \), not \( t \)
PROPHET INEQUALITY

\[ E[\text{reward}] = t(1 - q(t)) + \]
\[ \sum_i E[x_i - t|x_i \geq t \& x_j < t, \forall j \neq i] \cdot \Pr[x_i \geq t \& x_j < t, \forall j \neq i] \]
\[ = t(1 - q(t)) + \]
\[ \sum_i E[x_i - t|x_i \geq t] \cdot \Pr[x_i \geq t] \cdot \Pr[x_j < t, \forall j \neq i] \]
\[ = t(1 - q(t)) + \sum_i E[(x_i - t)^+] \cdot \Pr[x_j < t, \forall j \neq i] \]
\[ \geq t(1 - q(t)) + q(t) \sum_i E[(x_i - t)^+] \]

(we used that \( q(t) = \Pr[x_j < t, \forall j] \leq \Pr[x_j < t, \forall j \neq i] \) )
PROPHET INEQUALITY

\[ E[\text{reward}] \geq t(1 - q(t)) + q(t) \sum_i E[(x_i - t)^+] \]

\[ E[\max_i x_i] = E[t + \max_i (x_i - t)] \]
\[ = t + E[\max_i (x_i - t)] \]
\[ \leq t + E[\max_i (x_i - t)^+] \]
\[ \leq t + \sum_i E[(x_i - t)^+] \]

\[ t^* : q(t^*) = \frac{1}{2} \]

\[ E[\text{reward}] \geq \frac{t^*}{2} + \frac{1}{2} \sum_i E[(x_i - t^*)^+] \geq \frac{1}{2} E[\max_i x_i] \]
BACK TO AUCTIONS

- \( Rev = E[\sum_i \phi_i(v_i)x_i(v_i)] = E[\max_i \phi_i(v_i)^+] \)
- Pick \( t^* \) such that \( \Pr[\max_i \phi_i(v_i)^+ \geq t^*] = 1/2 \)
- Give item to bidder \( i \) if \( \phi_i(v_i) \geq t^* \)
- Prophet inequality gives
  \[
  E[\text{reward}] = E[\sum_i \phi_i(v_i)x_i(v_i) \geq \frac{1}{2} E[\max_i \phi_i(v_i)^+]]
  \]
- More concretely:
  - \( r_i = \phi_i^{-1}(t^*) \)
  - Remove all bidders with \( b_i < r_i \)
  - Run a second price with the remaining bidders
CRITIQUE #2: TOO MUCH DEPENDENCE ON THE DISTRIBUTION

• Optimal auction depends on the distribution
• Wasn’t the whole point of the Bayesian approach that this is unavoidable?
• We’ll assume that $v_i \sim D_i$ (in the analysis), but our auctions will not depend on the $D_i$s
  • “Prior independent” mechanism design
PRIOR INDEPENDENT MECHANISMS

• Sounds pretty optimistic...
• Existence of a good prior independent auction $A$ for (say) regular distributions implies that a single auction can compete with all the (uncountably many) optimal auctions, tailored to each distribution, simultaneously!
• Pretty wild!
• Any candidates?
  ◦ Second price auction!
BULOW-KLEMPERER THEOREM

• $OPT(n, D) =$ Expected revenue of optimal auction with $n$ i.i.d. buyers from $D$.
• $V(n, D) =$ Expected revenue of Vickrey with $n$ i.i.d. buyers from $D$.
• Theorem (1996): For all regular $D$ we have
  $$V(n + 1, D) \geq OPT(n, D)$$
• In more modern language: “The competition complexity of single-item auctions with regular distributions is 1”
  ◦ The competition complexity of $n$ bidders with additive valuations over $m$ independent, regular items is at least $\log m$ and at most $n + 2m - 2$ [EFFTW 17]
BULOW-KLEMPERER THEOREM

• Theorem (1996): For all regular $D$ we have
  \[ V(n + 1, D) \geq OPT(n, D) \]

• Intuitively: It is better to increase competition by a single buyer than invest in learning the underlying distribution!
BULOW-KLEMPERER THEOREM

Proof:

• Let $A$ be the following auction for $n + 1$ buyers from $D$:
  ◦ Run $OPT(n, D)$ on buyers $1, \ldots, n$
  ◦ If the item is not sold, give it for free to buyer $n + 1$

• Obvious observation 1: $Rev(A) = OPT(n, D)$
• Obvious observation 2: $A$ always allocates the item.
BULOW-KLEMPERER THEOREM

• Non obvious:
• The second price auction is the revenue maximizing auction over all auctions that always allocate the item.
  ◦ Why?
• Therefore

$$V(n + 1, D) \geq Rev(A) = OPT(n, D)$$
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  - Cremer-McLean auction for correlated buyers
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